Recurrent curvature tensors in Finsler spaces with generalised connection

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§ *1. Introduction.* There are several papers in which the recurrent Finsler spaces are examined. We note that recurrency in these papers is defined in different manner. There exist a great number of Finsler spaces in which metric connection may be defined. This possibility depends on the metric function $F(x, \dot{x})$. Some authors as R. B. MISHRA in 1 S. P. SINGH in [2] and others have considered some classes of Finsler spaces supplied with the metric connections which satisfy certain recurrency condition for a curvature tensor, as for example

$$R_{\alpha\beta\gamma\delta|\varkappa} = \Lambda_\varkappa R_{\alpha\beta\gamma\delta} \quad \text{or} \quad K_{\alpha\beta\gamma\delta|\varkappa|\varrho = \alpha_{\varkappa\tau}} K_{\alpha\beta\gamma\delta}.$$

The above equations can be considered as systems of partial differential equations for the metric function $F(x, \dot{x})$, because the tensors R and K are defined by the connection coefficients $\Gamma^{*\gamma}_{\alpha\beta}$, and they are functions of the metric tensor g which is determined by the $F(x, \dot{x})$ in the form of (2.1). Only those Finsler spaces which are constructed by the metric function $F(x, \dot{x})$ satisfying one of the above equations allow the recurrent curvature tensor.

In the present paper we shall examine recurrent curvature tensors in certain generalised Finsler spaces. A. MOÓR in [3] gave a generalisation of Finsler spaces in such a way that the metric tensor is not covariant constant, but only recurrent i.e. (2.6c) and (2.6d) are satisfied. For $\lambda_\delta = 0$ and $\mu_\delta = 0$ we obtain the classical Finsler spaces. In this paper we shall consider such kind of generalised Finsler space $\overline{F}_n$, which coincides with Moór's generalisation in the case of $\mu_\delta = 0$. The difference between the Moór's and our cases arises from the fact that $A_{\alpha\beta\gamma}$ in [3] is symmetric in its first two indices, but here in the first and last one. The purpose of this paper is to examine such classes of $\overline{F}_n$ in which one of the curvature tensors is recurrent with respect to$|\varkappa$ or$|\varkappa$ defined by (2.4) and (2.5). The explicit solution for $F(x, \dot{x})$ satisfying the recurrency condition for one of the curvature tensors K, R, P or S will not be given, we only obtain some conditions for the vectors λ, μ and the vectors of recurrency.

§ *2. Some definitions and notations.* We use the symbols and notations of [4] and [5] without any explanation. Let $F(x, \dot{x})$ be the metric function in the space $\overline{F}_n$, then the metric tensor g is defined by

$$g_{\alpha\beta}(x, \dot{x}) = 2^{-1} \dot{\partial}_\alpha \dot{\partial}_\beta F^2(x, \dot{x}) \tag{2.1}$$

as usual. If $\xi^\alpha(x, \dot{x})$ are coordinates of a vectorfield homogeneous of degree zero in $\dot{x}$, then

$$D\xi^{\alpha} = d\xi^{\alpha} + \Gamma^{*\alpha}_{\beta\gamma}\xi^{\beta}\, dx^{\gamma} + A^{\alpha}_{\beta\gamma}\xi^{\beta} Dl^{\gamma}, \tag{2.2}$$

$$Dl^{\gamma} = dl^{\gamma} + \Gamma^{*\alpha}_{0\gamma}\, dx^{\gamma} + A^{\alpha}_{0\gamma} Dl^{\gamma},$$

$$D\xi^{\alpha} = \xi^{\alpha}_{|\beta}\, dx^{\beta} + \xi^{\alpha}|_{\beta} Dl^{\gamma}, \tag{2.3}$$

$$\xi^{\alpha}_{|\beta} = \partial_{\beta}\xi^{\alpha} - \dot{\partial}_{\varkappa}\xi^{\alpha}\Gamma^{*\varkappa}_{\ \beta} + \Gamma^{*\alpha}_{\varkappa\beta}\xi^{\varkappa}, \tag{2.4}$$

$$\xi^{\alpha}|_{\beta} = F\dot{\partial}_{\varkappa}\xi^{\alpha}(\delta^{\varkappa}_{\beta} - A^{\varkappa}_{0\beta}) + A^{\alpha}_{\varkappa\beta}\xi^{\varkappa}. \tag{2.5}$$

