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Lindeberg type central limit theorems on one dimensional hypergroups

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Abstract. Let $(\mathbb{R}_+,*)$ be a Sturm-Liouville hypergroup and S_n the randomized sum of the n-th row of a triangular array $(X_{nj} : 1 \le j \le k_n)$ with values in the hypergroup and with independent rows. We suppose that the adapted variances $\sigma_n^2 := V_*(S_n)$ converge to ∞ and that the Lindeberg condition $\sigma_n^{-2}\sum_j E(X_{nj}^2 1_{\{X_{nj}>\varepsilon\sigma_n\}}) \to 0$ for all $\varepsilon > 0$ is satisfied. Then in the case of polynomial growth, S_n/σ_n converges in distribution to the Rayleigh distribution ρ_α if $\alpha := \lim_{x\to\infty} xA'(x)/A(x)$ exists (A is the Lebesgue density of a Haar measure). In the case of exponential growth S_n converges (after a suitable normalization) to the standard normal distribution if $\sum_j E(X_{nj})^2 = O(\sigma_n^2)$. The most important tool for the proofs is the Laplace representation $\varphi_\lambda(x) = \int_{-x}^{x} \exp(-t(\rho + i\lambda))\, d\nu_x(t)$ of the characters φ_λ of the hypergroup $(\mathbb{R}_+,*)$, which is shown to be valid for all (known) Sturm-Liouville hypergroups.

0. Introduction

Let $(X_{nj} : n \ge 1, 1 \le j \le k_n)$ be a triangular array of random variables with values in the halfline $\mathbb{R}_+ = [0,\infty[$, endowed with a hypergroup convolution $*$, such that the variables in each row are independent. Then the randomized sums $S_n := \Lambda\sum_{j=1}^{k_n} X_{nj}$ can defined as random variables having the distribution $P_{S_n} = P_{X_{n1}} * P_{X_{n2}} * \ldots * P_{X_{nk_n}}$. Central limit theorems, i.e. the question of the convergence in distribution of S_n after a suitable normalization, have been studied in a number of articles. Most of them are confined to the situation of a sequence $X_{nj} = X_j$, $k_n = n$, for example

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in the work of Kingman [11], Finckh [7], Eymard and Roynette [6], Zeuner [20], Gallardo [8], Voit [16], [17], [18], and Gallardo and Bouhaik [3] (some of the results are valid for different classes of hypergroups), but there are also important contributions in the general case, for example by Trimèche [13], [14] and Karpelevich, Tutubalin and Shur [10].

There are three types of central limit theorems with different kinds of limit distributions: The randomized sums of small random variables lead (without the necessity of a normalization) to *Gaussian* limit distributions which are characterized by the Fourier transform $\lambda \mapsto \exp(-c(\lambda^2 + \rho^2))$ and which are different for different hypergroups. In the case of polynomial growth of the hypergroup it can be shown that for a suitable choice of positive numbers σ_n, S_n/σ_n converges to a *Rayleigh* distribution ρ_α; here the parameter $\alpha \geq -\frac{1}{2}$ depends on properties of the hypergroup. The Bessel-Kingman hypergroups [11] belong to both of these classes. Third, in the case of exponential growth of the hypergroup the usual normalization $\sigma_n^{-1}(S_n - \mu_n)$ with suitable $\mu_n, \sigma_n > 0$ gives the convergence towards the *standard normal* distribution $\mathcal{N}_{0,1}$ on $\mathbb{R}$; here the limit distribution is the same for all hypergroups of this type (but the normalizing constants depend on the convolution $*$ on $\mathbb{R}_+$).

In the first section we study the most important tool for these limit theorems, the *Laplace representation* of the characters of the hypergroup $(\mathbb{R}_+, *)$ (an integral representation in terms of the characters $t \mapsto \exp(i\lambda t)$ of $(\mathbb{R}, +)$) which has been established by Chébli [4] for an important class of hypergroups on $\mathbb{R}_+$, and slightly extended in [21]. Using Voit's modification procedure [15] it is now possible to prove this result for all known hypergroups on $\mathbb{R}_+$. With this representation properties of the moment functions m_n can be derived which are characterized by the formula $\int m_n \, d\varepsilon_x * \varepsilon_y = \sum_{j=0}^{n} \binom{n}{j} m_j(x) m_{n-j}(y)$ and are used to define the adapted expectation and variance of random variables and the normalizing constants in the central limit theorems. In section 2 we prove in the case of exponential growth of the hypergroup a Lindeberg type central limit theorem for triangular arrays such that the Lindeberg condition and certain growth conditions for the moments of the array are valid. In particular these conditions are satisfied for a *sequence* of random variables. In the last section the case of polynomial growth is considered. Here the proof of a central limit theorem is much simpler and the only assumption except the Lindeberg condition is that $\sigma_n \to \infty$; this condition is necessary to exclude the convergence to the Gaussian distribution as in the situation treated in [13].

1. One-dimensional hypergroups and the Laplace representation of their characters

(1.1) Let $(\mathbb{R}_+, *)$ be a hypergroup in the sense of [9]. A function $\phi : \mathbb{R}_+ \to \mathbb{C}$ is called *multiplicative* if $\int \phi \, d\varepsilon_x * \varepsilon_y = \phi(x) \cdot \phi(y)$ for $x, y \in \mathbb{R}_+$, and it is called *character* of the hypergroup if it is in addition real valued and bounded. We call $(\mathbb{R}_+, *)$ a *Sturm-Liouville hypergroup* if there exists a function $\alpha \in \mathcal{C}^1(\mathbb{R}_+^*)$ with a possible singularity of first order at 0 such that all the characters ϕ are solutions of the Sturm-Liouville differential equation $\phi'' + \alpha\phi' = s\phi$, $\phi(0) = 1$ and $\phi'(0) = 0$ with $s \in \mathbb{R}$. In this case it is always possible to write $\alpha = A'/A$ with $A \in \mathcal{C}^1(\mathbb{R}_+)$ where A is the Lebesgue density of a Haar measure of this hypergroup.

