Finsler connection transformations associated to a general Finsler metrical structure of Miron type (I)

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§ 1. Introduction

Let M be an n-dimensional differentiable manifold of class C^∞ and let (x^i, y^i) be the canonical coordinates of a point $y \in T(M)$, where $T(M)$ is the tangent bundle of M [7] and let π be the canonical projection. The natural basis of $T(M)$ with respect to canonical coordinates is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$, and the mapping $N: y \in T(M) \to \to Ny \in T(M)_y$ is a regular distribution on $T(M)$, such that: $T(M)_y = Ny \oplus T(M)^v_y$. Let $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}$ be a local basis of the n dimensional local distribution N, where $N^i_k(x, y)$ are the coefficients of the non-linear connection defined by N. The notions and notations of M. MATSUMOTO [6] and R. MIRON [7] are used.

Let $F\Gamma = (N, F, C)$ be a Finsler connection with the coefficients $(N^i_j, F^i_{jk}, C^i_{jk})$, $\mathcal{T}$ the group of general Finsler connection transformations t: $(N, F, C) \to (\bar{N}, \bar{F}, \bar{C})$, and $\mathcal{T}_N = \{t \mid t \in \mathcal{T},\ t = t(0, B, D)\}$ the subgroup of $\mathcal{T}$, formed by the transformations t: $(N, F, C) \to (N, \bar{F}, \bar{C})$, which preserve the non-linear connection N. The transformations from $\mathcal{T}_N$ have the form:

$$\bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk} - B^i_{jk};\ \bar{C}^i_{jk} = C^i_{yk} - D^i_{jk} \tag{1.1}$$

where $B, D \in Z^1_2(M)$ are arbitrary Finsler tensor fields [7]. In the following we denote by $_|$, $|$ and $_{\|}$, $\|$ the h-and v-covariant derivatives relative to (N, F, C) and $(\bar{N}, \bar{F}, \bar{C})$ respectively.

Let $g = (g_{ij})$ be a metrical Finsler structure subordinated to a Finsler space $F_n = (M, L)$ [7], the corresponding metrical Finsler connection transformations were are studied by R. MIRON [7], [8] and M. HASHIGUCHI [3], [8]. All $t \in \mathcal{T}$ transformations contain terms of the general form ΩX, ΩY, where Ω is the Obata operator [7] and $X, Y \in Z^1_2(M)$ are arbitrary Finsler tensors. By Ω^*) we denote the second Obata operator.

In the following we insert the $t \in \mathcal{T}$ transformations in a more comprehensive class of transformations, called g-transformations, where the terms ΩX, ΩY are

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explicited by given. These classes of Finsler transformations contain also the conformal Finsler transformations [1], [2], [9]. The obtained results lead easy to the determination of special classes of classical Finsler connections and to their special invariants.

§ 2. The class of Finsler g-transformations

Definition 2.1. Let $F\Gamma=(N, F, C)$ be an arbitrary fixed Finsler connection and let $t\in\mathcal{T}$, t: $F\Gamma=(N, F, C)\to F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ be a general Finsler connection transformation; if

$$G_{ijk} \overset{\text{def}}{=} g_{ij|k} \quad \text{and} \quad g_{ijk} \overset{\text{def}}{=} g_{ij}|_k \tag{2.1}$$

are invariants of the transformation t, then t is called a g-transformation and is denoted by tg. If $N=\bar{N}$ it follows $t\in\mathcal{T}_N$ and the transformation t is denoted by $t_N g$.

Definition 2.2. If G_{ijk} and g_{ijk} are not invariants of the transformation $t\in\mathcal{T}$, then t is called a non-g-transformation.

Theorem 2.1. *A necessary and sufficient condition that $t\in\mathcal{T}_N$ be a g-transformation is that the transformation t has the form:*

$$\bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk}+\bar{T}^{*i}_{jk}-T^{*i}_{jk};\ \bar{C}^{*i}_{jk} = C^i_{jk}+\bar{S}^{*i}_{jk}-S^{*i}_{jk} \tag{2.2}$$

where T^i_{jk} is the h-torsion tensor of the connection $F\Gamma$, S^i_{jk} is the v-torsion tensor of $F\Gamma$, $\bar{T}, \bar{S}\in\mathfrak{T}^1_2$ are arbitrary skewsymmetric tensors: $\bar{T}^i_{jk}=-\bar{T}^i_{kj}$; $\bar{S}^i_{jk}=-\bar{S}^i_{kj}$ and T^*, $\bar{T}^*$, S^*, $\bar{S}^*$ are Finsler tensors of the form:

$$2T^{*i}_{jk} = T^i_{jk}+g_{sj}g^{ir}T^s_{kr}+g_{sk}g^{ir}T^s_{jr} \tag{2.3}$$

$$2S^{*i}_{jk} = S^i_{jk}+g_{sj}g^{ir}S^s_{kr}+g_{sk}g^{ir}S^s_{jr} \tag{2.4}$$

$$2\bar{T}^{*i}_{jk} = \bar{T}^i_{jk}+g_{sj}g^{ir}\bar{T}^s_{kr}+g_{sk}g^{ir}\bar{T}^*_{jr} \tag{2.5}$$

$$2\bar{S}^{*i}_{jk} = \bar{S}^i_{jk}+g_{sj}g^{ir}\bar{S}^s_{kr}+g_{sk}g^{ir}\bar{S}^s_{jr} \tag{2.6}$$

