# A characterization of exponential polynomials by a class of functional equations

By ZBIGNIEW GAJDA (Katowice)

## 1. *Introduction*

Let us consider a homogeneous linear differential equation

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f = 0, \tag{1}$$

where $a_0, \ldots, a_n$ are some complex constants and $f$ is an $n$-times differentiable complex-valued function defined on the real line. Suppose $\lambda_1, \ldots, \lambda_k$ to be all the distinct roots of the characteristic polynomial

$$P(\lambda) := a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0.$$

From the general theory of linear differential equations it follows that any solution $f$ of equation (1) has the form of an exponential polynomial

$$f(x) = \sum_{i=1}^{k} e^{\lambda_i x} P_i(x), \quad x \in \mathbf{R}, \tag{2}$$

$P_i$ being an arbitrary polynomial of degree less than the multiplicity of the root $\lambda_i$ $(i=1, \ldots, k)$.

Conversely, to each function $f$ of form (2) one can assign a homogeneous linear differential equation for which $f$ is a solution.

These facts seem to explain the importance of the concept of generalized exponential polynomial introduced and studied recently by many authors (see e.g. [3], [5], [6], and [8]).

In order to formulate the precise definition of this notion let us fix the necessary terminology. In the sequel $(G, +)$ will always denote an Abelian group and $X$ will stand for a linear space over a field $K$ of characteristic zero.

For a given element $h \in G$ we define the difference operator $\Delta_h$ with the span $h$ as follows:

$$\Delta_h f(x) := f(x+h) - f(x), \quad x \in G, \quad f \in X^G.$$

The superposition of operators $\Delta_{h_1}, \ldots, \Delta_{h_n}$ will be denoted by $\Delta_{h_1 \ldots h_n}$ and if $h_1 = \ldots = h_n = h$ then we shall shortly write $\Delta_h^n$ instead of $\underbrace{\Delta_{h \ldots h}}_{n \text{ times}}$.

A function $p\colon G\to X$ is called a polynomial function of degree less than $n$ ($n\in\mathbf{N}$) if and only if

$$\Delta_h^n p(x) = 0 \quad \text{for all } x,\quad h\in G. \tag{3}$$

Additionally, we assume that $\Delta_h^0 p := p$ and by a polynomial of degree less than zero we mean the function identically equal to zero.

It is well known (cf. [1], [4], and [7]) that any polynomial function $p$ of degree less than $n$ admits the unique representation

$$p = \sum_{i=0}^{n-1} \gamma_i, \tag{4}$$

where $\gamma_0$ is a constant and $\gamma_i$ $(i=1, \ldots, n-1)$ is a diagonalization of a symmetric $i$-additive function. The degree of the polynomial function $p$ is the highest index of a non-zero term in representation (4).

A function $m\colon G\to K\setminus\{0\}$ is said to be an exponential (or multiplicative) function if it is a homomorphism of the group $G$ into the multiplicative group of the field $K$, i.e.

$$m(x+y) = m(x)m(y), \quad x, y\in G.$$

Finally, we shall say that a function $f\colon G\to X$ is a generalized exponential polynomial if there exist exponential functions $m_1, \ldots, m_k\colon G\to K\setminus\{0\}$ and polynomial functions $p_1, \ldots, p_k\colon G\to X$ such that

$$f(x) = \sum_{i=1}^{k} m_i(x)p_i(x), \quad x\in G. \tag{5}$$

In connection with what has been mentioned at the beginning of this section it is natural to conjecture that the functions of form (5) should yield the general solution of a functional equation related in a certain sense to equation (1). In order to obtain a suitable equation one might try to replace all the derivatives in (1) by difference operators of the same order. This procedure leads, however, to an equation of the form

$$\sum_{i=0}^{n} \alpha_i f(x+ih) = 0, \quad x, h\in G$$

which has been examined extensively (cf. [2] and [7]) and fails to have more solutions than equation (3).

On the other hand, equation (1) may also be expressed in Heaviside's form

$$e^{\lambda_1 x} D^{n_1} e^{-\lambda_1 x} \ldots e^{\lambda_k x} D^{n_k} e^{-\lambda_k x} f(x) = 0, \quad x\in\mathbf{R} \tag{6}$$

in which $D^{n_i}$ $(i=1, \ldots, k)$ denotes the $n_i$-th iteration of the differential operator. We adopt here the convention that each operator $D^{n_i}$ acts on the whole expression standing on its right-hand side.

Now, let us replace all the differential operators occuring in (6) by difference operators and the multiplication by $e^{\lambda x}$ and $e^{-\lambda x}$-by operators $A_m$ and $B_m$ defined as follows:

$$A_m f(x) := m(x) f(x), \qquad x \in G, \quad f \in X^G;$$
$$B_m f(x) := m(x)^{-1} f(x), \quad x \in G, \quad f \in X^G,$$

$m$ being an exponential function. Transforming (6) in such a way we obtain the following functional equation:

$$(7) \qquad (A_{m_1} \Delta_{h_1}^{n_1} B_{m_1} \dots A_{m_k} \Delta_{h_k}^{n_k} B_{m_k}) f(x) = 0, \quad x, h_1, \dots, h_k \in G,$$

where the multiplication stands for the superposition of operators.

It turns out that generalized exponential polynomials can be characterized as solutions of equation (7). In what follows we are going to prove this result.