The connection coefficients in $\overline{F}_n$ are given in [4]. They satisfy the relations

$$\begin{aligned} &\text{a) } \Gamma^{*\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\gamma\beta} && \text{b) } A^{\alpha}_{\beta\gamma} = A^{\alpha}_{\gamma\beta} \\ &\text{c) } g_{\alpha\beta|\delta} = \lambda_{\delta} g_{\alpha\beta} && \text{d) } g_{\alpha\beta}|_{\delta} = \mu_{\delta} g_{\alpha\beta} \end{aligned} \tag{2.6}$$

and have the form

$$\begin{aligned} &\text{a) } \Gamma^{*\alpha}_{\beta\gamma} = \tilde{\Gamma}^{*\alpha}_{\beta\gamma} + T^{\alpha}_{\beta\gamma}(g, \lambda) \\ &\text{b) } A^{\alpha}_{\beta\gamma} = \tilde{A}^{\alpha}_{\beta\gamma} + Q^{\alpha}_{\beta\gamma}(g, \mu) \end{aligned} \tag{2.7}$$

where $\tilde{\Gamma}^{*\alpha}_{\beta\gamma}$ and $\tilde{A}^{\alpha}_{\beta\gamma}$ are connection coefficients of the Cartan connection deduced from $F(x, \dot{x})$, i.e. if $\lambda_{\delta}=0$, $\mu_{\delta}=0$. This space will be referred to as "ordinary Finsler space". $T^{\alpha}_{\beta\gamma}(g, \lambda)$, $Q^{\alpha}_{\beta\gamma}(g, \mu)$ are tensors which vanish for $\lambda_{\delta}=0$ and $\mu_{\delta}=0$ respectively. We shall use the relations [4]:

$$\begin{aligned} &\text{a) } l_{\beta|\varkappa} = 2^{-1} l_{\beta}\lambda_{\varkappa} && \text{b) } l^{\beta}_{|\varkappa} = -2^{-1} l^{\beta}\lambda_{\varkappa} \\ &\text{c) } l_{\beta}|_{\varkappa} = 2^{-1} l_{\beta}\mu_{\varkappa} + h_{\beta\varkappa} && \text{d) } l^{\beta}|_{\varkappa} = -2^{-1} l^{\beta}\mu_{\varkappa} + h^{\beta}_{\varkappa}, \end{aligned} \tag{2.8}$$

where

$$h_{\beta\varkappa} = g_{\beta\varkappa} - l_{\beta} l_{\varkappa} \quad h^{\beta}_{\varkappa} = \delta^{\beta}_{\varkappa} - l^{\beta} l_{\varkappa}.$$

The curvature tensors in $\overline{F}_n$ are defined by

$$2^{-1} K^{\ \ \beta}_{\alpha\gamma\delta} = \partial_{[\delta}\Gamma^{*\beta}_{|\alpha|\gamma]} - \dot{\partial}_{\iota}\Gamma^{*\beta}_{\alpha[\gamma}\Gamma^{*\iota}_{\delta]} + \Gamma^{*\varkappa}_{\alpha[\gamma}\Gamma^{*\beta}_{|\varkappa|\delta]}, \tag{2.9}$$

$$R^{\ \ \beta}_{\alpha\gamma\delta} = K^{\ \ \beta}_{\alpha\gamma\delta} + A^{\beta}_{\alpha\iota} K^{\ \ \iota}_{0\gamma\delta}, \tag{2.10}$$

$$P^{\ \ \beta}_{\alpha\gamma\delta} = F\dot{\partial}_{\varkappa}\Gamma^{*\beta}_{\alpha\gamma}(\delta^{\varkappa}_{\delta} - A^{\varkappa}_{0\delta}) - A^{\beta}_{\alpha\delta|\gamma} + A^{\beta}_{\alpha\iota}\dot{x}^{\varkappa}\dot{\partial}_{\gamma}\Gamma^{*\iota}_{\varkappa\delta}, \tag{2.11}$$

$$2^{-1} S^{\ \ \beta}_{\alpha\gamma\delta} = F\dot{\partial}_{\varkappa} A^{\beta}_{\alpha[\gamma}(\delta^{\varkappa}_{\delta]} - A^{\varkappa}_{|0|\delta]}) + A^{\varkappa}_{\alpha[\gamma} A^{\beta}_{|\varkappa|\delta]}. \tag{2.12}$$

These tensors are formed with connection coefficients of the recurrent Finsler space $\overline{P}_n$. In case of an ordinary Finsler space ($\lambda_{\gamma}=0$, $\mu_{\gamma}=0$) these connection coefficients reduce to the corresponding connection coefficients of the not recurrent Finsler space and the above defined curvature tensors become the wellknown curvature tensors in the ordinary Finsler space (where $A^{\varkappa}_{0\delta}=0$). The tensors defined by (2.9)—(2.12) satisfy the following relations [5]

$$-K_{\alpha\beta\gamma\delta}-K_{\beta\alpha\gamma\delta}-F\dot{\partial}_{\varkappa}g_{\alpha\beta}K^{\varkappa}_{0\gamma\delta}=2\lambda_{[\gamma|\delta]}g_{\alpha\beta}, \tag{2.13}$$