PROOF. We suppose that $t\in\mathcal{T}_N$ is of the form (2.2). Evaluating the h- and v-covariant derivatives of g_{ij} relative to $F\bar{\Gamma}$, we obtain:

$$g_{ij\|k} = g_{ij|k};\ g_{ij}\|_k = g_{ij}|_k \tag{2.7}$$

$$\text{i.e.} \quad \bar{G}_{ijk} = G_{ijk};\ \tilde{g}_{ijk} = g_{ijk}.$$

It means that $t_N=t_N g$.

Inversely, let $t_N\in\mathcal{T}_N$ the most general Finsler connection transformation

$$\bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk}+B^i_{jk};\ \bar{C}^i_{jk} = C^i_{jk}+D^i_{jk} \tag{2.8}$$

given in [7], where $B, D\in Z^1_2(M)$ are arbitrary Finsler tensors.

We have:

$$g_{ij\|k} = g_{ij|k}-B^s_{ik}g_{sj}-B^s_{jk}g_{si};\ g_{ij}\|_k = g_{ij}|_k-D^s_{ik}g_{sj}-D^s_{jk}g_{is} \tag{2.9}$$

It follows from (2.7) and (2.8)

$$B^s_{ik} g_{sj} + B^s_{jk} g_{is} = 0;\ D^s_{ik} g_{sj} + D^s_{jk} g_{is} = 0 \tag{2.10}$$

or equivalently:

$$\Omega^{*ir}_{sj} B^s_{rk} = 0;\ \Omega^{*ir}_{sj} D^s_{rk} = 0 \tag{2.11}$$

From (2.8) it follows:

$$\overline{T}^i_{jk} - T^i_{jk} = B^i_{jk} - B^i_{kj};\ \overline{S}^i_{jk} - S^i_{jk} = D^i_{jk} - D^i_{kj} \tag{2.12}$$

The relations (2.11) are equivalent with the relations:

$$\Omega^{ir}_{sj} B^s_{rk} = B^i_{jk};\ \Omega^{ir}_{sj} D^s_{rk} = D^i_{jk} \tag{2.13}$$

From (2.12) and (2.13) it follows:

$$\begin{cases} \Omega^{ir}_{sj} B^s_{rk} - \Omega^{ir}_{sk} B^s_{rj} = \overline{T}^i_{jk} - T^i_{jk} \\ \Omega^{ir}_{sj} D^s_{rk} - \Omega^{ir}_{sk} D^s_{ij} = \overline{S}^i_{jk} - S^i_{jk} \end{cases} \tag{2.14}$$

or:

$$\frac{1}{2} B^i_{jk} - g_{sj} g^{ir} B^s_{rk} - B^i_{kj} + g_{sk} g^{ir} B^s_{rj}) = \overline{T}^i_{jk} - T^i_{jk} \tag{2.15}$$

$$\frac{1}{2} (D^i_{jk} - g_{sj} g^{ir} D^s_{rk} - D^i_{kj} + g_{sk} g^{ir} D^s_{rj}) = \overline{S}^i_{jk} - S^i_{jk} \tag{2.16}$$

Then we have:

$$\Omega^{ir}_{sj} B^s_{rk} - \frac{1}{2} g_{sj} g^{ir} B^s_{kr} + \frac{1}{2} g_{sk} g^{ir} B^s_{jr} = g_{sj} g^{ir} (\overline{T}^s_{kr} - T^s_{kr}) \tag{2.17}$$

$$\Omega^{ir}_{sj} D^s_{rk} - \frac{1}{2} g_{sj} g^{ir} D^s_{kr} + \frac{1}{2} g_{sk} g^{ir} D^s_{jr} = g_{sj} g^{ir} (\overline{S}^s_{kr} - S^s_{kr}) \tag{2.18}$$

Inverting j and k we obtain by addition:

$$\Omega^{ir}_{sj} B^s_{rk} + \Omega^{ir}_{sk} B^s_{rj} = g_{sj} g^{ir} (\overline{T}^s_{kr} - T^s_{kr}) + g_{sk} g^{ir} (\overline{T}^s_{jr} - T^s_{jr}) \tag{2.19}$$

$$\Omega^{ir}_{sj} D^s_{rk} + \Omega^{ir}_{sk} D^s_{rj} = g_{sj} g^{ir} (\overline{S}^s_{kr} - S^s_{kr}) + g_{sk} g^{ir} (\overline{S}^s_{jr} - S^s_{jr}) \tag{2.20}$$

Summing (2.14) with (2.19) and (2.20) we obtain:

$$2\Omega^{ir}_{sj} B^s_{rk} = \overline{T}^{*i}_{jk} - T^{*i}_{jk};\ 2\Omega^{ir}_{sj} D^s_{rk} = \overline{S}^{*i}_{jk} - S^{*i}_{jk} \tag{2.21}$$

and also:

$$\mathfrak{T}^{*i}_{jk} = \mathfrak{T}^i_{jk} + g_{sj} g^{ir} \mathfrak{T}^s_{kr} + g_{sk} g^{ir} \mathfrak{T}^s_{jr} \tag{2.22}$$

where:

