On the product of distributions and the change of variable

By BRIAN FISHER (Leicester)

In the following we let N be the neutrix, see van der Corput [1], having domain $N'=\{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions linear sums of the functions $n^\lambda \ln^{i-1} n$, $\ln^i n$ for $\lambda>0$ and $i=1, 2, \ldots$, and all functions which converge to zero as n tends to infinity.

It follows that if

$$f(n) = n^3+2\ln n+2+n^{-1}$$

then the neutrix limit as n tends to infinity of $f(n)$ exists and

$$\mathrm{N}\!-\!\lim_{n\to\infty} f(n) = 2.$$

We will also use the neutrix R having domain $R'=\{1, 2, \ldots, r, \ldots\}$ and range R'' the real numbers, with negligible functions linear sums of the functions $r^\lambda \ln^{i-1} r$, $\ln^i r$ for $\lambda>0$ and $i=1, 2, \ldots$, and all functions which converge to zero as r tends to infinity.

Now let ϱ be a fixed infinitely differentiable function having the properties

(i) $\varrho(x) = 0$ for $|x| \geqq 1$,

(ii) $\varrho(x) \geqq 0$,

(iii) $\varrho(x) = \varrho(-x)$,

(iv) $\int_{-1}^{1} \varrho(x)\,dx = 1.$

We define the function δ_n by $\delta_n(x)=n\varrho(nx)$ for $n=1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a regular sequence converging to the Dirac delta-function δ.

The following definition for the product of two distributions was given in [3].

Definition 1. Let F and G be distributions and let $G_n=G*\delta_n$. We say that the neutrix product $F\circ G$ of F and G exists and is equal to the distribution H on the open interval (a, b) if

$$\mathrm{N}\!-\!\lim_{n\to\infty} (FG_n, \varphi) = (H, \varphi) \tag{1}$$

for all test functions φ with compact support contained in the interval (a, b).

Note that if we put $F_r = F*\delta_r$ and use the neutrix R we have

$$(FG_n, \varphi) = \mathrm{R}-\lim_{r\to\infty} (F_r G_n, \varphi)$$

and so equation (1) could be replaced by the equation

$$\mathrm{N}-\lim_{n\to\infty} [\mathrm{R}-\lim_{r\to\infty} (F_r G_n, \varphi)] = (H, \varphi). \tag{2}$$

The next definition for the change of variable in distributions was given in [4].

Definition 2. Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f)$ exists and is equal to the distribution h on the open interval (a, b) if

$$\mathrm{N}-\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\,dx = (h, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b), where $F_n = F*\delta_n$.

This definition was considered in [5] for the case where f is an infinitely differentiable function.

We now consider the problem of defining a neutrix product of the form $F(f)\circ G(g)$, where F and G are distributions and f and g are locally summable functions.

Let us suppose that the distributions $F(f)$ and $G(g)$ exist and are equal to F_1 and G_1 respectively on the open interval (a, b). Then if the neutrix product $F_1\circ G_1$ exists and is equal to H_1 on the open interval (a, b), we have on using equation (2)

$$\mathrm{N}-\lim_{n\to\infty} [\mathrm{R}-\lim_{r\to\infty} (F_{1r} G_{1n}, \varphi)] = (H_1, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b), where $F_{1r} = F_1*\delta_r$ and $G_{1n} = G_1*\delta_n$.

However, we should not necessarily expect the neutrix product $F_1\circ G_1$ to be equal to the neutrix product $F(f)\circ G(g)$. The distribution $F(f) = F_1$ is defined by the sequence $\{F_r(f)\}$, using the neutrix R, and the distribution $G(g) = G_1$ is defined by the sequence $\{G_n(g)\}$, using the neutrix N. On taking the the neutrix limits as r and n tend to infinity, we are just picking out the finite parts of the limits and omitting the divergent parts. These divergent parts should be taken into account when defining the neutrix product $F(f)\circ G(g)$. It is therefore more reasonable to use the sequence $\{F_r(f)\}$ rather than the sequence $\{F_{1r}\}$ to define the distribution $F(f) = F_1$ and the sequence $\{G_n(g)\}$ rather than the sequence $\{G_{1n}\}$ to define the distribution $G(g) = G_1$, when defining the neutrix product $F(f)\circ G(g)$.

This leads us to the following definition for the neutrix product $F(f)\circ G(g)$.

Definition 3. Let F and G be distributions, let f and g be locally summable functions and let $F_r = F*\delta_r$ and $G_n = G*\delta_n$. We say that the neutrix product $F(f)\circ G(g)$ of $F(f)$ and $G(g)$ exists and is equal to the distribution H on the open interval (a, b) if $F_r(f)G_n(g)$ is a locally summable function on the interval (a, b) and if

$$\text{N}\underset{n\to\infty}{-\lim}\,[\text{R}\underset{r\to\infty}{-\lim}\,(F_r(f)G_n(g), \varphi)] =$$

$$= \text{N}\underset{n\to\infty}{-\lim}\,[\text{R}\underset{r\to\infty}{-\lim}\int_a^b F_r(f(x))G_n(g(x))\varphi(x)\,dx] = (H, \varphi) \tag{3}$$

for all test functions φ with compact support contained in the interval (a, b).