$$\begin{aligned}-P_{\alpha\beta\gamma\delta}-P_{\beta\alpha\gamma\delta}-g_{\alpha\beta}\lambda_{\varkappa}A^{\varkappa}_{\gamma\delta}+\\ +F\dot{\partial}_{\varkappa}g_{\alpha\beta}[A^{\varkappa}_{0\delta|\gamma}-(\dot{x}^{\theta}\dot{\partial}_{\iota}\Gamma^{*\varkappa}_{\theta\gamma}+2^{-1}\lambda_{\gamma}\delta^{\varkappa}_{\iota})(\delta^{\delta}_{\iota}-A^{\iota}_{0\delta})]=\\ =(\lambda_{\gamma}|_{\delta}-\mu_{\delta|\gamma})g_{\alpha\beta},\end{aligned} \tag{2.14}$$

$$-S_{\alpha\beta\gamma\delta}-S_{\beta\alpha\gamma\delta}-2F^{2}\dot{\partial}_{\varkappa}g_{\alpha\beta}(\dot{\partial}_{[\delta}A^{\varkappa}_{|0|\gamma]}-\dot{\partial}_{\iota}A^{\varkappa}_{0[\gamma}A^{\iota}_{|0|\delta]}=2\mu_{[\gamma}|_{\delta]}g_{\alpha\beta}. \tag{2.15}$$

From the above equations using (2.1) after contraction with l^{α} we obtain

$$-K_{0\beta\gamma\delta}-K_{\beta0\gamma\delta}=(\lambda_{\gamma|\delta}-\lambda_{\delta|\gamma})l_{\beta}, \tag{2.16}$$

$$-P_{0\beta\gamma\delta}-P_{\beta0\gamma\delta}=l_{\beta}\lambda_{\varepsilon}A^{\varepsilon}_{\gamma\delta}+(\lambda_{\gamma}|_{\delta}-\mu_{\delta,\gamma})l_{\beta}, \tag{2.17}$$

$$-S_{0\beta\gamma\delta}-S_{\beta0\gamma\delta}=(\mu_{\gamma}|_{\delta}-\mu_{\delta}|_{\gamma})l_{\beta}. \tag{2.18}$$

The contraction of (2.16)—(2.18) with l^{β} yields:

$$-2K_{00\gamma\delta}=\lambda_{\gamma|\delta}-\lambda_{\delta|\gamma}, \tag{2.19}$$

$$-2P_{00\gamma\delta}=\lambda_{\varepsilon|\delta}A^{\varepsilon}_{\gamma\delta}+\lambda_{\gamma}|_{\delta}-\mu_{\delta|\gamma}, \tag{2.20}$$

$$-2S_{00\gamma\delta}=\mu_{\gamma}|_{\delta}-\mu_{\delta}|_{\gamma}. \tag{2.21}$$

§ 3. The $_{|}\varkappa$ recurrent curvature tensors in $\overline{F}_n$.

Definition 3.1. If for some tensor $T:::$ in $\overline{F}_n$ exists a vectorfield $a_{\varkappa}(x,\dot{x})$ homogeneous of degree zero in $\dot{x}$, such that

$$T:::_{|\varkappa}=a_{\varkappa}T::: \tag{3.1}$$

then the tensor $T:::$ is $_{|}\varkappa$ recurrent with vector of recurrency $a_{\varkappa}$.

Lemma 3.1. *If $T^{\beta}_{\alpha\gamma\delta}$ is a tensorfield in $\overline{F}_n$ homogeneous of degree zero in $\dot{x}$ and if it is $_{|}\varkappa$ recurrent with vector of recurrency $t_{\varkappa}$ i.e.*

$$T^{\beta}_{\alpha\gamma\delta|\varkappa}=t_{\varkappa}T^{\beta}_{\alpha\gamma\delta}, \tag{3.2}$$

then

$$T_{\alpha\beta\gamma\delta|\varkappa}=(t_{\varkappa}+\lambda_{\varkappa})T_{\alpha\beta\gamma\delta}, \tag{3.3}$$

$$T^{\beta}_{0\gamma\delta|\varkappa}=(t_{\varkappa}-2^{-1}\lambda_{\varkappa})T^{\beta}_{0\gamma\delta}, \tag{3.4}$$

$$T_{0\beta\gamma\delta|\varkappa}=(t_{\varkappa}+2^{-1}\lambda_{\varkappa})T_{0\beta\gamma\delta}, \tag{3.5}$$

$$T_{\beta0\gamma\delta|\varkappa}=(t_{\varkappa}+2^{-1}\lambda_{\varkappa})T_{\beta0\gamma\delta}, \tag{3.6}$$

$$T_{00\gamma\delta|\varkappa}=t_{\varkappa}T_{00\gamma\delta}. \tag{3.7}$$

PROOF. From (3.2), (2.6c) and

$$T_{\alpha\beta\gamma\delta}=g_{\varepsilon\beta}T^{\varepsilon}_{\alpha\beta\gamma} \tag{3.8}$$

we obtain (3.3). From (3.2), (2.8b) and

$$T^{\beta}_{0\gamma\delta|\varkappa} = T^{\beta}_{\alpha\gamma\delta|\varkappa} l^{\alpha} + T^{\beta}_{\alpha\gamma\delta} l^{\alpha}_{|\varkappa} \tag{3.9}$$

follows (3.4). From (3.3) and (2.8b) follow (3.5), (3.6) and (3.7).