$\mathfrak{T}^i_{jk}$ is equal to $\frac{1}{2} \overline{T}^i_{jk}$ or $\frac{1}{2} T^i_{jk}$, or $\frac{1}{2} \overline{S}^i_{jk}$ or $\frac{1}{2} S^i_{jk}$ and

corresponding $\mathfrak{T}^{*i}_{jk}$ is equal to $\frac{1}{2} \overline{T}^{*i}_{jk}$ or $\frac{1}{2} T^{*i}_{jk}$ or $\frac{1}{2} \overline{S}^{*i}_{jk}$ or $\frac{1}{2} S^{*i}_{jk}$.

Theorem 2.2. *Let $F\Gamma=(N, F, C)$ be a fixed Finsler connection, then the set of all Finsler connections with the properties* (2.7) *has the same multitude as the set* $\mathfrak{T}_2^1\times\mathfrak{T}_2^1$.

PROOF. The set $\mathscr{F}\Gamma=\{F\Gamma=(N, F, C)\}$ of the Finsler connections with a fixed non-linear connection N and with the properties (2.7), is given by (2.2), where $(\bar{T}, \bar{S})$ are the torsions of $F\bar{\Gamma}$ and (T, S) are the torsions of $F\Gamma$. The transformation (2.2) establishes an one-to-one correspondence between $\mathscr{F}\Gamma$ and $\mathfrak{T}_2^1\times\mathfrak{T}_2^1$. It follows that $\mathscr{F}\Gamma$ and $\mathfrak{T}_2^1\times\mathfrak{T}_2^1$ has the same multitude. We denote the set $\mathscr{F}\Gamma$ also by $\mathscr{F}\Gamma(N, g)$.

Theorem 2.3. *Let $\bar{T}, \bar{S}\in\mathfrak{T}_2^1$ be two given tensors, then exists an unique Finsler connection $F\bar{\Gamma}=(\bar{N}=N, \bar{F}, \bar{C})\in\mathscr{F}\Gamma$ with the torsion $(\bar{T}, \bar{S})$, and exists an unique g-transformation which transforms a Finsler connection $F\Gamma=(N, F, C)$ in the Finsler connection $F\bar{\Gamma}=(\bar{N}=N, \bar{F}, \bar{C})$ with the given torsion $(\bar{T}, \bar{S})\in\mathfrak{T}_1^2\times\mathfrak{T}_1^2$.*

Equivalently we have:

Theorem 2.4. *Let N be a fixed non-linear connection, then any Finsler connection $F\Gamma(N)$ with the property that its h- and v-covariant derivatives $g_{ij|k}$; $g_{ij}|_k$ are given by the fixed Finsler tensors G_{ijk}, g_{ijk}, is uniquely determined by its torsions (T, S).*

PROOF. Let $F\Gamma=(N, F, C)$ be a Finsler connection having the given torsion (T, S) and the fixed tensors $g_{ij|k}=G_{ijk}$, $g_{ij}|_k=g_{ijk}$, and let $F\bar{\Gamma}=(N, \bar{F}, \bar{C})$ be another Finsler connection with the same properties $g_{ij\|k}=G_{ijk}$; $g_{ij}\|_k=g_{ijk}$. It follows the relation (2.7) and $F\bar{\Gamma}(N)$ is obtained from $F\Gamma(N)$ by a g-transformation of the form (2.2), the torsion of $F\bar{\Gamma}(N)$ being $(\bar{T}, \bar{S})$. If exist two different Finsler connections $F\Gamma(N)\neq F\bar{\Gamma}(N)$ with the given properties and with the same torsion tensors $\bar{T}=T$, $\bar{S}=S$, then from (2.2) it follows $F=\bar{F}$ and $C=\bar{C}$, i.e. $F\Gamma(N)=F\bar{\Gamma}(N)$.

We obtain the following particular cases of Theorem 2.4:

1. If $F\Gamma(N)$ is a metrical Finsler connection, then $G_{ijk}=0$, $g_{ijk}=0$, and we obtain the classical results of R. Miron [7]:

Theorem 2.5. *If N is a fixed non-linear connection, then any metrical Finsler connection $F\Gamma=(N, F, C)$ is uniquely determinated by its torsion.*

Theorem 2.6. *If $F\Gamma(N)$ is an arbitrary fixed metrical Finsler connection, then any other metrical Finsler connection $F\bar{\Gamma}=(N, \bar{F}, \bar{C})$ is obtained from $F\Gamma(N)$ by a g-transformation* (2.2).

If $F\Gamma(N)=F\overset{c}{\Gamma}$ is the Cartan connection, i.e. the metrical tensor g is subordinated to a Finsler space $F_n=(M, L)$, then, because the connection is metrical and $T=0$, $S=0$, it follows that this is the unique connection with vanishing torsion, and any other metrical Finsler connection with the torsion $(\bar{T}, \bar{S})$ is given by (2.2) where $T=0$, $S=0$. We have in this way an other known result [7].