The largest class of functions α admitting a Sturm-Liouville hypergroup known up to date has been described in [21] and contains the hypergroups of Chébli-Trimèche and Levitan. Apart from minor technical conditions α belongs to this class if there exists a $\mathcal{C}^1$-function $\beta : \mathbb{R}_+ \to \mathbb{R}$ with $\beta(0) \geq 0$, $\beta \leq \alpha$ such that $\alpha - \beta$ and $2\beta' - \beta^2 + 2\alpha\beta$ are decreasing functions on $\mathbb{R}_+^*$. It has been proved in [21] that $\rho := \frac{1}{2} \lim_{x\to\infty} \frac{A'(x)}{A(x)} \geq 0$ exists and that the characters are the solutions of the initial value problem

$$\varphi_\lambda'' + \frac{A'}{A}\varphi_\lambda' = -(\rho^2 + \lambda^2)\varphi_\lambda$$

$$\varphi_\lambda(0) = 1, \quad \varphi_\lambda'(0) = 0 \text{ for } \lambda \in \hat{K} := i[0, \rho] \cup \mathbb{R}_+.$$

We first study the asymptotic behavior of the characters at ∞.

(1.2) Lemma. $\lim_{x\to\infty} \frac{\phi_{i\lambda}'(x)}{\phi_{i\lambda}(x)} = \lambda - \rho$ *for all* $\lambda \geq 0$.

Proof. It follows from [21], Proposition (4.2) that $\phi_{i\lambda} > 0$. The function $\psi := \phi_{i\lambda}'/\phi_{i\lambda}$ satisfies the differential equation $\psi' + \psi^2 + \frac{A'}{A}\psi = \lambda^2 - \rho^2$. Let $\varepsilon > 0$ and choose $x_0 > 0$ such that $|\frac{A'}{A} - 2\rho| < \varepsilon$ and $|(\frac{A'}{A})^2 - 4\rho^2| < \frac{\varepsilon^2}{2}$ on $[x_0, \infty[$. Then for $x \geq x_0$ it follows from $\psi(x) > \lambda - \rho + \varepsilon$ that $\psi'(x) < -\lambda\varepsilon - \frac{\varepsilon^2}{8}$ and so eventually ψ must stay smaller than $\lambda - \rho + \varepsilon$. A similar argument holds for a lower bound.

The following important representation of the multiplicative functions on a Sturm-Liouville hypergroup has been proved by Chébli under certain convexity conditions on the function A (see [21], (2.2) and (4.8)).

(1.3) Theorem. (Chébli, [4], Proposition 1-IV) *For every $x \in \mathbb{R}_+$ there exists a probability measure $\nu_x \in \mathcal{M}^1([-x,x])$ such that*

$$\varphi_\lambda(x) \;=\; \int e^{-t(\rho+i\lambda)}\nu_x(dt) \quad \textit{for all } \lambda \in \mathbb{C}.$$

If $\rho = 0$ then ν_x is symmetric.

Proof. We consider the case $\rho = 0$ first. Then Chébli's proof [4], Proposition 1-IV carries over to the more general situation in [21], (2.1). We will give a simplified version of his proof:

Let the function $A \in \mathcal{C}_+^1(\mathbb{R}_+)$ of the hypergroup satisfy $A' \geq 0$ and $\lim\limits_{x\to\infty} \frac{A'(x)}{A(x)} = 0$. It follows from $\rho = 0$ that for every $\lambda \in \mathbb{C}$ the function $u : {\mathbb{R}_+}^2 \to \mathbb{C}$ with $u(x,y) = \cos(\lambda x)\cdot\varphi_\lambda(y)$ satisfies $\ell[u] = 0$ where

$$\ell[u](x,y) := A(y)u_{xx}(x,y) - A(y)u_{yy}(x,y) - A'(y)u_y(x,y).$$

It follows from Chébli's maximum principle [4], Théorème (III-1) (which we may apply on ℓ because of $A' \geq 0$) that every solution u of this Cauchy problem with $u \geq 0$ and $u_y = 0$ on $\mathbb{R}_+ \times \{0\}$ is non negative in $\{(x,y) \in \mathbb{R}_+ : 0 \leq y \leq x\}$. Therefore for all (x,y) in this set there exists a positive measure μ_{xy} on $[x-y, x+y]$ with $u(x,y) = \int u(t,0)\,d\mu_{xy}(t)$ and in particular

$$\varphi_\lambda(y)\cdot\cos(\lambda x) = \int \cos(\lambda t)\,d\mu_{xy}(t)$$

for all $0 \leq y \leq x$.

Now let $*_\mathbb{R}$ be the usual convolution on $\mathbb{R}$ and $y \geq 0$. Then the (possibly signed) measure $\theta_y := \mu_{y,y} *_\mathbb{R} (\varepsilon_y + \varepsilon_{-y}) - \mu_{2y,y}$ has the support $[-y, 3y]$ and satisfies

$$\varphi_\lambda(y) = 2\varphi_\lambda(y)\cos^2(\lambda y) - \varphi_\lambda(y)\cos(2\lambda y) = \int \cos(\lambda t)\,d\theta_y(t).$$

Let $\sigma_1 \in \mathcal{M}^b(\mathbb{R}_+)$ be the measure θ_y shifted to the right by $3y$ and $\sigma_2 \in \mathcal{M}^b(\mathbb{R}_+)$ be the result of the shift of θ_y to the left by $3y$ and a subsequent reflection at the origin. Then we have $\int \cos(\lambda t)\,d(\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2)(t) = \varphi_\lambda(y)\cdot\cos(3\lambda y) = \int \cos(\lambda t)\,d\mu_{3y,y}(t)$ for all λ and it follows from the uniqueness theorem of the cosine transformation that $\frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2 = \theta_y$. Since σ_1

is supported by $[2y, 6y]$ and σ_2 by $[0, 4y]$ it follows that σ_1 cannot have mass on $]4y, 6y]$ and therefore the support of θ_y must be contained in $[-y, y]$. If ν_y denotes the symmetrization of θ_y then we still have $\varphi_\lambda(y) = \int \cos(\lambda t)\, d\nu_y(t) = \int \exp(-i\lambda t)\, d\nu_y(t)$. If we repeat the arguments from above we see that $\mu_{3y,y}$ is equal to the measure ν_y shifted to the right by $3y$ and so ν_y must be a probability measure.

We now consider the case $\rho > 0$. Then the multiplicative function ϕ_0 is positive by [21], (4.2) b) and we can construct the modified hypergroup $(\mathbb{R}_+, \star)$ with

$$\varepsilon_x \star \varepsilon_y := \frac{\phi_0}{\phi_0(x)\phi_0(y)} \cdot \varepsilon_x * \varepsilon_y$$

as in [15]. It has the Haar measure $\tilde{A} \cdot \lambda_{\mathbb{R}_+}$ where $\tilde{A} = \phi_0^2 \cdot A$. By [21], (4.6) and (2.11) it satisfies $\tilde{A}' \geq 0$. Moreover we have $\tilde{\rho} = \frac{1}{2} \lim_{x\to\infty} \tilde{A}'(x)/\tilde{A}(x)$ $= \lim_{x\to\infty} \phi_0'(x)/\phi_0(x) + \rho = 0$ by (1.2) and hence we can use the above to obtain a Laplace representation for the characters of $(\mathbb{R}_+ . *)$, $\tilde{\phi}_\lambda(x) = \int \exp(-i\lambda t)\, d\tilde{\nu}_x(t)$ for all $\lambda \in \mathbb{C}, x \in \mathbb{R}_+$. Since $\tilde{\phi}_\lambda = \varphi_\lambda/\phi_0$ we obtain $\varphi_\lambda(x) = \int \exp(-i\lambda t)\phi_0(x)\, d\tilde{\nu}_x(t)$.