Theorem 3.1. *If $K^{\beta}_{\alpha\gamma\delta}$ in $\bar{F}_n$ is $_|\varkappa$ recurrent, then also $\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}$ is $_|\varkappa$ recurrent with the same vector of recurrency $k\varkappa$.*

PROOF. From the condition of theorem we have

$$K^{\beta}_{\alpha\gamma\delta|\varkappa} = k_{\varkappa} K^{\beta}_{\alpha\gamma\delta}. \tag{3.10}$$

Using Lemma 3.1 we obtain

$$K_{0\beta\gamma\delta|\varkappa} = (k_{\varkappa} + 2^{-1}\lambda_{\varkappa}) K_{0\beta\gamma\delta}, \quad K_{\beta 0\gamma\delta} = (k_{\varkappa} + 2^{-1}\lambda_{\varkappa}) K_{\beta 0\gamma\delta}.$$

The sum of the above two equations gives

$$(K_{0\beta\gamma\delta} + K_{\beta 0\gamma\delta})_{|\varkappa} = (k_{\varkappa} + 2^{-1}\lambda_{\varkappa})(K_{0\beta\gamma\delta} + K_{\beta 0\gamma\delta}). \tag{3.11}$$

Using (2.16) we obtain from (3.11)

$$[(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}) l_{\beta}]_{|\varkappa} = (k_{\varkappa} + 2^{-1}\lambda_{\varkappa})(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}) l_{\beta}.$$

Using (2.8a) we get

$$(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})_{|\varkappa} l_{\beta} = k_{\varkappa}(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}) l_{\beta}.$$

Contraction of this equation with l^{β} gives

$$(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})_{|\varkappa} = k_{\varkappa}(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}). \tag{3.12}$$

A shorter proof is the following: (3.10)$\wedge$(3.7)$\Rightarrow$

$$K_{00\gamma\delta|\varkappa} = k_{\varkappa} K_{00\gamma\delta}, \tag{3.13}$$

(3.13) $\wedge$(2.19) $\Rightarrow$ (3.12).

Theorem 3.2. *If $P^{\beta}_{\alpha\gamma\delta}$ is $_|\varkappa$ recurrent, then also $\lambda_{\gamma}|_{\delta} - \mu_{\delta|\gamma} + \lambda_{\varepsilon} A^{\varepsilon}_{\gamma\delta}$ is $_|\varkappa$ recurrent with the same vector of recurrency $p_{\varkappa}$.*

PROOF. From the condition of the theorem we have

$$P^{\beta}_{\alpha\gamma\delta|\varkappa} = p_{\varkappa} P^{\beta}_{\alpha\gamma\delta}. \tag{3.14}$$

Substituting in Lemma 3.1 $T^{\beta}_{\alpha'\gamma\delta}$ and $t_{\varkappa}$ by $P^{\beta}_{\alpha\gamma\delta}$ and $p_{\varkappa}$, we obtain

$$(P_{0\beta\gamma\delta} + P_{\beta 0\gamma\delta})_{|\varkappa} = (p_{\varkappa} + 2^{-1}\lambda_{\varkappa})(P_{0\beta\gamma\delta} + P_{\beta 0\gamma\delta}). \tag{3.15}$$

Substituting (2.17) into (3.15) and using (2.8a) we get

$$(\lambda_{\gamma}|_{\delta} - \mu_{\delta|\gamma} + \lambda_{\varepsilon} A^{\varepsilon}_{\gamma\delta})_{|\varkappa} l_{\beta} = p_{\varkappa}(\lambda_{\gamma}|_{\delta} - \mu_{\delta|\gamma} + \lambda_{\varepsilon} A^{\varepsilon}_{\gamma\delta}) l_{\beta}. \tag{3.16}$$

Then contraction with l^{β} proves the theorem, i.e.

$$(\lambda_{\gamma}|_{\delta} - \mu_{\delta|\gamma} + \lambda_{\varepsilon} A^{\varepsilon}_{\gamma\delta})_{|\varkappa} = p_{\varkappa}(\lambda_{\gamma}|_{\delta} - \mu_{\delta|\gamma} + \lambda_{\varepsilon} A^{\varepsilon}_{\gamma\delta}).$$

Theorem 3.3. *If $S^{\beta}_{\alpha\gamma\delta}$ is $_{|}\varkappa$ recurrent, then also $\mu_\gamma|_\delta-\mu_\delta|_\gamma$ is $_{|}\varkappa$ recurrent with the same vector of recurrency $s_\varkappa$.*

PROOF. From the condition of the theorem we have

$$S^{\beta}_{\alpha\gamma\delta|\varkappa}=s_\varkappa S^{\beta}_{\alpha\gamma\gamma\delta}. \tag{3.17}$$