2. Let $F\Gamma=(N, F, C)$ be a Finsler connection with the torsion (T, S) and with

$$G_{ijk} = 2g_{ij}\omega_k;\ g_{ijk} = 2g_{ij}\lambda_k \tag{2.23}$$

where $\omega=\omega_k\, dx^k+\lambda_k \delta y^k$ is a fixed 1-form of $T(M)$, then $F\Gamma(N)$ is the conformal Finsler connection $F\Gamma(N, \omega)$. Any other conformal Finsler connection $F\bar{\Gamma}(N, \omega)$, with the same 1-form ω satisfies the relation (2.23) and consequently also the relation (2.7). It follows that any other Finsler connection $F\bar{\Gamma}(N, \omega)$ is given by (2.2). If the torsion $(\bar{T}, \bar{S})$ of $F\bar{\Gamma}(N, \omega)$ is the same as the torsion (T, S) of $F\Gamma(N, \omega)$, then from (2.2) it follows: $\bar{F}=F$, $\bar{C}=C$. As a particular case of Theorem 2.1 and of Theorem 2.2, we have:

Theorem 2.7. *If N is a fixed non-linear connection, then any conformal Finsler connection $F\Gamma(N, \omega)$ is uniquely determined by its torsion (T, S) and by the 1-form ω.*

Theorem 2.8. *If $F\Gamma(N, \omega)$ is a conformal Finsler connection with the 1-form ω, then any other conformal Finsler connection $F\bar{\Gamma}(N, \omega)$ with the same 1-form ω is obtained from $F\Gamma(N, \omega)$ by a g-transformation (2.2) and has the torsion $(\bar{T}, \bar{S})\in\mathfrak{T}^1_2\times\mathfrak{T}^1_2$.*

If we consider the conformal Weyl connection: $N=\overset{c}{N}$, $F=\overset{w}{F}$, $C=\overset{w}{C}$, denoted by $W\Gamma(\omega)$, then because of $T=0$, $S=0$ it follows that this connection is unique, and any other conformal Finsler connection $F\bar{\Gamma}(N, \omega)$ with the same 1-form ω can be obtained from $W\Gamma(\omega)$ by a g-transformation (2.2), where $T=0$, $S=0$. This is a theorem of R. MIRON and M. HASHIGUCHI [9].

§ 3. Finsler g-transformation having the invariants $\underset{1}{I}$ and $\underset{2}{I}$

We associate to the Finsler connection $F\Gamma=(N, F, C)$ the Finsler tensors:

$$\underset{1}{I}{}^i_{jk} = T^i_{jk}-\frac{1}{n-1}(\delta^i_j T_k-\delta^i_k T_j);\; \underset{2}{I}{}^i_{jk} = S^i_{jk}-\frac{1}{n-1}(\delta^i_j S_k-\delta^i_k S_j) \tag{3.1}$$

where: $T_k=T^i_{ik}$, $S_k=S^i_{ik}$ we can enunciate

Theorem 3.1. *The set of all Finsler connection transformations $t\in\mathcal{T}_N$, which have the invariants $\underset{1}{I}$, $\underset{2}{I}$ and are g-transformations, are given by:*

$$\bar{N}^i_j = N^i_j;\; \bar{F}^i_{jk} = F^i_{jk}+\delta^i_k\alpha_j-g_{jk}\alpha^i;\; \bar{C}^i_{jk} = C^i_{jk}+\delta^i_k\beta_j-g_{jk}\beta^i \tag{3.2}$$

have the invariants $\underset{1}{I}$, $\underset{2}{I}$ and are g-transformations, are given by:

w.

$$\bar{N}^i_j = N^i_j;\; \bar{F}^i_{jk} = F^i_{jk}+\delta^i_k\alpha_j-g_{jk}\alpha^i;\; \bar{C}^i_{jk} = C^i_{jk}+\delta^i_k\beta_j-g_{jk}\beta^i \tag{3.2}$$

PROOF. A g-transformation $t\in\mathcal{T}_{Ng}$ is of the form (2.2) imposing the conditions:

$$\underset{1}{\bar{I}}{}^i_{jk} = \underset{2}{I}{}^i_{jk};\quad \underset{2}{\bar{I}}{}^i_{jk} = \underset{2}{I}{}^i_{jk} \tag{3.3}$$

we get

$$\bar{T}^{*i}_{jk}-T^{*i}_{jk} = \delta^i_k\alpha_j-g_{jk}\alpha^i;\; \bar{S}^{*i}_{jk}-S^i_{jk} = \delta^i_k\beta_j-g_{jk}\beta^i \tag{3.4}$$

$$\text{where: } \alpha_j = -\frac{1}{n-1}(T_j-\bar{T}_j);\; \beta_j = -\frac{1}{n-1}(S_j-\bar{S}_j).$$

Replacing it into (2.2), it follows (3.2).

Inversely, (3.2) is a g-transformation, and it is easy to see that $\underset{1}{I}$ and $\underset{2}{I}$ are invariants of this transformation.