## 2. *Properties of the operator* $A_m \Delta_h B_m$

Let $m, n \colon G \to K \setminus \{0\}$ be two exponential functions and suppose $f \colon G \to X$ to be an arbitrary transformation. On account of the multiplicativity of $m$ and $n$ we get

$$(8) \qquad A_m \Delta_h B_m f(x) = m(x)[m(x+h)^{-1} f(x+h) - m(x)^{-1} f(x)] =$$
$$= m(h)^{-1} f(x+h) - f(x), \quad x, h \in G,$$

and hence

$$(9) \qquad (A_m \Delta_h B_m)(A_n \Delta_l B_n) f(x) = (A_m \Delta_h B_m)[n(l)^{-1} f(x+l) - f(x)] =$$
$$= m(h)^{-1} n(l)^{-1} f(x+h+l) - m(h)^{-1} f(x+h) - n(l)^{-1} f(x+l) + f(x),$$
$$x, h, l \in G.$$

As an immediate consequence of the symmetry of (9) we have

$$(10) \qquad (A_m \Delta_h B_m)(A_n \Delta_l B_n) f(x) = (A_n \Delta_l B_n)(A_m \Delta_h B_m) f(x), \quad x, h, l \in G.$$

Moreover, in view of the following obvious relation:

$$(11) \qquad (A_m \Delta_{h_1 \dots h_r} B_m) f(x) = (A_m \Delta_{h_r} B_m) \dots (A_m \Delta_{h_1} B_m) f(x), \quad x, h_1, \dots, h_r \in G,$$

applying (10) sufficiently many times, we deduce that

$$(12) \qquad (A_m \Delta_{h_1 \dots h_r} B_m)(A_n \Delta_{l_1 \dots l_s} B_n) f(x) = (A_n \Delta_{l_1 \dots l_s} B_n)(A_m \Delta_{h_1 \dots h_r} B_m) f(x),$$
$$x, h_1, \dots, h_r, l_1, \dots, l_s \in G.$$

In view of (12), if we are given a system of $k$ exponential functions $m_1, \dots, m_k$, then instead of

$$(A_{m_1} \Delta_{h_{1,1} \dots h_{1,n_1}} B_{m_1}) \dots (A_{m_k} \Delta_{h_{k,1} \dots h_{k,n_k}} B_{m_k}) f(x)$$

we can write

$$\left( \prod_{i=1}^{k} A_{m_i} \Delta_{h_{i,1} \dots h_{i,n_i}} B_{m_i} \right) f(x)$$

without paying attention to the succession of the operators

$$A_{m_i}\Delta_{h_{i,1}\dots h_{i,n_i}}B_{m_i} \quad (i=1,\dots,k).$$

**Lemma 1.** *If* $m_1,\dots,m_k\colon G\to K\setminus\{0\}$ *are given exponential functions, then the following two equations are equivalent:*

(i) $\left(\prod_{i=1}^{k} A_{m_i}\Delta_{h_i}^{n_i}B_{m_i}\right)f(x)=0$ *for all* $x, h_i\in G, \quad i=1,\dots,k;$

(ii) $\left(\prod_{i=1}^{k} A_{m_i}\Delta_{h_{i,1}\dots h_{i,n_i}}B_{m_i}\right)f(x)=0$ for all

$x, h_{i,1}\dots, h_{i,n_i}\in G, \quad i=1,\dots,k.$

Proof. It is enough to prove that (i) implies (ii). According to Theorem 2 from [1], for each $i=1,\dots,k$ one can select a finite set $J_i$, rational numbers $r_{i,j}$ and elements $u_{i,j}, v_{i,j}\in G$ (dependent on $h_{i,1},\dots,h_{i,n_i}$) such that

$$\Delta_{h_{i,1}\dots h_{i,n_i}}f(x)=\sum_{j\in J_i} r_{i,j}\Delta_{u_{i,j}}^{n_i}f(x+v_{i,j}), \quad x\in G.$$

Hence, with the aid of the notation

$$T_v f(x):=f(x+v), \quad x, v\in G, \tag{13}$$

it follows that

$$\begin{aligned}(A_{m_i}\Delta_{h_{i,1}\dots h_{i,n_i}}B_{m_i})f(x) &=\\ &= m_i(x)\sum_{j\in J_i} r_{i,j}\Delta_{u_{i,j}}^{n_i}\big(m_i(x+v_{i,j})^{-1}f(x+v_{i,j})\big)=\\ &= \sum_{j\in J_i} r_{i,j}m_i(v_{i,j})^{-1}[m_i(x)\Delta_{u_{i,j}}^{n_i}m_i(x)^{-1}f(x+v_{i,j})]=\\ &= \sum_{j\in J_i} r_{i,j}m_i(v_{i,j})^{-1}(A_{m_i}\Delta_{u_{i,j}}^{n_i}B_{m_i})T_{v_{i,j}}f(x), \quad x\in G.\end{aligned}$$

Furthermore, if $m$ is an exponential function, then

$$\begin{aligned}(A_m\Delta_h B_m)T_v f(x) &= m(h)^{-1}f(x+v+h)-f(x+v)=\\ &= T_v\big(m(h)^{-1}f(x+h)-f(x)\big)=T_v(A_m\Delta_h B_m)f(x), \quad x, v, h\in G.\end{aligned}$$