Using Lemma 3.1, and substituting $T^{\beta}_{\alpha\gamma\delta}$ and $t_\varkappa$ with $S^{\beta}_{\alpha\gamma\delta}$ and $s_\varkappa$ respectively, we obtain

$$(S_{\beta 0\gamma\delta}+S_{0\beta\gamma\delta})_{|\varkappa}=(s_\varkappa+2^{-1}\lambda_\varkappa)(S_{\beta 0\gamma\delta}+S_{0\beta\gamma\delta})). \tag{3.18}$$

Substituting (2.18) into (3.18), and using (2.8a) we get

$$(\mu_\gamma|_\delta-\mu_\delta|_\gamma)_{|\varkappa}l_\beta=s_\varkappa(\mu_\gamma|_\delta-\mu_\delta|_\gamma)l_\beta. \tag{3.19}$$

A contaction of (3.19) with l^β proves the theorem, i.e. (3.17)⇒

$$(\mu_\gamma|_\delta-\mu_\delta|_\gamma)_{|\varkappa}=s_\varkappa(\mu_\gamma|_\delta-\mu_\delta|_\gamma) \tag{3.20}$$

§ 4. *The $|\varkappa$ recurrent curvature tensors in $\overline{F}_n$.*

Definition 4.1. If for some tensor $T:::$ in $\overline{F}_n$ exists a vectorfield $\bar{a}_\varkappa(x,\dot{x})$ homogeneous of degree zero in $\dot{x}$, such that

$$(T:::)|_\varkappa=\bar{a}_\varkappa T:::, \tag{4.1}$$

then the tensor $T:::$ is $|\varkappa$ recurrent with the vector of recurrency $\bar{a}_\varkappa$.

Lemma 4.1. *If $T^{\beta}_{\alpha\gamma\delta}$ is a tensorfield in $\overline{F}_n$ homogeneous of degree zero in $\dot{x}$, and if it is $|\varkappa$ recurrent with vector of recurrency $\bar{t}_\varkappa$, i.e.*

$$T^{\beta}_{\alpha\gamma\delta}|_\varkappa=\bar{t}_\varkappa T^{\beta}_{\alpha\gamma\delta}, \tag{4.2}$$

then

$$T_{\alpha\beta\gamma\delta}|_\varkappa=(\bar{t}_\varkappa+\mu_\varkappa)T_{\alpha\beta\gamma\delta}, \tag{4.3}$$

$$T_{0\beta\gamma\delta}|_\varkappa=(\bar{t}_\varkappa+2^{-1}\mu_\varkappa-l_\varkappa)T_{0\beta\gamma\delta}+T_{\varkappa\beta\gamma\delta}, \tag{4.4}$$

$$T_{\beta 0\gamma\delta}|_\varkappa=(\bar{t}_\varkappa+2^{-1}\mu_\varkappa-l_\varkappa)T_{\beta 0\gamma\delta}+T_{\beta\varkappa\gamma\delta}, \tag{4.5}$$

$$T_{00\gamma\delta}|_\varkappa=(\bar{t}_\varkappa-2l_\varkappa)T_{00\gamma\delta}+T_{\varkappa 0\gamma\delta}+T_{0\varkappa\gamma\delta}. \tag{4.6}$$

PROOF. From (4.2) and (2.6d) follows (4.3). From (4.3) and (2.8d) follow (4.4) and (4.5). From (4.4), or (4.5) and (2.8d) follows (4.6).

Lemma 4.2. *If $T^{\beta}_{\alpha\gamma\delta}$ is $|\varkappa$ recurrent in $\overline{F}_n$, and if exist vectorfields $w_\varkappa(x,\dot{x})$ and $w'_\varkappa(x,\dot{x})$ homogeneous of degree zero in $\dot{x}$ such that*

$$T_{\varkappa\beta\gamma\delta}=w_\varkappa T_{0\beta\gamma\delta}, \tag{4.7}$$

$$T_{\beta\varkappa\gamma\delta}=w'_\varkappa T_{\beta 0\gamma\delta}, \tag{4.8}$$

then $T_{0\beta\gamma\delta}$ *and* $T_{\beta 0\gamma\delta}$ *are* $|\varkappa$ *recurrent and we have*

$$T_{0\beta\gamma\delta}|_\varkappa = (\bar{t}_\varkappa + 2^{-1}\mu_\varkappa - l_\varkappa + w_\varkappa)T_{0\beta\gamma\delta}, \tag{4.9}$$

$$T_{\beta 0\gamma\delta}|_\varkappa = (\bar{t}_\varkappa + 2^{-1}\mu_\varkappa - l_\varkappa + w'_\varkappa)T_{\beta 0\gamma\delta}, \tag{4.10}$$

$$T_{00\gamma\delta}|_\varkappa = (\bar{t}_\varkappa - 2l_\varkappa + w_\varkappa + w'_\varkappa)T_{00\gamma\delta}, \tag{4.11}$$

$$T_{\varkappa 0\gamma\delta} + T_{0\varkappa\gamma\delta} = (w_\varkappa + w'_\varkappa)T_{00\gamma\delta}. \tag{4.12}$$

PROOF. Substituting (4.7) and (4.8) into (4.4) and (4.5) we obtain (4.9) and (4.10). Contraction of (4.7) and (4.8) with l^β gives

$$T_{\varkappa 0\gamma\delta} = w_\varkappa T_{00\gamma\delta}, \tag{4.13}$$

$$T_{0\varkappa\gamma\delta} = w'_\varkappa T_{00\gamma\delta}. \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.6) we obtain (4.11). The sum of (4.13) and (4.14) gives (4.12).