We denote by $\mathscr{F}\Gamma^I$ the set of Finsler connections with the properties (3.3). Particularly we have the

Theorem 3.2. *If $F\Gamma(N)$ is a metrical Finsler connection ($G_{ijk}=0$, $g_{ijk}=0$), then the set of the metrical Finsler connections with the invariants $\underset{1}{I}$ and $\underset{2}{I}$ are given by* (3.2), *where α_i and β_i are arbitrary Finsler covectors.*

The set of the metrical Finsler connections having the properties (3.3) will be denoted by $\mathscr{F}\Gamma^I_g$.

As a particular case we find again the results from [10], where the general t_N transformations with the invariants $\underset{1}{I}$, $\underset{2}{I}$ [12] has been considered, imposing the condition that $F\Gamma$ and $F\bar{\Gamma}$ are metrical The following known results are also obtained as particular cases:

If $F\Gamma(N)$ is a metrical Finsler connection and $\underset{1}{I}=0$, $\underset{2}{I}=0$, then also $F\bar{\Gamma}(N)$ is a metrical Finsler connection with $\underset{1}{\bar{I}}=0$, $\underset{2}{\bar{I}}=0$.

Theorem 3.3. *By a Finsler g-transformation a semi-symmetric metrical Finsler connection $F\Gamma(N)$ is transformed also in a semi-symmetric metrical Finsler connection $F\bar{\Gamma}(N)$.*

Theorem 3.4. *The set of all semi-symmetric metrical Finsler connections $\mathscr{F}\Gamma^s_g$ or $\mathscr{F}\Gamma^{I=0}_g$ is given by* (3.2), *where $F\Gamma(N)$ is an arbitrary fixed semi-symmetric metrical Finsler connection.*

We have the relations: $\mathscr{F}\Gamma^s_g \subset \mathscr{F}\Gamma^I_g \subset \mathscr{F}\Gamma^I$, and we denote similarly the corresponding transformation groups too.

The Weyl type conformal invariants H^i_{jkl}, M^i_{jkl} and N^i_{jkl} of the group $\mathscr{F}\Gamma^s_g$ has been determined by R. Miron [9]. The projective Weyl invariants $\underset{1}{W}$, $\underset{2}{W}$, $\underset{3}{W}$ of the group $\mathscr{F}\Gamma^s_g$ has been determined in [11]. A special subgroup of $\mathscr{F}\Gamma^s_g$ is the group of S-concircular Finsler transformations. Their invariants has been established in [5].

It follows that the Theorems 3.2, 3.3, and 3.4 are special cases of Theorem 3.1 and all these theorems are obtained from the fundamental Theorem 2.2.

§ 4. Special classes of non-g Finsler transformations

The set $\mathscr{T}_N$ of the Finsler connection transformations with the same non-linear connection N is formed obviously from g-transformations and non-g-transformations.

The tensors B^i_{jk} and D^i_{jk} can be written as

$$(4.1)\quad B^i_{jk} = \frac{1}{2}(\bar{T}^i_{jk}-T^i_{jk})+\frac{1}{2}(B^i_{jk}+B^i_{kj});\ D^i_{jk} = \frac{1}{2}(\bar{T}^i_{jk}-T^i_{jk})+\frac{1}{2}(D^i_{jk}+D^i_{kj}).$$

We denote the last terms of B^i_{jk} and D^i_{jk} with $V^i_{jk}, \overline{V}^i_{jk}$. They are symmetric tensors. From (2.9) and (4.1) it follows

$$\overline{G}^i_{jk} = G^i_{jk} + \overline{T}^{*i}_{jk} - T^{*i}_{jk} - V^i_{jk};\ \bar{g}^i_{jk} = g^i_{jk} + \overline{S}^{*i}_{jk} - S^{*i}_{jk} - \overline{V}^1_{jk} \tag{4.2}$$

where we denote

$$G^i_{jk} = \frac{1}{2}\, g^{ia}(g_{aj|k} + g_{ak|j} - g_{jk|a});\ g^i_{jk} = \frac{1}{2}\, g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) \tag{4.3}$$

$$\overline{G}^i_{jk} = \frac{1}{2}\, g^{ia}(\overline{G}_{ajk} + \overline{G}_{akj} - \overline{G}_{jka});\ \bar{g}^i_{jk} = \frac{1}{2}\, g^{ia}(\bar{g}_{ajk} + \bar{g}_{akj} - \bar{g}_{jka}). \tag{4.4}$$

It follows $\overline{G}^i_{jk} \neq G^i_{jk}$ or(and) $\bar{g}^i_{jk} \neq g^i_{jk}$ from $\overline{G}_{ijk} \neq g_{ij|k}$ or(and) $\bar{g}_{ijk} \neq g_{ij}|_k$, and we have

Theorem 4.1. *Any non-g-transformation* $t_{N\,\mathrm{non}\,g} \in \mathscr{T}_N$ *is of the form:*

(4.5)

$$\overline{N}^i_j = N^i_j,\ \overline{F}^i_{jk} = F^i_{jk} + \overline{T}^{*i}_{jk} - T^{*i}_{jk} + \frac{1}{2}\, g^{ia}(g_{aj|k} + g_{ak|j} - g_{jk|a}) - \overline{G}^i_{jk};\ \overline{G}^i_{jk} \neq G^i_{jk}$$

$$\overline{C}^i_{jk} = C^i_{jk} + \overline{S}^{*i}_{jk} - S^{*i}_{jk} + \frac{1}{2}\, g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) - \bar{g}^i_{jk};\ \bar{g}^i_{jk} \neq g^i_{jk} \tag{4.6}$$

where $\overline{G}^i_{jk}$, $\bar{g}^i_{jk}$ *are two arbitrary symmetric Finsler tensors, which interpretation for a fixed* $F\overline{\Gamma}(\overline{N})$ *is given by* (4.4) *and* $\overline{T}, \overline{S} \in \mathfrak{T}^1_2$ *are two arbitrary tensors, where* $(\overline{T}, \overline{S})$ *is the torsion of* $F\overline{\Gamma}$.