Hence and from (11) we infer that for every positive integer $n$ the operators $A_m\Delta_h^n B_m$ and $T_v$ commute. As a result we derive

$$\begin{aligned}\left(\prod_{i=1}^{k} A_{m_i}\Delta_{h_{i,1}\dots h_{i,n_i}}B_{m_i}\right)f(x) &= \left(\prod_{i=1}^{k}\left(\sum_{j\in J_i} r_{i,j}m_i(v_{i,j})^{-1}A_{m_i}\Delta_{u_{i,j}}^{n_i}B_{m_i}T_{v_{i,j}}\right)\right)f(x)=\\ &= \sum_{j_1\in J_1}\dots\sum_{j_k\in J_k} r_{1,j_1}\dots r_{k,j_k}m_1(v_{1,j_1})^{-1}\dots m_k(v_{k,j_k})^{-1}\times\\ &\times T_{v_{1,j_1}}\dots T_{v_{k,j_k}}(A_{m_1}\Delta_{u_{1,j_1}}^{n_1}B_{m_1}\dots A_{m_k}\Delta_{u_{k,j_k}}^{n_k}B_{m_k})f(x)\end{aligned}$$

which ensures that (i) implies (ii), and completes the proof.

Let us note that for exponential functions $m, n\colon G\to K\setminus\{0\}$ and for an arbitrary transformation $\omega\colon G\to X$ the following formula holds:

$$m(h+l)[n(h)-m(h)]A_m\Delta_{h+l}B_m\omega(x)- \tag{14}$$

$$-m(h)[n(h+l)-m(h+l)]A_m\Delta_h B_m\omega(x) = m(h)n(h+l)(A_n\Delta_l B_n)(A_m\Delta_h B_m)\omega(x)-$$

$$-m(h+l)n(h)(A_n\Delta_h B_n)(A_m\Delta_l B_m)\omega(x), \quad x, h, l\in G.$$

Indeed, keeping in mind relations (8) and (9) we can perform the following calculations:

$$m(h+l)[n(h)-m(h)]A_m\Delta_{h+l}B_m\omega(x)-m(h)[n(h+l)-m(h+l)]A_m\Delta_h B_m\omega(x) =$$

$$= m(h+l)[n(h)-m(h)][m(h+l)^{-1}\omega(x+h+l)-\omega(x)] -$$

$$-m(h)[n(h+l)-m(h+l)][m(h)^{-1}\omega(x+h)-\omega(x)] =$$

$$= [n(h)-m(h)]\omega(x+h+l)+[m(h+l)-n(h+l)]\omega(x+h)+$$

$$+[m(h)n(h+l)-m(h+l)n(h)]\omega(x) =$$

$$= m(h)n(h+l)[m(h)^{-1}n(l)^{-1}\omega(x+h+l)-m(h)^{-1}\omega(x+h)-$$

$$-n(l)^{-1}\omega(x+l)+\omega(x)]-$$

$$-m(h+l)n(h)[m(l)^{-1}n(h)^{-1}\omega(x+h+l)-m(l)^{-1}\omega(x+l)-$$

$$-n(h)^{-1}\omega(x+h)+\omega(x)] =$$

$$= m(h)n(h+l)(A_n\Delta_l B_n)(A_m\Delta_h B_m)\omega(x)-$$

$$-m(h+l)n(h)A_n\Delta_h B_n)(A_m\Delta_l B_m)\omega(x),$$

which was to be shown.

Finally, observe that if the function $\omega$ is identically equal to a constant $c$, then

$$A_m\Delta_h B_m\omega(x) = m(h)^{-1}\omega(x+h)-\omega(x) = \big(m(h)^{-1}-1\big)c, \quad x\in G. \tag{15}$$

The foregoing formulas will play an essential role in the proof of our main result which we are going to present in the next section.

## 3. *A characterization of exponential polynomials*

**Theorem 1.** *Suppose that $m_1, \ldots, m_k\colon G\to K\setminus\{0\}$ are pairwise different exponential functions and let $n_1, \ldots, n_k$ be non-negative integers. Then a function $f\colon G\to X$ satisfies the equation*

$$\Big(\prod_{i=1}^{k} A_{m_i}\Delta_{h_i}^{n_i}B_{m_i}\Big)f(x) = 0, \quad x, h_i\in G, \quad i = 1, \ldots, k \tag{$(a)_{n_1\ldots n_k}$}$$

*if and only if f is of the form*

$(b)_{n_1\dots n_k}$ $$f=\sum_{i=1}^{k} m_i p_i,$$

*where* $p_i\colon G\to X$ $(i=1,\dots,k)$ *is a polynomial function of degree less then* $n_i$.

PROOF. First we shall prove that $(b)_{n_1\dots n_k}$ *implies* $(a)_{n_1\dots n_k}$.

If $n_1$ is a non-negative integer and $f=m_1p_1$ with a polynomial function $p_1$ of degree less than $n_1$, then we have

$$A_{m_1}\Delta_{h_1}^{n_1}B_{m_1}f(x)=m_1(x)\Delta_{h_1}^{n_1}p_1(x)=0,\quad x,h_1\in G.$$

This proves that $(b)_{n_1}$ implies $(a)_{n_1}$.