Lemma 4.3. *If* $K^\beta_{\alpha\gamma\delta}$ *is* $|\varkappa$ *recurrent, then also* $K_{00\gamma\delta}$ *is* $|\varkappa$ *recurrent with the same vector of recurrency* $\bar{k}_\varkappa$.

PROOF. Since

$$K^\beta_{\alpha\gamma\delta}|_\varkappa = \bar{k}_\varkappa K^\beta_{\alpha\gamma\delta} \tag{4.15}$$

it follows from Lemma 4.1 that (4.2)—(4.6) remain valid if we write K and $\bar{k}$ in place of T and $\bar{t}$ respectively. Now (4.4) has the form

$$K_{0\beta\gamma\beta}|_\varkappa = (\bar{k}_\varkappa + 2^{-1}\mu_\varkappa - l_\varkappa)K_{0\beta\gamma\delta} + K_{\varkappa\beta\gamma\delta}.$$

Contracting this equation with l^β, and using (2.8d), we obtain

$$K_{00\gamma\delta}|_\varkappa = (\bar{k}_\varkappa - 2l_\varkappa)K_{00\gamma\delta} + (K_{\varkappa 0\gamma\delta} + K_{0\varkappa\gamma\delta}).$$

With respect to (2.16) and (2.19) we get

$$K_{00\gamma\delta}|_\varkappa = \bar{k}_\varkappa K_{00\gamma\delta}. \tag{4.16}$$

So we proved that (4.15)⇒(4.16).

Lemma 4.4. *If* $K^\beta_{\alpha\gamma\delta}$ *is* $|\varkappa$ *recurrent with vector of recurrency* $\bar{k}_\varkappa$, *and there exist vectorfields* $\xi_\varkappa$ *and* $\xi'_\varkappa$ *such that*

$$K_{\varkappa\beta\gamma\delta} = \xi_\varkappa K_{0\beta\gamma\delta}, \tag{4.17}$$

$$K_{\beta\varkappa\gamma\delta} = \xi'_\varkappa K_{\beta 0\gamma\delta}, \tag{4.18}$$

then $K_{0\beta\gamma\delta}$ *and* $K_{\beta 0\gamma\delta}$ *are* $|\varkappa$ *recurrent:*

$$K_{0\beta\gamma\delta}|_\varkappa = (\bar{k}_\varkappa + 2^{-1}\mu_\varkappa - l_\varkappa + \xi_\varkappa)K_{0\beta\gamma\delta}, \tag{4.19}$$

$$K_{\beta 0\gamma\delta}|_\varkappa = (\bar{k}_\varkappa + 2^{-1}\mu_\varkappa - l_\varkappa + \xi'_\varkappa)K_{\beta 0\gamma\delta}, \tag{4.20}$$

and

$$\xi_\varkappa + \xi'_\varkappa = 2l_\varkappa. \tag{4.21}$$

PROOF. From Lemma 4.2, (4.17) and (4.18) follows that (4.7)—(4.14) are valid provided we substitute T, $\bar{t}$, w and w' with K, $\bar{k}$, ξ and ξ' respectively. (4.19) and (4.20) are the new form of (4.10) and (4.11). (4.12) has now the form:

$$K_{\varkappa 0\gamma\delta}+K_{0\varkappa\gamma\delta}=(\xi_\varkappa+\xi'_\varkappa)K_{00\gamma\delta}.$$

Using (2.16) and (2.19) we obtain

$$2l_\varkappa K_{00\gamma\delta}=(\xi_\varkappa+\xi'_\varkappa)K_{00\gamma\delta}$$

from which follows (4.21).

Theorem 4.1. *If $K^{\beta}_{\alpha\gamma\delta}$ in $\bar{F}_n$ is $|\varkappa$ recurrent, then also $\lambda_{\gamma|\delta}-\lambda_{\delta|\gamma}$ is $|\varkappa$ recurrent with the same vector of recurrency $\bar{k}_\varkappa$ i.e.* (4.15)$\Rightarrow$

$$(\lambda_{\gamma|\delta}-\lambda_{\delta|\gamma})|_\varkappa=\bar{k}_\varkappa(\lambda_{\gamma|\delta}-\lambda_{\delta|\gamma}). \tag{4.22}$$