Inversely: From (4.5) *and* (4.6) *it follows* $g_{ij\|k} = \overline{G}_{ijk} \neq g_{ij|k}$; $g_{ij}\|_k = \bar{g}_{ijk} \neq g_{ij}|_k$ *for any* $\overline{T}, \overline{S} \in \mathfrak{T}^1_2$. *Therefore* (4.5) *and* (4.6) *characterize the* $\mathscr{T}_{N\,\mathrm{non}\,g}$ *set of the non-g-transformations.*

Denoting by $\mathscr{T}_{Ng}$ the set of the g-transformations we obtain

Theorem 4.2. *(First Separation Theorem.)*

$$\mathscr{T}_N = \mathscr{T}_{Ng} \cup \mathscr{T}_{N\,\mathrm{non}\,g};\ \mathscr{T}_{Ng} \cap \mathscr{T}_{N\,\mathrm{non}\,g} = \emptyset.$$

The importance of this theorem is obvious in view of the determination of those special classes of Finsler connection transformations, where covariant derivatives or torsions of the connection satisfy certain conditions.

If we do not take into account the separation condition, then $\overline{G}^i_{jk} \neq G^i_{jk}$ or(and) $g^i_{jk} \neq g^i_{jk}$ and we obtain the global theorem of Separation, which is the principal theorem of this paper:

Theorem 4.3. *(Second Separation Theorem.) Let g be a general metrical Finsler structure of Miron type* [7], *then between two arbitrary Finsler connections* $F\Gamma(N)$,

$F\bar{\Gamma}(N)$ the following relations hold:

$$\bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk} + \bar{T}^{*i}_{jk} - T^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{aj|k} + g_{ak|j} - g_{jk|a}) - \bar{G}^i_{jk} \tag{4.7}$$

$$\bar{C}^i_{jk} = C^i_{jk} + \bar{S}^{*i}_{jk} - S^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) - \bar{g}^i_{jk} \tag{4.8}$$

where (T, S) and $(\bar{T}, \bar{S})$ are the torsions of $F\Gamma(N)$ and $F\bar{\Gamma}(N)$ respectively, and $\bar{G}^i_{jk}$, $\bar{g}^i_{jk}$ are given by (4.4).

If in the relations (4.5) and (4.6), $\bar{G}_{ijk}$, $\bar{g}_{ijk}$ are fixed and $(\bar{T}, \bar{S}) \in \mathfrak{T}^1_2 \times \mathfrak{T}^1_2$ are variable, then we obtain the

Theorem 4.4. *The set of Finsler connections $\mathscr{T}\Gamma(N)$ with the property that G_{ijk} and g_{ijk} are fixed, has the same multitude as the set $\mathfrak{T}^1_2 \times \mathfrak{T}^1_2$.*

Theorem 4.5. *Let $F\Gamma(N) \notin \mathscr{T}(N, g)$ be a fixed Finsler connection, then any Finsler connection $F\bar{\Gamma}(N)$ with fixed $\bar{G}_{ijk}$, $\bar{g}_{ijk}$ is uniquely determined through its given torsion $(\bar{T}, \bar{S})$, and is obtained from $F\Gamma(N)$ by means of a non-g-transformation* (4.5)—(4.6).

It follows that an arbitrary fixed Finsler connection $F\Gamma(N)$ and the set $\mathfrak{T}^1_2 \times \mathfrak{T}^1_2$ generate the class of Finsler connections $F\bar{\Gamma}(N)$, with the property that the h- and v-covariant derivative $\bar{G}_{ijk}$, $\bar{g}_{ijk}$ are fixed by certain properties.

Particularly for $\bar{G}_{ijk}=0$ and $\bar{g}_{ijk}=0$ we obtain the

Theorem 4.6. *The set of all metrical Finsler connections $F\bar{\Gamma}(N)$ obtained from a non-metrical Finsler connection $F\Gamma(N)$ is given by:*

$$\bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk} + \bar{T}^{*i}_{jk} - T^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{aj|k} - g_{ak|j} + g_{jk|a}) \tag{4.9}$$

$$\bar{C}^i_{jk} = C^i_{jk} + \bar{S}^{*i}_{jk} - S^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) \tag{4.10}$$

and any $F\bar{\Gamma}(N)$ is uniquely determined by means of its given torsion $(\bar{T}, \bar{S})$.