Now suppose that $(b)_{n_1\dots n_k}$ implies $(a)_{n_1\dots n_k}$ for some $k\geqq 1$ and all non-negative integers $n_1,\dots,n_k$. Choose arbitrarily integers $n_1,\dots,n_{k+1}\geqq 0$ and assume that

$$f=\sum_{i=1}^{k+1} m_i p_i,$$

where $m_1,\dots,m_{k+1}$ are exponential functions and

$$\Delta_h^{n_i}p_i(x)=0,\quad x,h_i\in G,\quad i=1,\dots,k+1.$$

For a fixed element $h_{k+1}\in G$ put

$$q_i(x):=\sum_{j=0}^{n_{k+1}}(-1)^{n_{k+1}-j}\binom{n_{k+1}}{j}\frac{m_i(h_{k+1})^j}{m_{k+1}(h_{k+1})^j}p_i(x+jh_{k+1}),$$
$$x\in G,\quad i=1,\dots,k,$$

and

$$g(x):=A_{m_{k+1}}\Delta_{h_{k+1}}^{n_{k+1}}B_{m_{k+1}}f(x),\quad x\in G.$$

It is readily seen that $q_i$ is a polynomial function of degree less than $n_i$ $(i=1,\dots,k)$ Moreover,

$$g(x)=m_{k+1}(x)\Delta_{h_{k+1}}^{n_{k+1}}m_{k+1}(x)^{-1}\Big(\sum_{i=1}^{k+1}m_i(x)p_i(x)\Big)=$$
$$=m_{k+1}(x)\Delta_{h_{k+1}}^{n_{k+1}}p_{k+1}(x)+m_{k+1}(x)\Delta_{h_{k+1}}^{n_{k+1}}\Big(\sum_{i=1}^{k}m_i(x)m_{k+1}(x)^{-1}p_i(x)=$$
$$=\sum_{i=1}^{k}m_{k+1}(x)\Delta_{h_{k+1}}^{n_{k+1}}m_i(x)m_{k+1}(x)^{-1}p_i(x)=$$
$$=\sum_{i=1}^{k}m_{k+1}(x)\sum_{j=0}^{n_{k+1}}(-1)^{n_{k+1}-j}\binom{n_{k+1}}{j}\frac{m_i(x+jh_{k+1}}{m_{k+1}(x+jh_{k+1})}p_i(x+jh_{k+1})=$$
$$=\sum_{i=1}^{k}m_i(x)\sum_{j=0}^{n_{k+1}}(-1)^{n_{k+1}-j}\binom{n_{k+1}}{j}\frac{m_i(h_{k+1})^j}{m_{k+1}(h_{k+1})^j}p_i(x+jh_{k+1})=$$
$$=\sum_{i=1}^{k}m_i(x)q_i(x),\quad x\in G.$$

Thus, by hypothesis, we have

$$(16)\qquad \Big(\prod_{i=1}^{k+1} A_{m_i}\Delta_{h_i}^{n_i}B_{m_i}\Big)f(x) = \Big(\prod_{i=1}^{k} A_{m_i}\Delta_{h_i}^{n_i}B_{m_i}\Big)g(x) = 0, \quad x, h_1, \dots, h_k \in G.$$

Since $h_{k+1}$ has been fixed arbitrarily, (16) yields $(a)_{n_1\dots n_{k+1}}$. Induction guarantees that $(b)_{n_1\dots n_k}$ implies $(a)_{n_1\dots n_k}$ for any $k\geqq 1$.

The proof of the converse implication will also run by induction.

Suppose $m_1$ to be an exponential function and let $n_1$ be such a non-negative integer that

$$A_{m_1}\Delta_{h_1}^{n_1}B_{m_1}f(x) = 0, \quad x, h_1 \in G.$$

Setting $p_1 := m_1^{-1}f$ we obtain $f = m_1 p_1$ and

$$\Delta_{h_1}^{n_1}p_1(x) = \Delta_{h_1}^{n_1}B_{m_1}f(x) = 0, \quad x, h_1 \in G,$$

which shows that $(a)_{n_1}$ implies $(b)_{n_1}$.

Further, suppose that the implication $(a)_{n_1\dots n_k} \Rightarrow (b)_{n_1\dots n_k}$ holds true for a $k\geqq 1$ and all $n_1, \dots, n_k \geqq 0$. As a result, $(a)_{n_1\dots n_{k+1}} \Rightarrow (b)_{n_1\dots n_{k+1}}$ is valid for all $n_1, \dots, n_k \geqq 0$ and $n_{k+1}=0$. In order to continue the induction with respect to $n_{k+1}$ let us assume that $(a)_{n_1\dots n_{k+1}}$ implies $(b)_{n_1\dots n_{k+1}}$ for arbitrarily chosen $n_1, \dots, n_k \geqq 0$ and $n_{k+1}=n$ with some $n\geqq 0$. Without loss of generality we may also assume that all the $n_i$'s $(i=1, \dots, k)$ are strictly positive, otherwise we would return to one of the previous cases.

Now, let $m_1, \dots, m_{k+1}$ be given pairwise different exponential functions and let $f\colon G\to X$ be a solution of the equation

$$(a)_{n_1\dots n_k, n+1}\qquad \big(A_{m_{k+1}}\Delta_{h_{k+1}}^{n+1}B_{m_{k+1}}\big)\Big(\prod_{i=1}^{k} A_{m_i}\Delta_{h_i}^{n_i}B_{m_i}\Big)f(x) = 0, \quad x, h_1, \dots, h_{k+1} \in G.$$

For $i=1, \dots, k$ put

$$D_i := \{h\in G\colon\ m_{k+1}(h) \neq m_i(h)\}.$$

Since, by hypothesis, $m_{k+1}$ does not coincide with $m_i$ for $i=1, \dots, k$, none of the sets $D_i$ is empty.

