Morita equivalence for a larger class of rings

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In [10] it was observed that two rings S and R with unities are Morita equivalent if and only if there exists a gamma ring with right and left unities such that its right and left operator rings are isomorphic to S and R. This has been extended to rings with local units [11]. In this paper we extend this result to rings satisfying the conditions i) $S^2=S$ and ii) $aS=0$ or $Sa=0$ implies that $a=0$, $a\in S$ which include rings with unities and rings with local units.*) We also prove that any ring satisfying the above conditions is equivalent to a division ring if and only if it is simple and completely reducible, thus obtaining an extension of Wedderburn theorem for simple Artinian rings.

1. Preliminaries

Let A and Γ be additive abelian groups. Then following BARNES [3] we say A is a Γ-ring if there exists a map $f\colon A\times\Gamma\times A\to A$ with $f(x,\alpha,y)=x\alpha y$ such that

i) f is additive in each variable, and

ii) $(x\alpha y)\beta z=x\alpha(y\beta z)$ for all $x,y,z\in A$ and $\alpha,\beta\in\Gamma$.

Consider the maps $[\alpha,x]\colon y\to y\alpha x$ and $[x,\alpha]\colon y\to x\alpha y$ $x\in A$, $\alpha\in\Gamma$ and for all $y\in A$. Clearly $[x,\alpha],[\alpha,x]$ belong to the endomorphism group $\operatorname{End}(A)$. The bilinearity of the map $\Gamma\times A\to\operatorname{End}A(A\times\Gamma\to\operatorname{End}A)$ given by $(\alpha,a)\to[\alpha,a]$ $((a,\alpha)\to[a,\alpha])$ gives rise to a linear map from $\Gamma\otimes_Z A\to\operatorname{End}A(A\otimes_Z\Gamma\to\operatorname{End}A)$ given by

$$\sum_i \alpha_i\otimes a_i\to\sum_i[\alpha_i,a_i]\Big(\sum_i a_i\otimes\alpha_i\to\sum_i[a_i,\alpha_i]\Big),$$

$\alpha_i\in\Gamma$ and $a_i\in A$. The image of $\Gamma\otimes_Z A(A\otimes_Z\Gamma)$ in $\operatorname{End}A$ is an associative ring denoted by $R(A,\Gamma)-L(A,\Gamma)$ and call it the right (left) operator ring of A. Ring multiplication in $R(A,\Gamma)$ and $L(A,\Gamma)$ is given by the rule,

$$\sum_i[\alpha_i,a_i]\sum_i[\beta_j,b_j]=\sum_{i,j}[\alpha_i,\alpha_i\beta_jb_j],$$

*) See about T. W. ANDERSON—K. R. FULLER, Rings and Categories of Modules, *Springer* (1973)

and

$$\sum_i [a_i, \alpha_i] \sum_j [b_j, \beta_j] = \sum_{i,j} [a_i \alpha_i b_j, \beta_j].$$

A is clearly a faithful $R(A, \Gamma) - L(A, \Gamma)$ bimodule.

For further details on Γ-rings and for literature on ring theory we refer to [3], [6], [7], [10] and [2], [5] respectively.

2. Throughout this paper we denote a Γ-ring by (A, Γ) or A, right and left operator rings by $R(A, \Gamma)$ or R and $L(A, \Gamma)$ or L respectively. All rings considered satisfy the gamma conditions namely,

(i) $S^2 = S$,

(ii) $Sa = 0$ or $aS = 0$ implies $a = 0$, $a \in S$.

Definition 2.1 [12]. A Γ-ring A is said to be weekly semiprime if $[x, \Gamma] \neq 0$ and $[\Gamma, x] \neq 0$ for all $0 \neq x \in A$ and $A\Gamma A = A$.

Remark. A is weakly semiprime implies $R(A, \Gamma)$, $L(A, \Gamma)$ satisfy the gamma conditions. Any ring with unity satisfies the gamma conditions, Fuller's rings with "enough idempotents" and more generally a ring with local units [1] satisfy the gamma conditions.

The following is a ring which satisfies the gamma conditions but does not contain unity or idempotent and hence has no local units.