For $\lambda := i\rho$ we have $1 = \int \exp(\rho t)\phi_0(x)\, d\tilde{\nu}_x(t)$ and hence the measure ν_x with density $t \mapsto \exp(\rho t)\phi_o(x)$ with respect to $\tilde{\nu}_x$ is a probability measure for every $x \in \mathbb{R}_+$. Therefore we have proved $\varphi_\lambda(x) = \int \exp(-t(\rho + i\lambda))\, d\nu_x(t)$.

(1.4) The moment functions $m_n : \mathbb{R}_+ \to \mathbb{R}_+$ of this hypergroup can be defined as $m_n(x) := \frac{\partial}{\partial \mu^n} \phi_{i(\rho+\mu)}(x)\big|_{\mu=0}$. They are the solutions of the differential equation

$$m_n'' + \frac{A'}{A} m_n' = 2n\rho m_{n-1} + n(n-1)m_{n-2}, \quad m_n(0) = m_n'(0) = 0 \text{ for } n \geq 1,$$

with $m_0 := 1, m_{-1} := 0$. The defining equation

$$\int m_n \, d\varepsilon_x * \varepsilon_y = \sum_{j=0}^{n} \binom{n}{j} m_j(x) m_{n-j}(y)$$

follows by Leibniz's rule. It should be noted that for $\rho = 0$ only the functions m_n with n even are non zero.

Using the Laplace representation it is now very easy to prove the following properties of the moment functions which will be important in the proof of the central limit theorems:

(1.5) Corollary. **a)** *$\|\varphi_\lambda\|_\infty = 1$ for all $\lambda \in \mathbb{C}$ with $|\Im\lambda| \le \rho$.*

b) *$m_n(x) = \int_{-x}^{x} t^n \, d\nu_x(t)$ for $n \ge 1$ and $x \in \mathbb{R}_+$. From this it follows that $m_k(x)^n \le m_{kn}(x) \le x^{kn}$ for $k, n \ge 1$ and $x \in \mathbb{R}_+$.*

c) *In the case $\rho > 0$ $m_1'(x)$ converges to 1 and $m_n(x)/x^n \to 1$ as $x \to \infty$.*

d) *Let $\rho = 0$. Then we have $1 - \frac{\lambda^2}{2} m_2 \le \varphi_\lambda \le 1 - \frac{\lambda^2}{2} m_2 + \frac{\lambda^4}{24} m_4$. Furthermore, if $\sup_{x>0} x A'(x)/A(x)$ is finite then there exists $\gamma > 0$ with $m_2(x) \ge \gamma x^2$ for all $x \ge 0$.*

Proof. Most of these properties have been proved in [19] (4.3), (5.7) and (6.5).

a) For these λ the modulus of the integrand in the Laplace representation (1.3) is bounded by $\exp(-t\rho)$, and the integral over this function gives $\phi_{i\rho} = 1$.

b) This follows from Jensen's inequality and the fact that the support of ν_x is contained in $[-x, x]$.

c) The last assertion is a consequence of the first and b).

d) The cosine function satisfies $1 - \frac{x^2}{2} \le \cos(x) \le 1 - \frac{x^2}{2} + \frac{x^4}{24}$ for all $x \in \mathbb{R}$. Therefore we have $1 - \frac{\lambda^2}{2} m_2(x) = \int (1 - \frac{\lambda^2}{2} t^2) \, d\nu_x(t) \le \int \cos(\lambda t) \, d\nu_x(t) = \varphi_\lambda(x) \le \int (1 - \frac{\lambda^2}{2} t^2 + \frac{\lambda^4}{24} t^4) \, d\nu_x(t) = 1 - \frac{\lambda^2}{2} m_2(x) + \frac{\lambda^4}{24} m_4(x)$.

(1.6) We can now use the moment functions m_1 and m_2 to define the adapted moments of a random variable X with values in the hypergroup $(\mathbb{R}_+, *)$:

$$E_*(X) := E(m_1(X))$$

and, in the case of $E_*(X) < \infty$,

$$\begin{aligned} V_*(X) &:= E(m_2(X)) - [E_*(X)]^2 \\ &= E\left(m_2(X) - 2m_1(X)E_*(X) + [E_*(X)]^2\right) \ge 0. \end{aligned}$$

Similar to the classical situation we have a Chebyshev inequality:

(1.7) Lemma. *If $E_*(X)$ is finite, then $P\{|m_1(X) - E_*(X)| \ge \varepsilon\} \le V_*(X)/\varepsilon^2$. In particular, if $m_1(a) > E_*(X)$ then we have*

$$P\{X \ge a\} \quad \le \quad \frac{V_*(X)}{(m_1(a) - E_*(X))^2} \ .$$

2. Moment conditions

(2.1) *Definition.* Let $(X_{nj} : n \geq 1, 1 \leq j \leq k_n)$ be a triangular array of random variables with values in $\mathbb{R}_+$ such that $E(X_{nj}^2) < \infty$ for all n, j. We assume that the random variables of each row are independent and denote by S_n their randomized sum $\Lambda\sum_{j=1}^{k_n} X_{nj}$ (for the definition of this random variable having the distribution $P_{X_{n1}} * \cdots * P_{X_{nk_n}}$ see [2], 7.1.5 or [21]). We use the notations

$$\begin{aligned}
\mu_{nj} &:= E_*(X_{nj}) = E(m_1(X_{nj})),\\
\mu_n &:= E_*(S_n) = \sum_{j=1}^{k_n} \mu_{nj},\\
\sigma_{nj}^2 &:= V_*(X_{nj}) = E(m_2(X_{nj})) - (E(m_1(X_{nj})))^2 \text{ and}\\
\sigma_n^2 &:= V_*(S_n) = \sum_{j=1}^{k_n} \sigma_{nj}^2
\end{aligned}$$

To avoid division by 0 we exclude the trivial case that all X_{nj} in some row are 0 almost surely; since $m_2 - m_1^2 > 0$ on $]0, \infty[$, this implies $\sigma_n > 0$ for all $n \geq 1$.