We define a function $\Gamma\colon G^{n+1}\times D_1^{n_1}\times\dots\times D_k^{n_k}\to X$ by

$$(17)\qquad \Gamma(x, y_1, \dots, y_n;\ h_{1,1}, \dots, h_{1,n_1};\ \dots;\ h_{k,1}, \dots, h_{k,n_k}) :=$$

$$= \frac{1}{n!}\Big(\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{m_i(h_{i,j})}{m_{k+1}(h_{i,j})-m_i(h_{i,j})}\Big)\times$$

$$\times\Delta_{y_1\dots y_n}B_{m_{k+1}}\Big(\prod_{i=1}^{k}\prod_{j=1}^{n_i}A_{m_i}\Delta_{h_{i,j}}B_{m_i}\Big)f(x),$$

$$x, y_1, \dots, y_n \in G, \quad i = 1, \dots, k, \quad j = 1, \dots, n_i,$$

with the agreement that for $n=0$, $\Gamma$ is a function of the variables $x$ and $h_{i,j}$ only.

It is clear that for $n\geqq 1$, $\Gamma$ is symmetric with respect to $y_1, \dots, y_n$. We shall show that $\Gamma$ is additive with respect to the first, and in view of the symmetry, in each of these variables. Indeed, let us fix elements $h_{i,j}\in D_i$, $i=1, \dots, k$, $j=1, \dots, n_i$ and let

$$a:=\frac{1}{n!}\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{m_i(h_{i,j})}{m_{k+1}(h_{i,j})-m_i(h_{i,j})};$$

$$\varphi(x):=B_{m_{k+1}}\Big(\prod_{i=1}^{k}\prod_{j=1}^{n_i}A_{m_i}\Delta_{h_{i,j}}B_{m_i}\Big)f(x),\quad x\in G.$$

Then

$$\Gamma(x, y_1, \dots, y_n;\ h_{1,1}, \dots, h_{1,n_1};\ \dots;\ h_{k,1}, \dots, h_{k,n_k}) = a\Delta_{y_1\dots y_n}\varphi(x), x, y_1, \dots, y_n\in G,$$

so the additivity of $\Gamma$ with respect to $y_1$ results from the fact that by $(a)_{n_1\dots n_k,n+1}$, Lemma 1 and (11) we have

$$\Delta_{y_1'+y_1'',y_2+y_n}\varphi(x)-\Delta_{y_1',y_2\dots y_n}\varphi(x)-\Delta_{y_1'',y_2\dots y_n}\varphi(x)=$$

$$=\Delta_{y_1',y_1'',y_2\dots y_n}\varphi(x)=\Delta_{y_1',y_1'',y_2\dots y_n}B_{m_{k+1}}\Big(\prod_{i=1}^{k}\prod_{j=1}^{n_i}A_{m_i}\Delta_{h_{i,j}}B_{m_i}\Big)f(x)=0,$$

$$x, y_1', y_1'', y_2, \dots, y_n\in G.$$

Next, observe that $\Gamma$ is constant as a function of $x$ with the remaining variables arbitrarily fixed. In fact, with the above notations the following relation holds:

$$\Delta_{y_1\dots y_n}\varphi(x+z)-\Delta_{y_1\dots y_n}\varphi(x)=\Delta_{y_1\dots y_n,z}\varphi(x)=$$

$$=\Delta_{y_1\dots y_n,z}B_{m_{k+1}}\Big(\prod_{i=1}^{k}\prod_{j=1}^{n_i}A_{m_i}\Delta_{h_{i,j}}B_{m_i}\Big)f(x)=0,\quad y_1, \dots, y_n, z\in G.$$

Finally, we are going to prove that $\Gamma$ does not depend on the variables $h_{i,j}$, $i=1, \dots, k, j=1, \dots, n_i$. To begin with, let us choose a pair $(r, s)$ such that $r\in\{1, \dots, k\}$, $s\in\{1, \dots, n_r\}$ and fix $h_{i,j}\in D_i$ for all $(i,j)$ with $1\leqq i\leqq k$, $1\leqq j\leqq n_i$, $(i,j)\neq(r,s)$. Introducing the following notations:

$$b:=\frac{1}{n!}\prod_{\substack{i=1\\(i,j)\neq(r,s)}}^{k}\prod_{j=1}^{n_i}\frac{m_i(h_{i,j})}{m_{k+1}(h_{i,j})-m_i(h_{i,j})};$$

$$\omega(x):=\Big(\prod_{\substack{i=1\\(i,j)\neq(r,s)}}^{k}\prod_{j=1}^{n_i}A_{m_i}\Delta_{h_{i,j}}B_{m_i}\Big)f(x),\quad x\in G$$

we can rewrite (17) in the form

$$\Gamma(x, y_1, \dots, y_n;\ h_{1,1}, \dots, h_{1,n_1};\dots;\ h_{k,1}, \dots, h_{k,n_k})=$$

$$=b\,\frac{m_r(h_{r,s})}{m_{k+1}(h_{r,s})-m_r(h_{r,s})}\Delta_{y_1\dots y_n}B_{m_{k+1}}(A_{m_r}\Delta_{h_{r,s}}B_{m_r})\omega(x),$$

$$x, y_1, \dots, y_n\in G,\quad h_{r,s}\in D_r.$$

To prove that $\Gamma$ does not depend on the variable $h_{r,s}$ it is sufficient to check that for arbitrarily fixed $x, y_1, \dots, y_n \in G$ the function $\Omega$ defined by