PROOF. From (4.15) and Lemma 4.2 follows

$$(K_{\alpha\beta\gamma\delta}+K_{\beta\alpha\gamma\delta})|_\varkappa=(\bar{k}_\varkappa+\mu_\varkappa)(K_{\alpha\beta\gamma\delta}+K_{\beta\alpha\gamma\delta}).$$

Substituting (2.13) into this equation we obtain

$$[(\lambda_{\gamma|\delta}-\lambda_{\delta|\gamma})g_{\alpha\beta}+F\dot{\partial}_\varepsilon\, g_{\alpha\beta}K^{\varepsilon}_{0\gamma\delta}]|_\varkappa=(\bar{k}_\varkappa+\mu_\varkappa)[(\lambda_{\gamma|\delta}-\lambda_{\delta|\gamma})g_{\alpha\beta}+F\dot{\partial}_\varepsilon\, g_{\alpha\beta}K^{\varepsilon}_{0\gamma\delta}].$$

Contracting with l^α and l^β, and using (2.8d), we arrive to (4.22).

Lemma 4.5. *If $P^{\beta}_{\alpha\gamma\delta}$ is $|\varkappa$ recurrent, then $P_{00\gamma\delta}$ is $|\varkappa$ recurrent with the same vector of recurrency $\bar{p}_\varkappa$.*

PROOF. In view of

$$P^{\beta}_{\alpha\gamma\delta}|_\varkappa=\bar{p}_\varkappa P^{\beta}_{\alpha\gamma\delta} \tag{4.23}$$

it follows from Lemma (4.1) that (4.2)—(4.6) remain valid if we write P and $\bar{p}$ in place of T and $\bar{t}$ respectively. (4.3) now has the form

$$P_{\alpha\beta\gamma\delta}|_\varkappa=(\bar{p}_\varkappa+\mu_\varkappa)P_{\alpha\beta\gamma\delta}.$$

Contracting the above equation first with l^α and then with l^β, and using (2.6d) we get

$$P_{00\gamma\delta}|_\varkappa=(\bar{p}_\varkappa-2l_\varkappa)P_{00\gamma\delta}+(P_{0\varkappa\gamma\delta}+P_{\varkappa 0\gamma\delta}$$

Substituting the corresponding values from (2.17) and (2.20) into the above equation we obtain

$$P_{00\gamma\delta}|_\varkappa=\bar{p}_\varkappa P_{00\gamma\delta}. \tag{4.24}$$

Lemma 4.6. *If $P^{\beta}_{\alpha\gamma\delta}$ is $|\varkappa$ recurrent with vector of recurrency $\bar{p}_\varkappa$, and if exist vectorfields $\eta_\varkappa$ and $\eta'_\varkappa$ such that*

$$P_{\varkappa\beta\gamma\delta}=\eta_\varkappa P_{0\beta\gamma\delta}, \tag{4.25}$$

$$P_{\beta\varkappa\gamma\delta}=\eta'_\varkappa P_{\beta 0\gamma\delta}, \tag{4.26}$$

then $P_{0\beta\gamma\delta}$ and $P_{\beta 0\gamma\delta}$ are also $|\varkappa$ recurrent:

$$P_{0\beta\gamma\delta}|_{\varkappa} = (\bar{p}_{\varkappa}+2^{-1}\mu_{\varkappa}-l_{\varkappa}+\eta_{\varkappa})P_{0\beta\gamma\delta}, \tag{4.27}$$

$$P_{\beta 0\gamma\delta}|_{\varkappa} = (\bar{p}_{\varkappa}+2^{-1}\mu_{\varkappa}-l_{\varkappa}+\eta'_{\varkappa})P_{\beta 0\gamma\delta}, \tag{4.28}$$

and

$$\eta_{\varkappa}+\eta'_{\varkappa} = 2l_{\varkappa}. \tag{4.29}$$

PROOF. (4.27) and (4.28) are consequence of (4.25), (4.26) and Lemma 4.2. Contracting (4.25) and (4.26) with l^{β}, and substituting these equations into (2.17) we get

$$-(\eta_{\varkappa}+\eta'_{\varkappa})P_{00\gamma\delta} = l_{\varkappa}(\lambda_{\varepsilon}A^{\varepsilon}_{\gamma\delta}+\lambda_{\gamma}|_{\delta}-\mu_{\delta|\gamma}).$$

Substituting the value of (2.20) into the above equation we obtain (4.29).