Let $B \subset \mathscr{C}[0, 1]$ be the set of all continuous real valued functions on the interval $[0, 1]$ which vanish in a neighbourhood of 0. B is a commutative ring under the pointwise addition and pointwise multiplication. The ring B can be easily checked to satisfy the gamma conditions.

Now we state the results which are needed for our purpose.

Lemma 2.2. [12]. *Let A be a weakly semiprime Γ-ring, L and R its operator rings. Then A is simple if and only if $R(L)$ is simple.*

Theorem 2.3. (Theorem 1 of [9].) *Let A be a weakly semiprime Γ-ring, L and R be its operator rings. Then L and R are Morita Equivalent.*

We now prove the converse. But first we need the following lemma

Lemma 2.4. *Let R and S be two associative rings satisfying the gamma conditions and R be Morita equivalent to S. Suppose that*

$$H: \mathscr{C}_R^{*)} \to \mathscr{C}_S \quad \text{and} \quad T: \mathscr{C}_S \to \mathscr{C}_R$$

are category equivalences with $HT \cong 1_{\mathscr{C}_S}$ and $TH \cong 1_{\mathscr{C}_R}$. Then ${}_S T(S)_R$ and ${}_R H(R)_S$ are canonical bimodules and

(1) *H and T are full and faithful;*

(2) *(H, T) and (T, H) are adjoint pairs of functors;*

(3) $H \cong (\mathrm{Hom}_R(T(S), -))S$, $T \cong (\mathrm{Hom}_S(H(R), -))R$;

(4) $S \cong (\mathrm{End}_R(T(S)))S$;

(5) *$T(S)$ is an R-generator in $\mathscr{C}_R$ and $H(R)$ is an S-generator in $\mathscr{C}_S$.*

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*) $\mathscr{C}_R$ denotes the subcategory of right R-modules with spanning conditions, $MR = M$ and $mR = 0$ implies $m = 0$, $m \in M$.

PROOF. 1) and (2) can be proved as in [2], (3) Let $M \in \mathscr{C}_R$. We define a map

$$\eta\colon M \to (\mathrm{Hom}_R(R, M))R \quad \text{by}$$

$$m \to \eta(m) \quad \text{such that} \quad (\eta(m))(r) = mr.$$

It is straightforward to prove that η is an R-module isomorphism.

Also $H(M) \in \mathscr{C}_S$. Hence using the above isomorphism we have

$$H(M) \cong (\mathrm{Hom}_S(S, H(M)))S \cong \mathrm{Hom}_S(T(S), M)S$$

So $H \cong (\mathrm{Hom}_R(T(S), -))S$. Similarly we prove $T \cong (\mathrm{Hom}_S(H(R), -))R$.

(4) By (iii) of above, we have,

$$H(T(S)) \cong (\mathrm{Hom}_R(T(S), T(S)))S = \mathrm{End}_R(T(S))S.$$

(5) Let $M \in \mathscr{C}_R$. We define a map

$$\omega\colon R^{(M)} \to M \quad \text{by}$$

$$r_{1m_1} + \ldots + r_{tm_t} \to m_1 r_1 + \ldots + m_t r_t.$$

Since $MR = M$, ω is an epimorphism and R generates $\mathscr{C}_R$.

Next let $N \in \mathscr{C}_S$, then there is an epimorphism $R^{(X)} \to T(N) \to 0$ in $\mathscr{C}_R$ where X is an indexing set. Since H is a category equivalence, it preserves epimorphisms and direct sums and hence we have an epimorphism $(H(R))^{(X)} \to H(T(N)) \to 0$ in $\mathscr{C}_S$. So $H(R)$ generates $\mathscr{C}_S$.

Similarly it can be shown that $T(S)$ is an R-generator.

This completes the proof of the lemma.