$$\Omega(h) := \frac{m_r(h)}{m_{k+1}(h) - m_r(h)} \Delta_{y_1 \dots y_n} B_{m_{k+1}} (A_{m_r} \Delta_h B_{m_r}) \omega(x), \quad h \in G$$

is constant on $D_r$. For, take elements $h, l \in G$ such that $h, h+l \in D_r$. Applying (14) with the substitution $m := m_r$, $n := m_{k+1}$ we obtain

$$\Omega(h+l) - \Omega(h) = \frac{1}{[m_{k+1}(h+l) - m_r(h+l)][m_{k+1}(h) - m_r(h)]} \times$$

$$\times \Delta_{y_1 \dots y_n} B_{m_{k+1}} [m_r(h+l)\big(m_{k+1}(h) - m_r(h)\big) A_{m_r} \Delta_{h+l} B_{m_r} \omega(x) -$$

$$- m_r(h)\big(m_{k+1}(h+l) - m_r(h+l)\big) A_{m_r} \Delta_h B_{m_r} \omega(x)] =$$

$$= \frac{m_r(h) m_{k+1}(h+l)}{[m_{k+1}(h+l) - m_r(h+l)][m_{k+1}(h) - m_r(h)]} \times$$

$$\times \Delta_{y_1 \dots y_n} \Delta_l B_{m_{k+1}} A_{m_r} \Delta_h B_{m_r} \omega(x) - \frac{m_r(h+l) m_{k+1}(h)}{[m_{k+1}(h+l) - m_r(h+l)][m_{k+1}(h) - m_r(h)]} \times$$

$$\times \Delta_{y_1 \dots y_n} \Delta_h B_{m_{k+1}} A_{m_r} \Delta_l B_{m_r} \omega(x).$$

On account of $(a)_{n_1 \dots n_k, n+1}$, Lemma 1 and (11), both the minuend and subtrahend in the last difference vanish.

According to what has been shown so far, a function $\gamma: G \to X$ may unambiguously be determined by the following formula:

$$\gamma(y) := \Gamma(x, \underbrace{y, \dots, y}_{n}; h_{1,1}, \dots, h_{1,n_1}; \dots; h_{k,1}, \dots h_{k,n_k}), \quad y \in G,$$

which is independent of a particular choice of elements $x \in G$, $h_{i,j} \in D_i$, $i = 1, \dots, k$, $j = 1, \dots, n_i$. If $n = 0$, $\gamma$ is simply a constant, whereas for $n \geqq 1$ $\gamma$ is a diagonalization of a symmetric $n$-additive function.

Put

$$g(x) := f(x) - m_{k+1}(x) \gamma(x), \quad x \in G.$$

Our next aim is to show that $g$ satisfies equation $(a)_{n_1 \dots n_k, n}$ which, in virtue of Lemma 1 and (11), is equivalent to

$$A_{m_{k+1}} \Delta^n_{h_{k+1}} B_{m_{k+1}} \Big( \prod_{i=1}^{k} \prod_{j=1}^{n_i} A_{m_i} \Delta_{h_{i,j}} B_{m_i} \Big) g(x) = 0, \tag{18}$$

$$x, h_{k+1}, h_{i,j} \in G, \quad i = 1, \dots, k, \quad j = 1, \dots, n_i.$$

First we confine ourselves to the case where $h_{i,j} \in D_i$, $i = 1, \dots, k$, $j = 1, \dots, n_i$. Setting

$$m_i'(x) := m_i(x) m_{k+1}(x)^{-1}, \quad x \in G, \quad i = 1, \dots, k$$

and making use of (15) and Lemma 2 from [1], we infer that

$$\Delta^n_{h_{k+1}} B_{m_{k+1}} \Big( \prod_{i=1}^{k} \prod_{j=1}^{n_i} A_{m_i} \Delta_{h_{i,j}} B_{m_i} \Big( m_{k+1}(x)\gamma(x)\Big) =$$

$$= \Delta^n_{h_{k+1}} \Big( \prod_{i=1}^{k} \prod_{j=1}^{n_i} B_{m_{k+1}} A_{m_i} \Delta_{h_{i,j}} B_{m_i} A_{m_{k+1}} \Big) \gamma(x) =$$

$$= \Big( \prod_{i=1}^{k} \prod_{j=1}^{n_i} A_{m_i'} \Delta_{h_{i,j}} B_{m_i'} \Big) \Delta^n_{h_{k+1}} \gamma(x) =$$

$$= \prod_{i=1}^{k} \prod_{j=1}^{n_i} \big( m_i'(h_{i,j})^{-1} - 1 \big) n! \gamma(h_{k+1}) = \prod_{i=1}^{k} \prod_{j=1}^{n_i} \Big( \frac{m_{k+1}(h_{i,j})}{m_i(h_{i,j})} - 1 \Big) n! \times$$

$$\times \Gamma(x, \underbrace{h_{k+1}, \dots, h_{k+1}}_{n}; h_{1,1}, \dots, h_{1,n_1}; \dots; h_{k,1}, \dots, h_{k,n_k}) =$$

$$= \Delta^n_{h_{k+1}} B_{m_{k+1}} \Big( \prod_{i=1}^{k} \prod_{j=1}^{n_i} A_{m_i} \Delta_{h_{i,j}} B_{m_i} \Big) f(x).$$

This implies (18) under the assumption that $h_{i,j} \in D_i$, $i=1, \dots, k$, $j=1, \dots, n_i$. It remains to check that this restriction is actually redundant.