Theorem 4.2. *If $P^{\beta}_{\alpha\gamma\delta}$ is $|\varkappa$ recurrent, then also $\lambda_{\varepsilon}A^{\varepsilon}_{\gamma\delta}+\lambda_{\gamma}|_{\delta}-\mu_{\delta|\gamma}$ is $|\varkappa$ recurrent with the same vector of recurrency $\bar{p}_{\varkappa}$.*

PROOF. From (4.23) and Lemma 4.1 we obtain

$$(P_{\alpha\beta\gamma\delta}+P_{\beta\alpha\gamma\delta})|_{\varkappa} = (\bar{p}_{\varkappa}+\mu_{\varkappa})(P_{\alpha\beta\gamma\delta}+P_{\beta\alpha\gamma\delta}).$$

Contracting the above equation with l^{α}, and using (2.8d) we get

$$(P_{0\beta\gamma\delta}+P_{\beta 0\gamma\delta})|_{\varkappa}-(P_{\varkappa\beta\gamma\delta}-P_{\beta\varkappa\gamma\delta}) = (\bar{p}_{\varkappa}+2^{-1}\mu_{\varkappa}-l_{\varkappa})(P_{0\beta\gamma\delta}+P_{\beta 0\gamma\delta}).$$

Contracting again with l^{β}, and using (2.8d), (2.17) and (2.20) we arrive to

$$(\lambda_{\varepsilon}A^{\varepsilon}_{\gamma\delta}+\lambda_{\gamma}|_{\delta}-\mu_{\delta|\gamma})|_{\varkappa} = \bar{p}_{\varkappa}(\lambda_{\varepsilon}A^{\varepsilon}_{\gamma\delta}+\lambda_{\gamma}|_{\delta}-\mu_{\delta|\gamma}), \tag{4.30}$$

i.e. (4.23)⇒(4.30)

Lemma 4.7. *If $S^{\beta}_{\alpha\gamma\delta}$ is $|\varkappa$ recurrent, then also $S_{00\gamma\delta}$ is $|\varkappa$ recurrent with the same vector of recurrency $\bar{s}_{\varkappa}$.*

PROOF. Since

$$S^{\beta}_{\alpha\gamma\delta}|_{\varkappa} = \bar{s}_{\varkappa}S^{\beta}_{\alpha\gamma\delta} \tag{4.31}$$

we have from Lemma 4.1

$$S_{00\gamma\delta}|_{\varkappa} = (\bar{s}_{\varkappa}-2l_{\varkappa})S_{00\gamma\delta}+(S_{\varkappa 0\gamma\delta}+S_{0\varkappa\gamma\delta}). \tag{4.32}$$

Substituting from (2.18) and (2.21) into this equation we obtain

$$S_{00\gamma\delta}|_{\varkappa} = \bar{s}_{\varkappa}S_{00\gamma\delta}. \tag{4.33}$$

Lemma 4.8. *If $S^{\beta}_{\alpha\gamma\delta}$ in $\bar{F}_n$ is $|\varkappa$ recurrent with vector of recurrency $\bar{s}_{\varkappa}$, and if there exist vectorfields $\zeta_{\varkappa}$ and $\zeta'_{\varkappa}$ such that*

$$S_{\varkappa\beta\gamma\delta} = \zeta_{\varkappa}S_{0\beta\gamma\delta}, \tag{4.34}$$

$$S_{\beta\varkappa\gamma\delta} = \zeta'_{\varkappa}S_{\beta 0\gamma\delta}, \tag{4.35}$$

then $S_{0\beta\gamma\delta}$, $S_{\beta 0\gamma\delta}$ *are* $|\varkappa$ *recurrent i.e.*

(4.36) $$S_{0\beta\gamma\delta}|_{\varkappa}=(\bar{s}_{\varkappa}+2^{-1}\mu_{\varkappa}-l_{\varkappa}+\zeta_{\varkappa})S_{0\beta\gamma\delta},$$

(4.37) $$S_{\beta 0\gamma\delta}|_{\varkappa}=(\bar{s}_{\varkappa}+2^{-1}\mu_{\varkappa}-l_{\varkappa}+\zeta'_{\varkappa})S_{\beta 0\gamma\delta}$$

and

(4.38) $$\zeta_{\varkappa}+\zeta'_{\varkappa}=2l_{\varkappa}.$$

PROOF. (4.36) and (4.37) are consequences of (4.34), (4.35) and Lemma 4.2 (formulae (4.10) and (4.11)). Contracting (4.34) and (4.35) with l^{β}, and substituting these equations into (2.18), we get

$$-(\zeta_{\varkappa}+\zeta'_{\varkappa})S_{00\gamma\delta}=(\mu_{\gamma}|_{\delta}-\mu_{\delta}|_{\gamma})l_{\varkappa}.$$

Substituting (2.21) into this equation we obtain (4.38).

Theorem 4.3. *If* $S^{\beta}_{\alpha\gamma\delta}$ *is* $|\varkappa$ *recurrent, then also* $\mu_{\gamma}|_{\delta}-\mu_{\delta}|_{\gamma}$ *is* $|\varkappa$ *recurrent with the same vector of recurrency* $\bar{s}_{\varkappa}$.

PROOF. Substituting (2.18) and(2.21) into (4.32) we obtain

(4.39) $$(\mu_{\gamma}|_{\delta}-\mu_{\delta}|_{\gamma})|_{\varkappa}=\bar{s}_{\varkappa}(\mu_{\gamma}|_{\delta}-\mu_{\delta}|_{\gamma}),$$

i.e. (4.31)$\Rightarrow$(4.39).

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