From (3) of Lemma 2.4, we have $T(S) \cong (\mathrm{Hom}_S(H(R), S))R$ as right R-modules and $H(R) \cong \mathrm{Hom}_R(S, H(R))S$ as right S-modules. We observe that $H(R) \cong \cong \mathrm{Hom}_R(S, H(R))S$ as left R-modules also. To see this, we define $(rf)(s) = rf(s)$, $r \in R, f \in \mathrm{Hom}_R(S, H(R))S$ and if $\Theta\colon H(R) \xrightarrow{\cong} (\mathrm{Hom}_S(S, H(R)))S$ denotes the isomorphism as abelian groups, given by $h \to \Theta(h)$, then $(rh) \to \Theta(rh)$ such that $(\Theta(rh))(s) = = (rh)(s) = r(hs) = r(\Theta(h))(s)$. Hence $H(R) \cong (\mathrm{Hom}_S(S, H(R)))S$ as left R-modules.

Again we have $\psi\colon \mathrm{Hom}_R(T(S), R) \to \mathrm{Hom}_S(S, H(R))$ is an abelian group isomorphism given by $\psi(f) = H(f)\,\eta_S$ where $f \in \mathrm{Hom}_R(T(S), R)$ and $\eta_S\colon S \to HT(S)$ is an isomorphism. We show that ψ is a left R-module isomorphism. We define, for $f \in \mathrm{Hom}_R(T(S), R)$, $g \in \mathrm{Hom}_S(S, H(R))$, $r \in R$,

$$(rf)(x) = r(f(x)), \quad x \in T(S) \quad \text{and} \quad (rg)(s) = H(\varrho_r)\,g(s),$$

$s \in S$ and $\varrho_r\colon R \to R$ is left multiplication. Combining these we get $H(R) \overset{\varphi}{\cong} \overset{\varphi}{\cong} (\mathrm{Hom}_R(T(S), R))S$ is a left R and right S bimodule isomorphism.

We are ready to prove the converse.

Theorem 2.5. *Let R and S be two rings, satisfying the gamma conditions, which are Morita equivalent. Then there exists a weakly semiprime gamma ring such that its right and left operator rings are isomorphic to R and S respectively.*

PROOF. In the notations of Lemma 2.4, we denote the canonical bimodules ${}_ST(S)_R$ and $(\mathrm{Hom}_R(T(S), R))S$ by ${}_SA_R={}_ST(S)_R$ and ${}_R\Gamma_S=(\mathrm{Hom}_R(T(S), R))S$. We give a Γ-ring structure to A by defining

$$A\times\Gamma\times A \to A \quad \text{by}$$
$$(a, f, a') \to af(a').$$

Similarly we define a map from

$$\Gamma\times A\times\Gamma \to \Gamma \quad \text{by}$$
$$(f, a, f') \to f(a)f' \quad .$$

This defines on Γ, the structure of a A-ring. Clearly (A, Γ) and (Γ, A) are weakly semiprime gamma rings.

Now we prove that R is isomorphic to the right operator ring of the gamma ring (A, Γ). But first we show that $RH(R)=R$. To see this let $\varrho_r\colon R\to R$ denote the left multiplication by elements of R, for every $r\in R$. Now we have epimorphisms

$$\bigoplus_{r\in R} R \xrightarrow{\oplus\varrho_r} R \to 0 \quad \text{and} \quad \bigoplus_{r\in R} H(R) \xrightarrow{\oplus H(\varrho_r)} H(R) \to 0,$$

since H is a category equivalence. So every $x\in H(R)$ can be written as $\sum_i r_i x_i'$, $r_i\in R$, $x_i'\in H(R)$. Hence $RH(R)=H(R)$. This in view of the bimodule isomorphism

$$\varphi\colon H(R) \to (\mathrm{Hom}_R(T(S), R))S \quad \text{implies that} \quad R\Gamma = \Gamma.$$

Similarly it can be proved $ST(S)=T(S)$.

We define a map

$$\sigma\colon [\Gamma, T(S)] \to R \quad \text{by}$$
$$\sigma\big(\sum_i [f_i d_i, x_i]\big) = \sum_i (f_i d_i)(x_i) = \sum_i f_i(d_i x_i).$$

Clearly σ is a well defined ring homomorphism. If $\sum_i f_i(d_i x_i)=0$, then $\sum_i (f_i d_i)(x_i)=0$. This implies $\sum_i [f_i d_i, x_i]=0$. Hence σ is injective.