To start with, suppose that for some $r \in \{1, \dots, k\}$ and $s \in \{1, \dots, n_r\}$, $h_{r,s}$ belongs to $G \setminus D_r$, while $h_{i,j} \in D_i$ for $i=1, \dots, k$, $j=1, \dots, n_i$, $(i,j) \neq (r,s)$. Take an arbitrary element $h' \in D_r$ and put $h'' := h_{r,s} - h'$. Then

$$\frac{m_{k+1}(h'')}{m_r(h'')} = \frac{m_{k+1}(h_{r,s}-h')}{m_r(h_{r,s}-h')} = \frac{m_{k+1}(h_{r,s})}{m_r(h_{r,s})} \frac{m_r(h')}{m_{k+1}(h')} = \frac{m_r(h')}{m_{k+1}(h')} \neq 1,$$

whence $h'' \in D_r$ and $h_{r,s} = h' + h''$. If we choose an $h_{k+1} \in G$ and put

$$\psi(x) := A_{m_{k+1}} \Delta^n_{h_{k+1}} B_{m_{k+1}} \Big( \prod_{\substack{i=1 \\ (i,j) \neq (r,s)}}^{k} \prod_{j=1}^{n_i} A_{m_i} \Delta_{h_{i,j}} B_{m_i} \Big) g(x), \quad x \in G,$$

then the left-hand side of equation (18) is equal to

$$(A_{m_r} \Delta_{h_{r,s}} B_{m_r}) \psi(x) = m_r(x) \Delta_{h'+h''} \big( m_r(x)^{-1} \psi(x) \big) =$$

$$= m_r(x) \Delta_{h'} \big( m_r(x+h'')^{-1} \psi(x+h'') \big) + m_r(x) \Delta_{h''} \big( m_r(x)^{-1} \psi(x) \big) =$$

$$= m_r(h'')^{-1} m_r(x+h'') \Delta_{h'} \big( m_r(x+h'')^{-1} \psi(x+h'') \big) + m_r(x) \Delta_{h''} \big( m_r(x)^{-1} \psi(x) \big) =$$

$$= m_r(h'')^{-1} T_{h''} (A_m \Delta_{h'} B_{m_r}) \psi(x) + (A_{m_r} \Delta_{h''} B_{m_r}) \psi(x), \quad x \in G,$$

where both summands in the last sum vanish identically on account of what has been established before.

Now it becomes apparent that, by induction with respect to the number of elements $h_{i,j}$ lying outside $D_i$, one can prove that (18) is valid for all $h_{i,j} \in G$, $i=1, \dots, k$, $j=1, \dots, n_i$.

Since, by hypothesis, $(a)_{n_1\dots n_k,n}$ implies $(b)_{n_1\dots n_k,n}$, there exist polynomial functions $p_1, \dots, p_k$ of degrees less than $n_1, \dots, n_k$, respectively, and a polynomial function $p_{k+1}$ of degree less than $n$ such that

$$g = \sum_{i=1}^{k+1} m_i p_i.$$

Consequently,

$$f = g + m_{k+1}\gamma = \sum_{i=1}^{k} m_i p_i + m_{k+1}(p_{k+1}+\gamma),$$

where $p_{k+1}+\gamma$ is a polynomial function of degree less than $n+1$. Thus we arrive at a decomposition $(b)_{n_1\dots n_k,n+1}$ of the function $f$, which completes induction on $n_{k+1}$. Induction with respect to $k$ ends the proof of the whole theorem.

If $m_1, \dots, m_k$ are exponential functions, then by (11) and Lemma 1 equation $(a)_{n_1\dots n_k}$ is equivalent to

$$(a')_{n_1\dots n_k} \qquad (\prod_{i=1}^{k}\prod_{j=1}^{n_i} A_{m_i}\Delta_{h_{i,j}}B_{m_i}f(x) = 0.$$
$$x, h_{i,j}\in G, \quad i = 1, \dots, k, \quad j = 1, \dots, n_i.$$

Moreover, on account of (8), equation $(a')_{n_1\dots n_k}$ may be written in the following equivalent form:

$$(a'')_{n_1\dots n_k} \qquad (\prod_{i=1}^{k}\prod_{j=1}^{n_i}(m_i(h_{i,j})^{-1}T_{h_{i,j}}-I))f(x) = 0,$$
$$x, h_{i,j}\in G, \quad i = 1, \dots, k, \quad j = 1, \dots, n_i.$$

where $T_h$ is defined by (13) and $I$ denotes the identity operator.

In general, neither equation $(a)_{n_1\dots n_k}$ nor $(a')_{n_1\dots n_k}$ force the functions $m_1, \dots, m_k$ to be multiplicative without assuming this from the very beginning. For example the equation

$$A_m\Delta_h B_m f(x) = 0, \quad x, h\in G$$

is fulfilled by any $m\colon G\to K\setminus\{0\}$ and $f\colon G\to X$ such that $\frac{f}{m}$ is a constant function. In this respect equation $(a'')_{n_1\dots n_k}$ appears to have an essential advantage.