Then we say that the *Lindeberg condition* is satisfied if

$$(L) \quad \sigma_n^{-2} \sum_{j=1}^{k_n} E\Big(X_{nj}^2 \cdot 1_{\{X_{nj} \geq \varepsilon\sigma_n\}}\Big) \;\to\; 0 \qquad \text{as } n \to \infty \text{ for every } \varepsilon > 0.$$

(2.2) *Remark.* This condition—and the classical Lindeberg condition, which limits the deviation from the expected value $E_*(X_{nj})$, all the less—is not sufficient to imply the convergence in distribution of $\sigma_n^{-1}(S_n - \mu_n)$ towards a standard normal distribution $\mathcal{N}_{0,1}$. On the one hand it follows from the results of Trimèche [13] that if σ_n, μ_n and some higher moments remain small, S_n itself converges in distribution to the Gaussian distribution of the hypergroup which in general is different from $\mathcal{N}_{0,1}$.

On the other hand consider the Sturm-Liouville hypergroup with $A(x) = \cosh(x)^2$ where the randomized sum $x \overset{\Lambda}{+} y$ takes the values $|x - y|$ and $x + y$ with probabilities $\frac{\cosh(x-y)}{2\cosh(x)\cosh(y)}$ and $\frac{\cosh(x+y)}{2\cosh(x)\cosh(y)}$ respectively. Then $m_1(x) = x\tanh x$, $m_2(x) = x^2$ and $V_*(x) = x^2/\cosh(x)^2$. If X_{nj}

is the constant random variable 2^j (for $1 \leq j \leq k_n = n$) then $\sigma_n^2 = \sum_{j=1}^n \left(\frac{2^j}{\cosh 2^j}\right)^2$ is bounded and $\mu_n = \sum_{j=1}^n 2^j \tanh 2^j \sim 2^{n+1}$. The classical Lindeberg condition is clearly satisfied (although (L) is not) but it is not hard to show that $S_n - 2^{n+1}$ converges to the distribution

$$(1-e^{-2})^{-1} \sum_{j=0}^{\infty} e^{-2j} \cdot \varepsilon_{-2j-1}$$

on $\mathbb{R}$.

If for the same convolution we set $X_{nj} := \frac{1}{2} \ln j$ for $1 \leq j \leq n$ then we have $\mu_n \sim \frac{n}{2} \ln n$, $\sigma_n^2 \sim \frac{1}{3} (\ln n)^3$, and it is clear that the Lindeberg condition (L) in (2.1) is satisfied. It is not hard to show that in this case $\frac{2}{\sqrt{n} \ln n}(S_n - \frac{1}{4} n (\ln n)^2)$ converges in distribution to the standard normal law $\mathcal{N}_{0,1}$. Therefore $\sigma_n^{-1}(S_n - \mu_n)$ cannot converge in distribution. (This phenomenon that the central limit theorem is valid with different norming constants also happens in the classical situation (see for example [5], exercise 7.2.10).)

It is therefore necessary to impose some additional conditions on the random variables of the triangular array (but none on the hypergroup):

(2.3) *Definition.* Let $(X_{nj} : 1 \leq n, 1 \leq j \leq k_n)$ be a triangular array and μ_{nj}, μ_n and σ_n be defined as in (2.1) . Then we impose the following conditions:

$$(I_1) \qquad \sigma_n \quad \to \quad \infty \qquad \text{for } n \to \infty,$$

and

$$(I_2) \qquad \sum_{j=1}^{k_n} \mu_{nj}^2 \quad = \quad O(\sigma_n^2) \qquad \text{for } n \to \infty.$$

(2.4) *Remark.* These conditions, including the Lindeberg condition (L), are all satisfied if $X_{nj} = X_j$ comes from a sequence of identically distributed random variables since $\sum_{j=1}^n \mu_{nj}^2$ and σ_n^2 are proportional to $k_n = n$ and by [1], Example 51.1. More generally, in the case that the triangular array consists of the beginning parts of an arbitrary independent sequence, condition (I_1) is clearly fulfilled if (L) is valid since the sum in (L) increases with n.

In order to prove a central limit theorem we need some auxiliary results. Some of them are the same as in the classical situation, for example that the Lindeberg condition implies the Feller condition:

(2.5) Lemma. *Suppose that the array (X_{nj}) satisfies the Lindeberg condition (L). Then*

a) $\sigma_n^{-1} \max_{j \le k_n} \mu_{nj}$ *converges to 0 for* $n \to \infty$.

b) $\sigma_n^{-1} \max_{j \le k_n} \sigma_{nj}$ *converges to 0 for* $n \to \infty$.

Proof. Let $\varepsilon > 0$ be arbitrary. For every j we have

$$\begin{aligned} \mu_{nj}^2 + \sigma_{nj}^2 &\le E(m_2(X_{nj}) 1_{\{X_{nj} < \varepsilon\sigma_n\}}) + E(m_2(X_{nj}) 1_{\{X_{nj} \ge \varepsilon\sigma_n\}}) \\ &\le m_2(\varepsilon\sigma_n) + \sum_{k=1}^{k_n} E(m_2(X_{nk}) 1_{\{X_{nk} \ge \varepsilon\sigma_n\}}). \end{aligned}$$

Therefore $\sigma_n^{-2} \max_{j=1,\dots,k_n} (\mu_{nj}^2 + \sigma_{nj}^2) \le \varepsilon^2 + \sigma_n^{-2} \sum_{k=1}^{k_n} E(X_{nk} 1_{\{X_{nk} \ge \varepsilon\sigma_n\}})$; the first summand can be made arbitrary small and the second converges to 0 for every $\varepsilon > 0$ by (L). This implies a) and b).

(2.6) *Remark.* We have seen in (2.2) that the conditions (I_1), (I_2) and (L) are essential for the validity of the central limit theorem. It should also be noted that these conditions are independent from each other. Let us consider the hypergroup with $A := \sinh^2$. It is an easy calculation that $V_*(x) = 1 - \frac{x^2}{\sinh^2(x)}$ and this converges to 1 as $x \to \infty$. Therefore if we define the sequence $X_n := k \ln k$ if $n = k^3$ and $X_n := 1$ for all other n, we have $\sigma_n \sim c \cdot \sqrt{n}$ and $\sum_{j=1}^n \mu_{nj}^2 \sim \frac{n}{27} (\ln n)^2$ whence all conditions except I_2 are satisfied (and even the stronger Lyapounov condition).