Note that for the particular case $f(x)=g(x)=x$, equations (2) and (3) are identical. Definition 3 then reduces to definition 1.

In the following x_+^λ is the locally summable function defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x \leqq 0 \end{cases}$$

for $\lambda > -1$.

Example 1. Let $F = x_+^{1/2}$, $G = \delta'$, $f = x_+^2$ and $g = x_+$. Then

$$F(f)\circ G(g) = -\frac{1}{2}\delta. \tag{4}$$

Proof. We notice that the sequence $\{F_r(x) = x_+^{1/2} * \delta_r(x)\}$ is a sequence of continuous functions which is pointwise convergent to the continuous function $x_+^{1/2}$. It follows that the sequence $\{F_r(x^2)\}$ is a sequence of continuous functions which is pointwise convergent to the continuous function $F(f) = x_+$.

The function $G_n(x_+)$ is the continuous function defined by

$$G_n(x_+) = \begin{cases} \delta_n'(x), & x > 0, \\ 0, & x \leqq 0. \end{cases}$$

The function $F_r(x^2)\,G_n(x_+)$ is therefore continuous and so locally summable. Further, the sequence $\{F_r(x^2)G_n(x_+)\colon r = 1, 2, \ldots\}$ is pointwise convergent to the continuous function $x_+\,G_n(x_+)$.

Now let φ be an arbitrary test function with compact support. Then

$$(F_r(x^2)G_n(x_+), \varphi) = \int_0^{1/n} F_r(x^2)\delta_n'(x)\varphi(x)\,dx$$

and so

$$\text{R}\underset{r\to\infty}{-\lim}\,(F_r(x^2)G_n(x_+), \varphi) = \lim_{r\to\infty}\int_0^{1/n} F_r(x^2)\delta_n'(x)\varphi(x)\,dx =$$

$$= \int_0^{1/n} x\delta_n'(x)\varphi(x)\,dx = -\int_0^{1/n}[\varphi(x) + x\varphi'(x)]\delta_n(x)\,dx =$$

$$= -\int_0^{1/n}[\varphi(0) + 0(1/n)]\delta_n(x)\,dx.$$

Thus

$$\mathrm{N}-\lim_{n\to\infty} [\mathrm{R}-\lim_{r\to\infty} (F_r(x^2)G_n(x_+), \varphi)] = -\lim_{n\to\infty} \int_0^{n/1} [\varphi(0)+0(1/n)]\delta_n(x)\,dx =$$

$$= -\frac{1}{2}\varphi(0) = -\frac{1}{2}(\delta, \varphi)$$

and equation (4) follows.

We have noted in this example that $F(f)=F_1=x_+$. Further $G(g)=G_1=\delta'(x_+)=\frac{1}{2}\delta'$, see Theorem 5 of [4]. Thus

$$F_1\circ G_1 = \frac{1}{2}x_+\circ\delta' = -\frac{1}{2}\delta,$$

see Theorem 5 of [2]. It follows that in this example

$$F(f)\circ G(g) = F_1\circ G_1.$$

Example 2. Let $F=x_+^{-1/2}$, $G=\delta, f=x$ and $g=x_+^{1/2}$. Then

$$F(f)\circ G(g) = \delta. \tag{5}$$

PROOF. The sequence $\{F_r(x)=F_r(f)=x_+^{-1/2}*\delta_r(x)\}$ is a sequence of continuous functions which is pointwise convergent to the locally summable function $x_+^{-1/2}$ for $x\neq 0$.

The function $G_n(x_+^{1/2})$ is the continuous function defined by

$$G_n(x_+^{1/2}) = \begin{cases} \delta_n(x_+^{1/2}), & x>0, \\ n\varrho(0), & x\leqq 0. \end{cases}$$

The function $F_r(x)\,G_n(x_+^{1/2})$ is therefore a locally summable function and the sequence $\{F_r(x)\,G_n(x_+^{1/2})\colon r=1, 2, \ldots\}$ is pointwise convergent to the summable function $x_+^{-1/2}G_n(x_+^{1/2})$, for $x\neq 0$.

Now let φ be an arbitrary test function with compact support. Then

$$\big(F_r(x)G_n(x_+^{1/2}), \varphi\big) = \int_0^{1/n^2} F_r(x)\delta_n(x^{1/2})\varphi(x)\,dx$$

and so

$$\mathrm{R}-\lim_{n\to\infty} \big(F_r(x)G_n(x_+^{1/2}), \varphi\big) = \lim_{r\to\infty} \int_0^{1/n^2} F_r(x)\delta_n(x^{1/2})\varphi(x)\,dx =$$

$$= \int_0^{1/n^2} x_+^{-1/2}\delta_n(x^{1/2})\varphi(x)\,dx = 2\int_0^{1/2} \delta_n(y)\varphi(y^2)\,dy =$$

$$= 2\int_0^{1/2} [\varphi(0)+O(n^{-2})]\delta_n(y)\,dy,$$

where the substitution $x=y^2$ has been made. Thus

$$\mathrm{N}\underset{n\to\infty}{-\lim}\,[\mathrm{R}\underset{r\to\infty}{-\lim}(F_r(x)G_n(x_+^{1/2}),\varphi)] = 2\lim_{n\to\infty}\int_0^{1/n}[\varphi(0)+O(n^{-2})]\delta_n(y)\,dy =$$

$$= \varphi(0) = (\delta,\varphi)$$

and equation (5) follows.