This special case is studied in [14], starting from the Sanini transformations [8]. Comparing the relations (4.9), (4.10) and the Sanini transformations:

$$\begin{cases} \bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk} + \frac{1}{2} g^{ia} g_{aj|k} + \Omega^{ir}_{sj} X^s_{rk}; \\ \bar{C}^i_{jk} = C^i_{jk} + \frac{1}{2} g^{ia} g_{aj}|_k + \Omega^{ir}_{sj} Y^s_{rk} \end{cases} \tag{4.11}$$

where X, Y are arbitrary Finsler tensors, we obtain directly that

$$\Omega^{ir}_{sj} X^s_{rk} = \bar{T}^{*i}_{jk} - T^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{ak|j} - g_{jk|a}) \tag{4.12}$$

$$\Omega^{ir}_{sj} Y^s_{rk} = \bar{S}^{*i}_{jk} - S^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{ak}|_j - g_{jk}|_a) \tag{4.13}$$

Hence the last terms of the relations (4.11) are uniquely determinated by means of $(\bar{T}, \bar{S})\in\mathfrak{T}_2^1\times\mathfrak{T}_2^1$.

What about the form of X and Y in this case? Since we have:

$$\Omega_{sj}^{*ir}\, U_{rk}^s = 0;\; \Omega_{sj}^{*ir}\,\bar{U}_{rk}^s = 0 \tag{4.14}$$

where U and $\bar{U}$ are the Finsler tensors from the right-hand member of (4.12) and (4.13), it follows that the tensor systems (4.12)—(4.13) are compatible, and has the general solution:

$$\begin{cases} X_{jk}^i = \bar{T}_{jk}^{*i} - T_{jk}^{*i} + \dfrac{1}{2}\, g^{ia}(g_{ak|j} - g_{jk|a}) + \Omega_{bj}^{*ic}\,\tilde{X}_{ck}^b \\ Y_{jk}^i = \bar{S}_{jk}^{*i} - S_{jk}^{*i} + \dfrac{1}{2}\, g^{ia}(g_{ak}|_j - g_{jk}|_a) + \Omega_{bj}^{*ic}\,\tilde{Y}_{ck}^b \end{cases} \tag{4.15}$$

where $\tilde{X}$, $\tilde{Y}$ are arbitrary Finsler tensors with null effect in (4.12) and (4.13), since $\Omega_{sj}^{ir}\Omega_{br}^{*sk}=0$. Hence in (4.12)—(4.13) the tensors X and Y must be fixed as in (4.15) for a fixing of $F\bar{\Gamma}(N)$, otherwise the right-hand member of (4.12) and (4.13) is not compatible with the left-hand member for a fixed $F\bar{\Gamma}(N)$.

We suppose that $F\bar{\Gamma}(N)$ is fixed from the condition, that it has not the same torsion as $F\Gamma(N)$. If X and Y are completely arbitrary, then considering

$$X_{rk}^s = \frac{1}{2}\, g^{as}(g_{ak|r} - g_{rk|a});\; Y_{rk}^s = \frac{1}{2}\, g^{as}(g_{ak|r} - g_{rk}|_a) \tag{4.16}$$

it follows that $F\bar{\Gamma}(N)$ has the same torsion as $F\Gamma(N)$. This is a contradiction. Thus can be considered the Sanini transformations are explicitly expressed also in this way too as a particular case of the *Theorem* 4.5.

If $F\Gamma(N)$ and $F\bar{\Gamma}(N)$ are metrical Finsler connections, then the explicit form of the Sanini transformations is

$$\Omega_{sj}^{ir} X_{rk}^s = \bar{T}_{jk}^{*i} - T_{jk}^{*};\; \Omega_{sj}^{ir} Y_{rk}^i = \bar{S}_{jk}^{*i} - S_{jk}^{*i} \tag{4.17}$$

$$X_{jk}^i = \bar{T}_{jk}^{*i} - T_{jk}^{*i} + \Omega_{sj}^{ir}\tilde{X}_{rk}^s;\; Y_{jk}^i = \bar{S}_{jk}^{*i} - S_{jk}^{*i} + \Omega_{sj}^{*ir}\tilde{Y}_{rk}^s \tag{4.18}$$

namely we obtain (2.2), and these transformations depend on the two skew-symmetric tensors $\bar{T}_{jk}^i, \bar{S}_{jk}^i\in\mathfrak{T}_2^1$ only. For the case of the Cartan connection $\overset{c}{F\Gamma}(N)\overset{c}{F\bar{\Gamma}}(N)$, which is metrical with $T=0$ and $S=0$, the explicit determination follows from (2.2) as given in Theorem 4.6 of R. MIRON [7].