On the other hand, if we consider the sequence $X_n := 2^{k/2}$ if $n = 2^k$ and $X_n := 1$ for n which are not of this form, then $\sigma_n \sim c \cdot \sqrt{n}$ and $\sum_{j=1}^n \mu_{nj}^2 = O(n)$. Therefore (I_1) and (I_2) are valid. Since the Feller condition (2.1) a) is violated, (L) is also not valid.

Finally, if $X_{nj} = \frac{1}{n}$ and $k_n := n^2$, then $\sigma_n \to \frac{1}{\sqrt{3}}$ and (I_2) and (L) are satisfied, but (I_1) is not.

The following properties of the moments μ_{nj} and σ_{nj} will be used in the proof of the central limit theorem.

(2.7) Lemma. a) *If (I_2) is satisfied then $\mu_n = O(\sigma_n^2)$.*

b) *Let (I_2) and the Feller conditions (2.5) be valid. Then*

$$\lim_{n \to \infty} \sigma_n^{-k-l} \cdot \sum_{j=1}^{k_n} \mu_{nj}^k \sigma_{nj}^l = 0$$

for all $k, l \in \mathbb{N}$ such that $k + l \ge 3$.

Proof. **a)** Only the case $\rho > 0$ has to be proved. Since m_1, m_2 and their first derivatives at 0 vanish whereas the second derivatives are positive, the function m_2/m_1 is continuous and strictly positive on $\mathbb{R}_+$ and converges to ∞ by (1.5) c). Therefore there exists a lower bound $c > 0$, i.e. $m_2 \geq c \cdot m_1$ on $\mathbb{R}_+$. This implies $\sigma_{nj}^2 = E(m_2(X_{nj})) - \mu_{nj}^2 \geq \mu_{nj}(c - \mu_{nj})$. For every n, j with $\mu_{nj} \leq c/2$ it follows $\mu_{nj} \leq \frac{2}{c} \cdot \sigma_{nj}^2$, and for the others we have $\mu_{nj} < \frac{2}{c} \cdot \mu_{nj}^2$. Combining these inequalities we obtain $\sum_j \mu_{nj} \leq \frac{2}{c} \cdot \left(\sum_j \sigma_{nj}^2 + \sum_j \mu_{nj}^2\right)$. By (I_2) there exists a number $K > 0$ with $\sum_j \mu_{nj}^2 \leq K\sigma_n^2$ and hence $\sum_j \mu_{nj} \leq \frac{2}{c} \cdot (K+1) \cdot \sigma_n^2$.

b) If $k \geq 2$ then

$$\sigma_n^{-k-l} \cdot \sum_j \mu_{nj}^k \sigma_{nj}^l \leq \sigma_n^{-2} \sum_j \mu_{nj}^2 \cdot \sigma_n^{-k+2} \max\{\mu_{nj}^{k-2}\} \cdot \sigma_n^{-l} \max\{\sigma_{nj}^l\}$$

and by the assumptions all of the factors are bounded and at least one of them converges to 0. A similar but simpler argument holds in the case $l \geq 2$ since $\sigma_n^{-2} \sum_j \sigma_{nj}^2 = 1$.

3. The case of exponential growth

We now consider the case $\rho > 0$ of exponential growth. The next result has been proved in [20] in the special case of Chébli hypergroups. The possibility that A'/A can now be smaller than 2ρ (this happens, for example, in the case $A = \cosh^2$) makes the general situation more difficult.

(3.1) Lemma. *Let A satisfy $\rho > 0$. Then for every $a \geq 0$ we have*

$$\sup_{x \in [0, r^2 a]} \left| \phi_{i\rho - \lambda/r}(x) - \exp\left[\frac{i\lambda}{r} \cdot m_1(x)\right] \right| \;\to\; 0 \quad \textit{for } r \to \infty.$$

Proof. Let $\zeta_r(x) := \phi_{i\rho-\lambda/r}(rx) \cdot \exp\left[-\frac{i\lambda}{r} \cdot m_1(rx)\right]$ for $x \in \mathbb{R}$, $r > 1$. It follows from (1.5) a) that $\|\zeta_r\|_\infty = \|\phi_{i\rho-\lambda/r}\|_\infty = 1$ and therefore as in the proof of [20], Lemma (4.1) we obtain from the differential equation for ζ_r that

$$|\zeta_r'(x)|^2 + 2r \cdot \int_0^x \frac{A'(rt)}{A(rt)} |\zeta'(t)|^2 \, dt \quad \leq \quad 2\lambda^2 \cdot \int_0^x |1 - m_1'(rt)^2| \cdot |\zeta_r'(t)| \, dt$$

and since $m_1'(x)$ converges to 1 as $x \to \infty$ (see (1.5) c)), the right hand side is not larger than $c_1 \cdot \int_0^x |1 - m_1'(rt)| \cdot |\zeta_r'(t)|\, dt$ where $c_1 := 2\lambda^2(1 + \|m_1'\|_\infty)$.

We now use the notations $M_r(x) := \int_0^x |1 - m_1'(rt)|^2\, dt$ and $Z_r(x) := \int_o^x |\zeta_r'(t)|^2\, dt$. Using Hölder's inequality we obtain

$$|\zeta_r'(x)|^2 + 2r \cdot \int_0^x \frac{A'(rt)}{A(rt)} |\zeta'(t)|^2\, dt \quad \leq \quad c_1 \cdot \sqrt{M_r(x) Z_r(x)}.$$

It follows from [21], (2.1) that $A'(x)/A(x)$ converges to 2ρ for $x \to \infty$. Therefore we can choose $x_0 > 0$ with $A'/A \geq \rho$ on $[x_0, \infty[$. In the case $t \leq x_0/r$ we have $|\zeta_r'(t)|^2 \leq c_1 \sqrt{M_r(t) Z_r(t)} \leq c_1 \sqrt{M_r(x_0/r) Z_r(x_0/r)}$ and hence $r \int_0^{x_0/r} |\zeta_r'(t)|^2\, dt \leq r \cdot \frac{x_0}{r} \cdot c_1 \sqrt{M_r(x_0/r) Z_r(x_0/r)}$. On the other hand, for $x > x_0/r$ we have $r \int_{x_0/r}^x |\zeta_r'(t)|^2\, dt \leq \frac{r}{\rho} \int_{x_0/r}^x \frac{A'(rt)}{A(rt)} \cdot |\zeta_r'(t)|^2\, dt \leq \frac{c_1}{2\rho} \sqrt{M_r(x) Z_r(x)}$. Combining these two inequalities we obtain for all $x \geq x_0 > x_0/r$

