Commutativity of generalized Boolean rings

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We know that a ring R satisfying $x^2=x$, for every $x\in R$ is Boolean which is necessarily commutative. Recently in a paper [7] we have weakened the condition for a semi-prime ring R and proved that if $(xy)^2-xy$ is central for all x, y in R, then R must be commutative. In Section I of the present paper we generalize the above result which is an extension of the theorem of Herstein [2] which inturns generalizes the famous theorem of JACOBSON [5, Theorem 11]. In fact, we prove the following:

Theorem A. *Let $n>1$ be a fixed positive integer and R be a semi-prime ring in which $(xy)^n-xy$ is central, for every x, y in R, then R is commutative.*

It is also well known that every ring with unity 1 satisfying the identity $x^{n+1}=x^n$ is a Boolean ring and thus commutative. In Section II of this paper we deal with the commutativity of the rings in which $x^{n+1}-x^n$ is central, for all $x\in R$, n being a positive integer. This, at the same time generalizes the above referred result and includes the result due to HERSTEIN [2] for the case $n=1$. Indeed we prove the following:

Theorem B. *Let n be a fixed positive integer and R be a ring with unity 1 in which $x^{n+1}-x^n$ is central, for all $x\in R$, then R is commutative.*

In the end we provide two examples to show that existence of unity in the ring of the above theorem is rather essential.

In what follows, $[x, y]$ stands for commutator $xy-yx$ and $Z(R)$ denotes the centre of an associative ring R.

Section I

In preparation for the proof of our Theorem A, we first establish the following lemmas:

Lemma 1.1. *Let $n>1$ be a fixed positive integer and R be a prime ring in which $(xy)^n-xy\in Z(R)$, for all $x, y\in R$, then R contains no nonzero zerodevisors.*

PROOF. It suffices to show that R is a reduced ring. Let a be an element of R such that $a^2=0$. Using the hypothesis of theorem for any $y\in R$ we get $\{(ay)^n-ay\}y=$ $=y\{(ay)^n-ay\}$. With $y=ya$, we have $ayaya=0$ i.e. $(ay)^3=0$, for all $y\in R$. Thus $a=0$ by Lemma 1.1 of [4].

Lemma 1.2. *Let $n>1$ be a fixed positive integer and R be a division ring in which $(xy)^n-xy\in Z(R)$, for all $x, y\in R$, then R is commutative.*

PROOF. Using the hypothesis of the lemma, with $x=xy^{-1}$ we get $(xy^{-1}\cdot y)^n-$ $-xy^{-1}\cdot y\in Z(R)$, which implies that,

$$[x^n, y]-[x, y] = 0 \tag{1}$$

Again on replacing y by $x^{-1}y$ in the identity $(xy)^n-xy\in Z(R)$ and combining (1), we get $[x^n, y]-[x, y^n]=0$. By Kaplansky's theorem [6], R is finite dimensional over its centre $Z(R)$. Since $[x^n, y]-[x, y^n]=0$, for any $c\in Z(R)$ we have

$$(c^n-c)[x^n, y] = [c^n x^n, y]-[cx, y^n] = [(cx)^n, y]-[cx, y^n] = 0$$

If $[x^n, y]=0$, then the result follows from (1). If $[x^n, y]\neq 0$, then $(c^n-c)[x^n, y]=0$ implies $c^n=c$, for all $c\in Z(R)$. Obviously $Z(R)$ is finite and then R is also finite. Hence R is commutative.

PROOF OF THEOREM A. Since R is semi-prime, in which $(xy)^n-xy$ is central, then R is isomorphic to a subdirect sum of prime rings R_α each of which as a homomorphic image of R satisfies the hypothesis placed on R. Hence it is sufficient to prove the theorem in the case when R is prime in which $(xy)^n-xy$ is central. Now by Lemma 1.1, R is reduced. As is well known prime reduced ring R is completely prime. According to S. A. AMITSUR [1], R can be embedded in a division ring satisfying the same polynomial identity. Hence we can assume that R is a division ring in which $(xy)^n-xy$ is central. By Lemma 1.2, R is commutative.

Section II

The following lemma is due to HERSTEIN [3] which will be extensively used in the proof of our Theorem B.

Lemma 2.1. *Let R be a ring and for every $x, y\in R$ there exists a polynomial $P_{x,y}(t)$ with integer coefficients which depends on x and y such that $[x^2 P_{x,y}(x)-x, y]=0$. The R is commutative.*

PROOF OF THEOREM B. Using the hypothesis of Theorem B, for any $y\in R$ we have

$$[x^n, y]-[x^{n+1}, y] = 0 \tag{1}$$

Now replace x by $(1+x)$ in (1), to get

$$[(1+x)^n, y]-[(1+x)^{n+1}, y] = 0 \tag{2}$$

But since

$$[(1+x)^n, y] = n[x, y]+\sum_{i=2}^{n-1}\binom{n}{i}[x^i, y]+[x^n, y]$$

and

$$[(1+n)^{n+1}, y] = (n+1)[x, y]+\sum_{j=2}^{n}\binom{n+1}{j}[x^j, y]+[x^{n+1}, y].$$

Thus (2), becomes

$$\left[\sum_{i=2}^{n-1}\binom{n}{i}x^i-\sum_{j=2}^{n}\binom{n+1}{j}x^j, y\right]-[x, y]=0$$

i.e. $[x^2P(x)-x, y]=0$, where $P(x)$ is the polynomial with integer coefficients. Hence by Lemma 2.1, R is commutative.

The following examples show that the ring in the hypothesis of Theorem B must contain unity.

Example 1. Let R be the subring generated by the matrices,

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

in the ring of all 3×3 matrices over Z_2, the ring of integer modulo 2. For all integer $n>1$ and for all $x\in R$, $x^{n+1}-x^n\in Z(R)$. But R is not commutative.

Example 2. Let $R=\left\{\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}/a, b, c, \text{ are integers}\right\}$. For all $n\geqq 2$ and all $x\in R$, $x^{n+1}-x^n\in Z(R)$. However R is not commutative.

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