Now let us consider the distribution $\delta(x_+^{1/2})$. Then for arbitrary test function φ we have

$$(\delta_n(x_+^{1/2}),\varphi) = \int_0^{1/n^2}\delta(x^{1/2})\varphi(x)\,dx = 2\int_0^{1/n} y\delta_n(y)\varphi(y^2)\,dy = O(n^{-1}).$$

Thus

$$\mathrm{N}\underset{n\to\infty}{-\lim}\,(\delta_n(x_+^{1/2}),\varphi) = 0$$

and so $\delta(x_+^{1/2})=0$. It follows that in this example

$$F(f)\circ G(g) \neq F_1\circ G_1$$

even though $f(x)=x$.

We do however havethe following theorem.

Theorem. *Let F and G be distributions, let f be a locally summable function and let g be an infinitely differentiable function. Then if the distributions $F(f)=F_1$ and $G(g)=G_1$ exist and the product $F(f)\circ G(g)$ exists on the open interval (a,b)*

$$F(f)\circ G(g) = F_1\circ G(g) \tag{6}$$

on the interval (a,b). In particular, if $g(x)=x$

$$F(f)\circ G(g) = F_1\circ G_1 \tag{7}$$

on the interval (a,b).

Proof. Let φ be an arbitrary test function with compact support contained in the interval (a,b). Then from Definitions 2 and 3 we have

$$\mathrm{R}\underset{r\to\infty}{-\lim}\,(F_r(f),\varphi) = (F(f),\varphi) = (F_1,\varphi),$$

$$\mathrm{N}\underset{n\to\infty}{-\lim}\,(G_n(g),\varphi) = (G(g),\varphi) = (G_1,\varphi),$$

$$\mathrm{N}\underset{n\to\infty}{-\lim}\,[\mathrm{R}\underset{r\to\infty}{-\lim}\,(F_r(f)G_n(g),\varphi)] = (F(f)\circ G(g),\varphi). \tag{8}$$

Since g is an infinitely differentiable function, $G_n(g)$ is an infinitely differentiable function and so $G_n(g)\varphi$ is a test function with compact support contained in the interval (a,b). Thus

$$\mathrm{R}\underset{r\to\infty}{-\lim}(F_r(f)G_n(g),\varphi) = \mathrm{R}\underset{r\to\infty}{-\lim}\,(F_r(f),G_n(g)\varphi) =$$

$$= (F(f),G_n(g)\circ\varphi) = (F_1G_n(g),\varphi).$$

Equation (8) implies that

$$\operatorname*{N-lim}_{n\to\infty} \left(F_1 G_n(g), \varphi\right)$$

exists and

$$\operatorname*{N-lim}_{n\to\infty} \left(F_1 G_n(g), \varphi\right) = F\big((f)\circ G(g), \varphi\big). \tag{9}$$

Now let us consider the product $F_1 \circ G(g)$. Putting $F_{1r} = F_1 * \delta_r$ and using Definition 3 we have

$$\begin{aligned}\left(F_1 \circ G(g), \varphi\right) &= \operatorname*{N-lim}_{n\to\infty} \left[\operatorname*{R-lim}_{r\to\infty} \left(F_{1r} G_n(g), \varphi\right)\right] = \\ &= \operatorname*{N-lim}_{n\to\infty} \left(F_1 G_n(g), \varphi\right) = \left(F(f)\circ G(g), \varphi\right)\end{aligned}$$

on using equation (9). Equation (6) now follows.

When $g(x)=x$, we have $G(g)=G_1$ and equation (7) follows. This completes the proof of the theorem.

References

[1] J. G. VAN DER CORPUT, Introduction to the neutrix calculus, *J. Analyse Math.* **7** (1959—60), 291—398.

[2] B. FISHER, On defining the product of distributions, *Math. Nachr.* **99** (1980), 239—249.

[3] B. FISHER, A non-commutative neutrix product of distributions, *Math. Nachr.* **108** (1982), 117—127.

[4] B. FISHER, On defining the distribution $\delta^{(r)}\big(f(x)\big)$ for summable f, *Publ. Math. (Debrecen)* **32** (1985), 233—243.

[5] B. FISHER and Y. KURIBAYASHI, Changing the variable in distributions, *Dem. Math.*, **17** (1984), 499—514.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY
LEICESTER
LE1 7RH

(Received November 5, 1985)