$$\begin{aligned} r Z_r(x) &\leq x_0 c_1 \sqrt{M_r(x_0/r) Z_r(x_0/r)} + \frac{c_1}{2\rho} \sqrt{M_r(x) Z_r(x)} \\ &\leq c_1 (x_0 + \frac{1}{2\rho}) \sqrt{M_r(x) Z_r(x)} \end{aligned}$$

which implies

$$r^2 Z_r(x) \leq c_2 M_r(x) \quad \text{for all } x \geq x_0.$$

From this it follows using Hölder's inequality in the same way as in the proof of [20], Lemma (4.1), that $|1 - \zeta_r(t)| \leq [x \cdot Z_r(x)]^{1/2} \leq \sqrt{c_2} \sqrt{\frac{x}{r}} \sqrt{\frac{M_r(x)}{r}}$ and hence

$$\begin{aligned} &\sup_{t \in [0, r^2 a]} \left| \phi_{i\rho - \lambda/r}(t) - \exp\left[\frac{i\lambda}{r} \cdot m_1(t)\right] \right| \\ &\qquad = \sup_{x \in [0, ra]} |\zeta_r(x) - 1| \quad \leq \quad \sup_{x \leq ra} \sqrt{c_2} \sqrt{\frac{x}{r}} \sqrt{\frac{M_r(x)}{r}} \\ &\qquad \leq \sqrt{c_2 a} \sqrt{\frac{M_r(ra)}{r}}. \end{aligned}$$

In order to prove the assertion it is therefore sufficient to show $M_r(ra)/r \to 0$ for every $a \geq 0$. We have $\frac{M_r(ra)}{r} = \frac{1}{r} \int_0^{ra} |1 - m_1'(rx)|^2\, dx = \int_0^a |1 - m_1'(r^2 t)|^2\, dt$. The integrand is bounded and converges to 0 for all $t > 0$ ((1.5) c)) and the assertion follows from Lebesgue's theorem of bounded convergence.

The next Lemma is a well known fact from calculus (see the inequality after (51.12) in [1]).

(3.2) Lemma. *Let $a_{nj}, b_{nj} \in \mathbb{C}$ satisfy $\lim\limits_{n\to\infty} \sum_j |a_{nj} - b_{nj}| = 0$ and*

$$\prod_j \max(a_{nj}, 1), \prod_j \max(b_{nj}, 1) \leq K \qquad \text{for all } n.$$

Then $\lim\limits_{n\to\infty} \prod_j a_{nj}$ exists if and only $\lim\limits_{n\to\infty} \prod_j b_{nj}$ exists and in this case the limits are equal.

(3.3) Theorem. *Let $(X_{nj} : n \geq 1, 1 \leq j \leq k_n)$ be a triangular array, and $S_n := \Lambda \sum\limits_{j=1}^{k_n} X_{nj}$. Suppose that (I_1), (I_2) and the Lindeberg condition (L) are satisfied and $\tilde{\mu}_n := m_1^{-1}(\mu_n)$. Then $\sigma_n^{-1}(S_n - \tilde{\mu}_n)$ converges in distribution to the standard normal distribution $\mathcal{N}_{0,1}$.*

Proof. In the first part we will prove that the distribution of $\sigma_n^{-1}(m_1(S_n) - \mu_n)$ converges to $\mathcal{N}_{0,1}$ by using Lévy's continuity theorem on $\mathbb{R}$ and showing the convergence of the characteristic functions for each $\lambda \in \mathbb{R}$. In the following equations the expression $a_n \approx b_n$ means that the sequences $(a_n)_n$ and $(b_n)_n$ have the same limit; we will first give an outline of the proof and fill in the details for each $\approx$ afterwards:

$$\begin{aligned}
&E\Big(\exp\Big(i\lambda \frac{m_1(S_n) - \mu_n}{\sigma_n}\Big)\Big) \\
&\quad= \exp(-i\lambda\mu_n/\sigma_n) \cdot E\Big(\exp\Big(\frac{i\lambda}{\sigma_n} \cdot m_1(S_n)\Big)\Big) \\
&\quad\approx \exp(-i\lambda\mu_n/\sigma_n) \cdot E(\phi_{i\rho-\lambda/\sigma_n}(S_n)) \\
&\quad= \prod_{j=1}^{k_n} \exp(-i\lambda\mu_{nj}/\sigma_n) \cdot E(\phi_{i\rho-\lambda/\sigma_n}(X_{nj})) \\
&\quad\approx \prod_{j=1}^{k_n} \Big(1 - \frac{i\lambda\mu_{nj}}{\sigma_n} - \frac{\lambda^2\mu_{nj}^2}{2\sigma_n^2}\Big) \cdot E(\phi_{i\rho-\lambda/\sigma_n}(X_{nj}))
\end{aligned}$$

$$\approx \prod_{j=1}^{k_n}\Big(1-\frac{i\lambda\mu_{nj}}{\sigma_n}-\frac{\lambda^2\mu_{nj}^2}{2\sigma_n^2}\Big)\cdot\Big(1+\frac{i\lambda\mu_{nj}}{\sigma_n}-\frac{\lambda^2(\mu_{nj}^2+\sigma_{nj}^2)}{2\sigma_n^2}\Big)$$
$$\approx \prod_{j=1}^{k_n}\Big(1-\frac{\lambda^2\sigma_{nj}^2}{2\sigma_n^2}\Big)$$
$$\approx \prod_{j=1}^{k_n}\exp\Big(-\frac{\lambda^2\sigma_{nj}^2}{2\sigma_n^2}\Big)$$
$$= \exp(-\lambda^2/2)$$
$$= \widehat{\mathcal{N}_{0,1}}(\lambda).$$

We will now give the arguments for the validity of each $\approx$:

$\approx_1$: Let $r>0$. Then $\big|E\big(\exp\big(\frac{i\lambda}{\sigma_n}m_1(S_n)\big)\big)-E(\phi_{i\rho-\lambda/\sigma_n}(S_n))\big| \leq 2P\{S_n>r\sigma_n^2\}+E\big(\big|\exp\big(\frac{i\lambda}{\sigma_n}m_1(S_n)\big)-\phi_{i\rho-\lambda/\sigma_n}(S_n)\big|1_{\{S_n\leq r\sigma_n^2\}}\big)$. The first term is smaller than $2\sigma_n^2/(m_1(r\sigma_n^2)-\mu_n)^2$ by (1.7) as soon as $m_1(r\sigma_n^2)-\mu_n>0$; but because of (2.7) a) this can be achieved for large enough $r>0$. Furthermore this upper bound can be made arbitrary small if we choose r big. On the other hand, by (3.1) the second term converges to 0 for every $r>0$ and hence the first $\approx$ is valid.