Considering as 1-form on $T(M)$: $\omega=\omega_k\, dx^k+\lambda_k\delta y^k$ and the relations

$$\bar{G}_{ijk} = 2g_{ij}\,\omega_k;\quad \bar{g}_{ijk} = 2g_{ij}\,\lambda_k \tag{4.19}$$

we arrive to

Theorem 4.7. *The set of all conformal Finsler connections $F\bar{\Gamma}(N;\omega)$ obtained from an arbitrary fixed Finsler connection $F\Gamma(N)$ is given by*

$$(4.20)\qquad \bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk} + \bar{T}^{*i}_{jk} - T^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{aj|k} + g_{ak|j} - g_{jk|a}) - \\ - \delta^i_j \omega_k - \delta^i_k \omega_j + g_{jk}\omega^i$$

$$(4.21)\qquad \bar{C}^i_{jk} = C^i_{jk} + \bar{S}^{*i}_{jk} - S^{*i}_{jk} + \frac{1}{2} g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) - \delta^i_j \lambda_k - \delta^i_k \lambda_j + g_{jk}\lambda^i.$$

Therefore we obtain as a particular case of (4.5) and (4.6) the results from [2] for $N=\bar{N}$ and an explicit determination of ΩX and ΩY.

For a fixed $\omega=\omega_0$ it follows

Theorem 4.8. *Let $F\Gamma(N)$ be a fixed Finsler connection, then the set of all conformal Finsler connections with the same 1-form ω_0, denoted by $F\bar{\Gamma}(N,\omega_0)$, has the multitude of the $\mathfrak{T}^1_2\times\mathfrak{T}^1_2$ set.*

§ 5. Non-g Finsler transformations which have the invariants $\underset{1}{I}$ and $\underset{2}{I}$

From the relations (3.1), (3.3) and (3.4) we have the

Theorem 5.1. *A necessary and sufficient condition for a non-g-transformation to have the invariants $\underset{1}{I}$ and $\underset{2}{I}$ is that it be of the form:*

$$(5.1)\qquad \bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk} + \delta^i_k \alpha_j - g_{jk}\alpha^i + \frac{1}{2} g^{ia}(g_{aj|k} + g_{ak|j} - g_{jk|a}) - \bar{G}^i_{jk}$$

$$(5.2)\qquad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_k \beta_j - g_{jk}\beta^i + \frac{1}{2} g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) - \bar{g}^i_{jk}$$

where α_k, β_k are arbitrary Finsler covectors, and $\bar{G}^i_{jk}$, $\bar{g}^i_{jk}$ are arbitrary symmetric Finsler tensors $\bar{G}^i_{jk}$ and $\bar{g}^i_{jk}$ are given by (4.4).

For $\bar{G}^i_{jk}=0$, $\bar{g}^i_{jk}=0$ we get the

Theorem 5.2. *A necessary and sufficient condition that the Finsler transformatton $t\in\mathcal{T}_N$, which transforms a fixed general Finsler connection $F\Gamma(N)$ in a metrical Finsler connection $F\bar{\Gamma}(N)$, has the invariants $\underset{1}{I}$ and $\underset{2}{I}$ is that it be of the form:*

$$\begin{cases} (5.3)\qquad \bar{N}^i_j = N^i_j;\ \bar{F}^i_{jk} = F^i_{jk} + \delta^i_k \alpha_j - g_{jk}\alpha^i + \frac{1}{2} g^{ia}(g_{aj|k} + g_{ak|j} - g_{jk|a}) \\ (5.4)\qquad \bar{C}^i_{jk} = C^i_{jk} + \delta^i_k \beta_j - g_{jk}\beta^i + \frac{1}{2} g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) \end{cases}$$

where α_j and β_j are arbitrary Finsler covectors.

We obtain as a special case the transformation from [11].

Theorem 5.3. *The transformations* (5.3)—(5.4) *with $\underset{1}{I}=0$, $\underset{2}{I}=0$ form the class of the Finsler connection transformations $t\in\mathcal{T}_N$, which transform an arbitrary fixed semi-symmetric Finsler connection in a semi-symmetric metrical Finsler connection.*

Analogously from (4.20) and (4.21) we have the:

Theorem 5.4. *The class of the transformations* t: $F\Gamma(N) \to F\bar{\Gamma}(N, \omega)$ *which have the invariants* $\underset{1}{I}$ *and* $\underset{2}{I}$ *are characterized by:*

$$\begin{cases} (5.5) \quad \bar{N}^i_{jk} = N^i_{jk};\ \bar{F}^i_{jk} = F^i_{jk} - \delta^i_j \omega_k + \delta^i_k \xi_j - g_{jk}\xi^i + \frac{1}{2} g^{ia}(g_{aj|k} + g_{ak|j} - g_{jk|a}) \\ (5.6) \quad \bar{C}^i_{jk} = C^i_{jk} - \delta^i_j \lambda^k + \delta^i_k \eta_j - g_{jk}\eta^i + \frac{1}{2} g^{ia}(g_{aj}|_k + g_{ak}|_j - g_{jk}|_a) \end{cases}$$

where ξ_k *and* η_k *are arbitrary Finsler covectors.*

In this way the importance of the First Separation Theorem and of the explicit determination of the tensors $\Omega^{ir}_{sj} X^s_{rk}$ and $\Omega^{ir}_{sj} Y^s_{rk}$ from the Sanini transformations is accentuated, these transformations being obtained as a particular case of the g-transformations and of the non-g-transformations.

In a forthcoming paper we shall give a Separation Theorem also for the set of general Finsler connection transformations $\mathcal{T} = \{t \mid t\colon (N, F, C) \to (\bar{N}, \bar{F}, \bar{C})\}$ with a variable non-linear connection N.

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