Next, since $T(S)$ is an R-generator in $\mathscr{C}_R$ an element $r\in R$ can be written in the form $r=\sum_i f_i(t_i)$ where $f_i\in\mathrm{Hom}_R(T(S), R)$ and $t_i\in T(S)$. Again we have $ST(S)=T(S)$. Hence $t_i=\sum_j \delta_j y_j$, $\delta_j\in S$, $y_j\in T(S)$. So

$$r = \sum_i f_i\big(\sum_j \delta_j y_j\big) = \sum_{i,j} f_i(\delta_j y_j) = \sum_{i,j} (f_i\delta_j)(y_j).$$

Thus $r\in R$ corresponds to the element $\sum_{i,j} [f_i\delta_j, y_j]$ in $[\Gamma, T(S)]$. Hence σ is an epimorphism. That is $R\cong R(T(S), \Gamma)$.

Before we proceed to establish the isomorphism between $L(T(S), \Gamma)$ and S, we show that $S\cong(\mathrm{End}\, T(S)_R)S$ as rings. We define a map

$$\Theta\colon S \to \mathrm{Hom}_R(T(S), T(S))\, S$$

by

$$\Theta(s) = T(\eta_S^{-1})T(\varrho_{\eta_S(s)}),$$

where $s \in S$ and η_S: $S \to HT(S)$ is a right S-module isomorphism and $\varrho_{\eta_S(s)}$ is left multiplication.

If $s_1, s_2 \in S$, then

$$\varrho_{\eta_S(s_1)} \eta_S^{-1} \varrho_{\eta_S(s_2)} = \varrho_{\eta_S(s_1)} \varrho_{s_2}.$$

To see this, let $s' \in S$. Then

$$(\varrho_{\eta_S(s_1)} \eta_S^{-1} \varrho_{\eta_S(s_2)})(s') = \varrho_{\eta_S(s_1)}(\eta_S^{-1}(\eta_S(s_2)s')) = \eta_S(s_1)s_2 s'$$

and

$$(\varrho_{\eta_S(s_1)} \varrho_{s_2})(s') = \eta_S(s_1)s_2 s'.$$

So

$$T(\eta_S^{-1} \varrho_{\eta_S(s_1)} \varrho_{s_2}) = T(\eta_S^{-1} \varrho_{\eta_S(s_1)}) T(\eta_S^{-1} \varrho_{\eta_S(s_2)})$$

and hence Θ is a ring homomorphism.

This together with Lemma 2.4 (4) establishes the required ring isomorphism between S and $(\text{End } T(S)_R)S$.

The inverse isomorphism

$$\Theta^{-1}: (\text{End } T(S)_R)S \to S \quad \text{is given by}$$

$$\Theta^{-1}(\sum_i f_i s_i) = \sum_i \eta_S^{-1} H(f_i) \eta_S(s_i).$$

Now we define a map

$$\lambda: [T(S), \Gamma] \to S \quad \text{as the composition of}$$

$$[T(S), \Gamma] \xrightarrow{\mu} (\text{End } T(S)_R)S \xrightarrow{\Theta^{-1}} S$$

$$\mu(\sum_i [t_i, g_i s_i]) \to \mu(\sum_i [t_i, g_i s_i]) \to \sum_i \eta_S^{-1} H(\mu([t_i, g_i])) \eta_S(s_i)$$

where

$$(\sum_i [t_i, g_i s_i])(t') = \sum_i [t_i, g_i s_i] t', \ t' \in T(S).$$

That is, $\lambda(\sum_i [t_i, g_i s_i]) = \Theta^{-1} \mu(\sum_i [t_i, g_i s_i]) = \sum_i \eta_S^{-1} H(\mu([t_i, g_i])) \eta_S(s_i)$.

$$\text{But} \quad \mu(\sum_i [t_i, g_i])(t') = \sum_i [t_i, g_i] t' = \sum_i t_i g_i(t') = \sum_i \varrho_{t_i} g_i(t')$$

$$\text{and hence} \quad \lambda(\sum_i [t_i, g_i s_i]) = \sum_i \eta_S^{-1} H(\varrho_{t_i}) H(g_i) \eta_S(s_i).$$

This is clearly a well defined ring monomorphism.

To see that λ is onto it suffices to check on generators.