$\approx_2$: $\sum_j\big|\exp\big(-\frac{i\lambda\mu_{nj}}{\sigma_n}\big)-\big(1-\frac{i\lambda\mu_{nj}}{\sigma_n}-\frac{\lambda^2\mu_{nj}^2}{2\sigma_n^2}\big)\big| \leq \frac{\lambda^3}{3\sigma_n^3}\sum_j\mu_{nj}^3$ as soon as $\max_j\sigma_n^{-1}\lambda\mu_{nj}\leq 2$ (which happens for large n by (2.5) a)). But this upper bound converges to 0 by (2.7) b). The second condition of (3.2) follows from $\prod_j\big|1-\frac{i\lambda\mu_{nj}}{\sigma_n}-\frac{\lambda^2\mu_{nj}^2}{2\sigma_n^2}\big|=\prod_j\big(1+\frac{\lambda^4\mu_{nj}^4}{4\sigma_n^4}\big)^{1/2}\leq\exp\big(\frac{\lambda^4}{8\sigma_n^4}\sum_j\mu_{nj}^4\big)$ and again using (2.7) b).

$\approx_3$: We have $\sum_j\big|E(\phi_{i\rho-\lambda/\sigma_n}(X_{nj}))-\big(1+\frac{i\lambda}{\sigma_n}\mu_{nj}-\frac{\lambda^2}{2\sigma_n^2}(\mu_{nj}^2+\sigma_{nj}^2)\big)\big| \leq \sum_j E(|r_3(\lambda/\sigma_n,X_{nj})|)$ where r_3 is the remainder term in the Taylor expansion $\phi_{i\rho-\delta}(x)=1+i\delta m_1(x)-\frac{\delta^2}{2}m_2(x)+r_3(\delta,x)$. From [19], 4.3 we conclude that $|r_3(\lambda/\sigma_n,x)|\leq\frac{\lambda^3}{6\sigma_n^3}m_3(x)$ and hence we obtain the upper bound $\sum_j E\big(\frac{\lambda^3}{6\sigma_n^3}X_{nj}m_2(X_{nj})\cdot 1_{\{X_{nj}\leq\varepsilon\sigma_n\}}\big)+\sum_j E\big(r_3(\lambda/\sigma_n,X_{nj})\cdot 1_{\{X_{nj}>\varepsilon\sigma_n\}}\big)$ where $\varepsilon>0$ can be chosen arbitraryly. The first number is smaller than $\frac{\lambda^3\varepsilon}{6\sigma_n^2}\sum_j(\mu_{nj}^2+\sigma_{nj}^2)$ which is bounded by a multiple of ε because of (I_2) and hence can be made arbitrarily small. The second number is not larger than $\sum_j E\big(\big(2+\frac{\lambda}{\sigma_n}m_1(X_{nj})+\frac{\lambda^2}{2\sigma_n^2}m_2(X_{nj})\big)1_{\{X_{nj}>\varepsilon\sigma_n\}}\big) \leq \sigma_n^{-2}\big(\frac{2}{\varepsilon^2}+\frac{\lambda}{\varepsilon}+\frac{\lambda^2}{2}\big)\cdot\sum_j E(X_{nj}^2 1_{\{X_{nj}>\varepsilon\sigma_n\}})$, and this converges to 0 by the Lindeberg condition (L). We can therefore apply (3.2).

$\approx_4$: Since the terms of order ≤ 2 cancel, the condition to check in (3.2) leads to $\sum_j \frac{\lambda^3}{2\sigma_n^3}\mu_{nj}(2\mu_{nj}^2+\sigma_{nj}^2)+\frac{\lambda^4}{4\sigma_n^4}\mu_{nj}^2(\mu_{nj}^2+\sigma_{nj}^2)$. Here we can again apply (2.7) b).

$\approx_5$: Finally $\sum_j\big|\big(1-\frac{\lambda^2\sigma_{nj}^2}{2\sigma_n^2}\big)-\exp\big(-\frac{\lambda^2\sigma_{nj}^2}{2\sigma_n^2}\big)\big|\leq\sum_j 2\frac{\lambda^4\sigma_{nj}^4}{8\sigma_n^4}\to 0$ by similar arguments as for $\approx_2$.

Now let $\Delta(x):=m_1^{-1}(x)-x$. In the final part of the proof we will show that the difference $\sigma_n^{-1}(\Delta(m_1(S_n))-\Delta(\mu_n))$ between $\sigma_n^{-1}(S_n-\tilde{\mu}_n)$ and $\sigma_n^{-1}(m_1(S_n)-\mu_n)$ converges to 0 in probability. This implies the assertion of the theorem.

We have $\Delta'(x)\to 0$ as $x\to\infty$ by (1.5) c). Therefore for every $\varepsilon>0$ we can choose $x_\varepsilon>0$ such that $|\Delta(x)-\Delta(y)|\leq\varepsilon^{3/2}\cdot|x-y|$ for all $x,y\geq x_\varepsilon$. With a similar argument as in $\approx_1$ we see that $P\{m_1(S_n)<2x_\varepsilon\}$ converges to 0 as $n\to\infty$. Hence $\mu_n=E_*(S_n)=E(m_1(S_n))\geq x_\varepsilon$ if n is large enough. But then by (1.7)

$$\begin{aligned}
&P\{\sigma_n^{-1}|\Delta(m_1(S_n))-\Delta(\mu_n)|\geq\varepsilon\}\\
&\quad\leq P\{m_1(S_n)<x_\varepsilon\}+P\{m_1(S_n)\geq x_\varepsilon,\ |\Delta(m_1(S_n))-\Delta(\mu_n)|\geq\sigma_n\varepsilon\}\\
&\quad\leq P\{m_1(S_n)<x_\varepsilon\}+P\{|m_1(S_n)-\mu_n|\geq\sigma_n\varepsilon^{-1/2}\}\\
&\quad\leq P\{m_1(S_n)<x_\varepsilon\}+\varepsilon.
\end{aligned}$$

4. The case of polynomial growth

In this part we suppose that $A'(x)/A(x)\to 0$ as $x\to\infty$. Since in the case of hypergroups there is no dichotomy between exponential and polynomial growth as in the group case, and since in the case of subexponential but not polynomial growth the asymptotical behavior of random walks is much more complicated (see [18], Theorem 3.5) we have to impose the additional assumption that

$$\lim_{x\to\infty} x\cdot A'(x)/A(x)=2\alpha+1$$

where $\alpha\geq-\frac{1}{2}$. This number (which is used instead the more natural number $2\alpha+1\geq 0$ because of its significance as a parameter of the Bessel functions) determines the limit distribution in the central limit theorem. The corresponding hypergroups are of polynomial growth since the Haar measure ω of these hypergroups satisfies $\omega([0,x]) = o(x^{\beta+1})$ as $x \to \infty$ for every $\beta > 2\alpha + 1$.