A solution $f\colon G\to X$ of equation $(a'')_{n_1\dots n_k}$ will be called reducible if there exist integers $n'_1, \dots, n'_k$, $0\le n'_1\le n_1, \dots, 0\le n'_k\le n_k$ such that $n'_i<n_i$ for at least one $i\in\{1, \dots, k\}$ and $f$ satisfies equation $(a'')_{n'_1\dots n'_k}$. Otherwise, we say that the solution is irreducible.

In terms of irreducible solutions of equation $(a'')_{n_1\dots n_k}$ we are able to give a complete characterization of the class of generalized exponential polynomials.

**Theorem 2.** *Fix positive integers $n_1, \dots, n_k$. Let a function $f\colon G\to X$ and some pairwise different functions $m_1, \dots, m_k\colon G\to K\setminus\{0\}$ satisfy equation $(a'')_{n_1\dots n_k}$. Suppose $f$ to be an irreducible solution of this equation. Then the functions $m_1, \dots, m_k$ are multiplicative and there exist polynomial functions $p_1, \dots, p_k\colon G\to X$ of degrees $n_1-1, \dots, n_k-1$, respectively, such that*

$$f = \sum_{i=1}^{k} m_i p_i.$$

PROOF. It suffices to check the multiplicativity of the function $m_1$, for instance.

Since $f$ is an irreducible solution of the equation considered, one can find elements $x^0, h^0_{i,j}\in G$, $i=1, \dots, k$, $j=1, \dots, n_i$, $(i,j)\neq(1,1)$ such that

$$\Big(\prod_{\substack{i=1\\(i,j)\neq(1,1)}}^{k}\prod_{j=1}^{n_i}\big(m_i(h^0_{i,j})^{-1}T_{h^0_{i,j}}-I\big)\Big)f(x^0)\neq 0.$$

With the use of the notation

$$\varphi(x):=\Big(\prod_{\substack{i=1\\(i,j)\neq(1,1)}}^{k}\prod_{j=1}^{n_i}\big(m_i(h^0_{i,j})^{-1}T_{h^0_{i,j}}-I\big)\Big)f(x),\quad x\in G$$

we have

(19) $$\varphi(x^0)\neq 0$$

and

(20) $$m_1(h)^{-1}\varphi(x+h)-\varphi(x)=0,\quad x,h\in G.$$

Hence, in particular,

$$\varphi(x^0+h)=m_1(h)\,\varphi(x^0),\quad h\in G,$$

or, equivalently,

(21) $$\varphi(h)=m_1(h-x^0)\,\varphi(x^0),\quad h\in G.$$

Substituting (21) to (20) we derive

$$\big(m_1(x+h-x^0)-m_1(h)\,m_1(x-x^0)\big)\,\varphi(x^0)=0,\quad x,h\in G,$$

which, in view of (19), means that $m_1$ is multiplicative.

The rest of the assertion of Theorem 2 is now a consequence of Theorem 1 and the fact that $f$ is an irreducible solution of the equation considered.

## 4. *Concluding remarks*

Let us restrict ourselves for a moment to the case where $G:=\mathbf{R}$, $X=K:=\mathbf{C}$ and

$$m_i(x):=e^{\lambda_i x},\quad x\in\mathbf{R},\quad i=1,\dots,k.$$

Then, either of equations $(a)_{n_1\dots n_k}$ and $(a')_{n_1\dots n_k}$ is equivalent to equation (6) in the class of $n$-times differentiable solutions $f\colon \mathbf{R}\to\mathbf{C}$, where $n:=n_1+\dots+n_k$. This fact results on differentiating both sides of equation $(a')_{n_1\dots n_k}$ with respect to each of the variables $h_{i,j}$, $i=1,\dots,k$, $j=1,\dots,n_i$, successively. From this point of view, the theory of equations dealt with in Section 3 contains the theory of homogeneous linear differential equations.

On the other hand, the equations proposed in our paper do not impose any differentiability properties on their solutions; it is known that there exist generalized exponential polynomials from $\mathbf{R}$ into $\mathbf{C}$ which are discontinuous and even non-measurable. Therefore these equations provide a method of introducing a substitute for homogeneous linear differential equations in abstract spaces without any differential structure.

## References


[1] D. Z. DJOKOVIĆ, A representation theorem for $(X_1-1)\dots(X_n-1)$ and its applications, *Ann. Polon. Math.* **22** (1969), 189—198.

[2] D. GIROD, J. H. B. KEMPERMAN, On the functional equation $\sum_{j=1}^{n} a_j f(x+T_j y)=0$, *Aequationes Math.* 4 (1970), 230.

[3] P. G. LAIRD, On characterizations of exponential polynomials, *Pacific J. Math.* **80** (1979), 503—507.

[4] S. MAZUR, W. ORLICZ, Grundlegende Eigenschaften der polynomischen Operationen I, II, *Studia Math.* **5** (1934), 50—68, 179—189.

[5] L. REICH, J. SCHWAIGER, On polynomials in additive and multiplicative functions, Functional equations: History, Applications and Theory, *D. Reidel Publishing Company,* 1984, 127—160.

[6] L. REICH, J. SCHWAIGER, Über algebraische Relationen zwischen additiven und multiplikativen Funktionen, *Aequationes Math.* **27** (1984), 114—134.

[7] L. SZÉKELYHIDI, On a class of linear functional equations, *Publ. Math. Debrecen* **29** (1982), 19—28.

[8] L. SZÉKELYHIDI, On the zeros of exponential polynomials, C. R. *Math. Rep. Acad. Sci. Canada* **4** (1982), 189—194.


*(Received November 22, 1985)*