Since $H(R)$ is an S-generator and $R\Gamma = \Gamma$, $s \in S$ is given by finite sum of elements of the form $fr(gs)$, $fr \in \text{Hom}_S(\Gamma, S)R$ and $gs \in \Gamma = \text{Hom}_R(T(S), R)S$. Now if ψ denotes the right R-module homomorphism $T(S) \to \text{Hom}_S(\text{Hom}_R(T(S), R)S, S)R$, then for any $fr \in \mathrm{Hom}_S(\mathrm{Hom}_R(T(S), R)S, S)R$, there exists a $t \in T(S)$ such that $\psi(t)(gs) = fr(gs)$. But by definition,

$$\psi(t)(gs) = \eta_S^{-1} H(\varrho_t) H(g) \eta_S(s),$$

which by the definition of λ implies that $\lambda([t, gs]) = fr(gs)$. Hence $fr(gs) \in S$ corresponds to $[t, gs] \in [T(S), \Gamma]$ and so λ is an epimorphism and hence an isomorphism. This completes the proof.

Combining Theorems 2.3 and 2.5, we have the main theorem of this paper.

Theorem 2.6. *Let R and S be two rings satisfying the gamma conditions. Then R is Morita equivalent to S if and only if there exists a weakly semiprime gamma ring such that its right and left operator rings are isomorphic to R and S respectively.*

The first Wedderburn theorem for rings with unities can be interpreted in Morita language as "A ring is simple Artinian if and only if it is Morita equivalent to a division ring". As an application of Theorem 2.6 we have an extension of the Wedderburn theorem as follows.

Theorem 2.7. *Any ring satisfying the gamma conditions is Morita equivalent to a division ring if and only if it is simple and completely reducible.*

Proof. Since a division ring has unity it satisfies the gamma conditions. So any ring Morita equivalent to it is Γ-context equivalent to it by Theorem 2.6. Suppose S is the ring Γ context equivalent to the division ring D and (A, Γ) be the gamma ring such that $L(A, \Gamma) \cong S$ and $R(A, \Gamma) \cong D$. If I is a nonzero left ideal of A, then $I^* = \{d \in D / Ad \subseteq I\}$ is a nonzero left ideal of D and hence $I^* = D$. It follows that $A = I$. Hence A is a faithful irreducible S-module. That is S is primitive. Now $S \subseteq \mathrm{End}(A_D)$. Let $l = \sum_i [x_i, \alpha_i] \in S$. Then $l(a) = \sum_i [x_i, \alpha_i'] a = \sum_i x_i [\alpha_i, a]$ for every $a \in A$. This implies that lA is finite dimensional. Hence S contains a nonzero linear transformation of finite rank [5] and since S is simple by Lemma 2.2, S must coincide with Soc. S, the socle of S.

Conversely suppose S is simple and completely reducible and $S = \oplus I_\alpha$ where each I_α is a minimal left ideal of S. Then $I_\alpha = Se_\alpha$ for some idempotent $e_\alpha \in I_\alpha$.

Se_α can be given a Γ-ring structure $(\Gamma = e_\alpha S)$ and it can be easily checked that $R(Se_\alpha, e_\alpha S) \cong D \cong e_\alpha Se_\alpha$, a division ring.

It remains to show that $L(Se_\alpha, e_\alpha S) \cong S$. Define a map

$$\sigma\colon L(Se_\alpha, e_\alpha S) \to S \quad \text{by}$$

$$\sigma\Big(\sum_i [l_i' e_\alpha, e_\alpha l_i]\Big) = \sum_i l_i' e_\alpha l_i.$$

σ is clearly a ring monomorphism. We show σ is onto. Since Se_α and Se_β are minimal and S is simple we have $Se_\alpha \cong Se_\beta$ and given by $\varphi_\beta(xe_\alpha) = e_\beta$, $x \in S$. It follows that $e_\beta = xe_\alpha ye_\beta$ for some $y \in S$. Hence $le_\beta = le_\beta^2 = (e_\beta xe_\alpha)(e_\alpha ye_\beta)$. Hence σ is onto.

Se_α can be easily checked to be a weakly semiprime gamma ring. Se_α is a faithful S-module follows from the simplicity of S. So by Theorem 2.6, S is Morita equivalent to a division ring. This completes the proof.

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