The most important cases of hypergroups of polynomial growth are the *Bessel-Kingman hypergroups* with parameter $\alpha \geq -\frac{1}{2}$. They are determined by the characters $\varphi_\lambda(x) = j_\alpha(\lambda x)$ where $j_\alpha(x) := \Gamma(\alpha + 1)(2/x)^\alpha \mathcal{J}_\alpha(x)$ are *modified Bessel functions*. Here $A(x) = x^{2\alpha+1}$ (and so the function $xA'(x)/A(x)$ from above is constant). As limit distribution in the central limit theorem we will obtain the Rayleigh distribution ρ_α which has the density $x \mapsto c_\alpha \cdot x^{2\alpha+1} \exp(-x^2/2)$ on $\mathbb{R}_+$. This is the Gaussian distribution of this hypergroup; its Fourier transform (which is the Hankel transform in this case) is the function $\lambda \mapsto \int_0^\infty j_\alpha(\lambda x)\, d\rho_\alpha(x) = \exp(-\lambda^2/2)$.

(4.1) Lemma. *If* $2\alpha + 1 := \lim_{x\to\infty} xA'(x)/A(x)$ *exists then for every* $a > 0$

$$\sup_{x\in[0,ra]} \left|\phi_{1/r}(x) - j_\alpha(x/r)\right| \to 0 \qquad \text{as } r \to \infty.$$

Proof. It is easily checked that the proof of [20], Lemma 5.3 generalizes to this situation since $A'(x) \geq 0$ (see [21], Corollary (2.11)).

(4.2) Theorem. *Let* $\lim_{x\to\infty} xA'(x)/A(x) = 2\alpha + 1$, *and* $(X_{nj} : 1 \leq j \leq k_n)$ *be a triangular array such that* $\sigma_n \to \infty$ *and the Lindeberg condition (L) in (2.1) are satisfied. Then the randomized sums* S_n *of each row satisfy* $\sigma_n^{-1} S_n \to \rho_\alpha$ *in distribution.*

Proof. Let $\varepsilon > 0$. Then we can choose $r > 0$ such that $P\{S_n > r\sigma_n\} < \varepsilon$ and by Lemma (4.1) and $\sigma_n \to \infty$ we find $n_0 \geq 1$ such that

$$\left|j_\alpha\Big(\frac{\lambda x}{\sigma_n}\Big) - \phi_{\lambda/\sigma_n}(x)\right| < \varepsilon \qquad \text{for all } x \in [0, r\sigma_n], n \geq n_0.$$

Thus

$$\begin{aligned}
&\left|E(j_\alpha(\frac{\lambda}{\sigma_n} S_n)) - E(\phi_{\lambda/\sigma_n}(S_n))\right| \\
&\qquad \leq \quad E\Big(\left|j_\alpha(\frac{\lambda}{\sigma_n} S_n) - \phi_{\lambda/\sigma_n}(S_n)\right| \cdot 1_{\{S_n \leq r\sigma_n\}}\Big) \;+\; 2P\{S_n \geq r\sigma_n\} \\
&\qquad < \quad 3\varepsilon\ .
\end{aligned}$$

This implies

$$\lim_{n\to\infty} E\big(j_\alpha\big(\frac{\lambda}{\sigma_n}S_n\big)\big) = \lim_{n\to\infty} E(\phi_{\lambda/\sigma_n}(S_n)) = \lim_{n\to\infty}\prod_{j=1}^{k_n} E(\phi_{\lambda/\sigma_n}(X_{nj})).$$

It follows from (1.5) d) that for all $\varepsilon > 0$

$$\begin{aligned}\sum_{j=1}^{k_n}\Big|E\big(\phi_{\lambda/\sigma_n}(X_{nj})\big) - 1 + \frac{\lambda^2}{2\sigma_n^2}E\big(m_2(X_{nj})\big)\Big| \\ \le \sum_{j=1}^{k_n} E\Big(\frac{\lambda^4}{24\sigma_n^4}m_4(X_{nj})\cdot 1_{\{X_{nj}\le\varepsilon\sigma_n\}}\Big) \\ + \sum_{j=1}^{k_n} E\Big(\frac{\lambda^2}{2\sigma_n^2}m_2(X_{nj})\cdot 1_{\{X_{nj}>\varepsilon\sigma_n\}}\Big).\end{aligned}$$

By the Lindeberg condition the last sum converges to 0. It follows from (1.5) that $m_4(x) \le x^4 \le x^2 m_2(x)/\gamma$ and hence the first sum on the right hand side is not larger than

$$\sum_{j=1}^{k_n} E\Big(\frac{\lambda^4}{24\gamma\sigma_n^4}\varepsilon^2\sigma_n^2 m_2(X_{nj})\Big) = \frac{\lambda^4\varepsilon^2}{24\gamma\sigma_n^2}E(m_2(S_n)) = \frac{\lambda^4\varepsilon^2}{24\gamma}.$$

This can be made arbitrary small and hence

$$\sum_{j=1}^{k_n}\big|E(\phi_{\lambda/\sigma_n}(X_{nj})) - 1 + \frac{\lambda^2}{2\sigma_n^2}E(m_2(X_{nj}))\big| \to 0.$$

It follows from (3.2) that

$$\lim_{n\to\infty}\prod_{j=1}^{k_n} E(\phi_{\lambda/\sigma_n}(X_{nj})) = \lim_{n\to\infty}\prod_{j=1}^{k_n}\left(1 - \frac{\lambda^2}{2\sigma_n^2}E(m_2(X_{nj}))\right).$$

By (2.5) b), condition (i) in [5], Lemma 7.1 is satisfied and thus the above limit is equal to $\prod_{j=1}^{k_n}\exp\big(-\frac{\lambda^2\sigma_{nj}^2}{2\sigma_n^2}\big) = \exp\big(-\frac{\lambda^2}{2}\big)$. Therefore we have proved

$$\lim_{n\to\infty} E\Big(j_\alpha\Big(\frac{\lambda}{\sigma_n}S_n\Big)\Big) = \exp(-\lambda^2/2)$$

and this is the Hankel transform of the Rayleigh distribution ρ_α at λ. By the continuity theorem of this transform the assertion follows.

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