A general minimax inequality and its consequences

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Two real-valued functions f, g given on the Cartesian product $X \times Y$ of two nonempty sets X, Y are said to satisfy minimax inequality if the following holds:

$$\text{(MMI)} \qquad \inf_{y \in Y} \sup_{x \in X} g(x, y) \leqq \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

In the case when g equals f (MMI) is known as the statement of any generalization of the celebrated minimax theorem due to von Neumann, namely the following minimax equality:

$$\text{(MME)} \qquad \inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

Since (MMI) holds if and only if for every positive real numbers c

$$\inf_{y \in Y} \sup_{x \in X} g(x, y) < \sup_{x \in X} \inf_{y \in Y} f(x, y) + c$$

holds, that is for any $c > 0$ there exists (x, y) in $X \times Y$ such that

$$\sup_{u \in X} g(u, y) < \inf_{v \in Y} f(x, v) + c$$

or, what is the same, for any $c > 0$ there exists (x, y) in $X \times Y$ with

$$0 \leqq f(x, v) - g(u, y) + c$$

for every (u, v) in $X \times Y$.

By introducing, as in [3] the sets in $X \times Y$ as follows

$$K_{u,v}^c := \{(x, y) \in X \times Y : 0 \leqq f(x, u) - g(u, y) + c\}$$

we see that (MMI) is nothing else that $\{K_{u,v}^c : (u, v) \in X \times Y\}$ is a family of subsets of $X \times Y$ with common point for every $c > 0$. Our approach, as an improved one of [1], [2], [3] is the following: we give conditions on f and g, sufficient for the sets $K_{u,v}^c$ having the finite intersection property for any $c > 0$ and also sufficient for a topology with respect to which these sets are closed and one of them is even compact, for any $c > 0$ as before. The well-known principle of Riesz for the existence of a common point, for every $c > 0$, then applies. We note that the usual concave-convex type and continuity requirements besides some compactness assumptions in general implies our ones. So we prove a rather general minimax theorem some consequence of which are presented here. It is of natural to ask wther the "fixed point" type minimax theorems also follows from ours or not. For minimax inequalities see S. SIMMONS [4], Z. SEBESTYÉN [2], [3].

Theorem 1. *Let f, g be real-valued functions on $X\times Y$ satisfying the following three properties:*

$$\min_{(u,v)\in G}\sum_{(x,y)\in F}\lambda(x,y)[f(x,v)-g(u,y)] \leqq \sup_{(x,y)\in X\times Y}\min_{(u,v)\in G}[f(x,v)-g(u,y)] \tag{1}$$

for every pair F, G of finite sets in $X\times Y$ and a discrete probability measure λ on F;

$$0 \leqq \inf_{(u,v)\in X\times Y}\sup_{(x,y)\in X\times Y}[f(x,v)-g(u,y)] \leqq \sup_{(x,y)\in X\times Y}\sum_{(u,v)\in G}\mu(u,v)[f(x,v)-g(u,y)] \tag{2}$$

for every finite subset G of $X\times Y$ and a discrete probability measure μ on G;

$$\textit{for every } c>0 \textit{ there exists } (u_c, v_c) \textit{ in } X\times Y \tag{3}$$

such that if $D\subset X\times Y$ has the property: for any (x,y) in $K^c_{u_c,v_c}$ there exists (u,v) in D with $(x,y)\notin K^c_{u,v}$ then the same property holds for some finite subset of D.

Then the minimax inequality (MMI) *holds for f, g on $X\times Y$.*

PROOF. Let $\{X\times Y\setminus K^c_{u,v}\colon (u,v)\in X\times Y\}$, the complements in $X\times Y$ of the sets $K^c_{u,v}$, be the open subbase of a topology τ_c for a fixed $c>0$. Property (3) then assures, besides the closedness of $K^c_{u,v}$'s, that $K^c_{x_c,y_c}$ is even compact by Alexander's subbase lemma. Thus only the finite intersection property of $\{K^c_{u,v}\colon (u,v)\in X\times Y\}$ remains to be proved. Let now $(u_1,v_1),\ldots,(u_n,v_n)$ be given points in $X\times Y$. We prove in the following that $\bigcap_{i=1}^{n} K^c_{u_i,v_i}$ is nonempty. For if the contrary holds for any (x,y) in $X\times Y$ there exists i, $1\leqq i\leqq n$ such that $f(x,v_i)-g(u_i,y)+c<0$, i.e.

$$\min_i\,[f(x,v_i)-g(u_i,y)+c) < 0 \quad \text{for any} \quad (x,y) \quad \text{in} \quad X\times Y \tag{4}$$

holds. As a consequence, the range $\varphi_c(X\times Y)$ of the function φ_c, defined on $X\times Y$ (and taking values in $\mathbf{R}^n$) by

$$\varphi_c(x,y) = \big(f(x,v_1)-g(u_1,y)+c,\ldots,f(x,v_n)-g(u_n,y)+c\big) \quad (x,y\in X\times Y) \tag{5}$$

does not meet the positive cone $\mathbf{R}^n_+=\{(\lambda_1,\ldots,\lambda_n)\colon \lambda_i\geqq 0\ (i=1,\ldots,n)\}$ in $\mathbf{R}^n$. We state a little more: $\operatorname{co}\varphi_c(X\times Y)$, the convex hull of the range, also doesn't meet $\operatorname{int}\mathbf{R}^n_+=\{(\lambda_1,\ldots,\lambda_n)\colon \lambda_i>0\ (i=1,\ldots,n)\}$. Indeed, the assumption of the contrary implies the existence of a finite set F in $X\times Y$ and a discrete probability measure λ on that such that

$$0 < \sum_{(x,y)\in F}\lambda(x,y)[f(x,v_i)-g(u_i,y)]+c, \quad i=1,2,\ldots,n$$

holds, which is equivalent to holding with $G=\{(u_i,v_i)\}_{i=1}^n$

$$-c < \min_{(u,v)\in G}\sum_{(x,y)\in F}\lambda(x,y)[f(x,v)-g(u,y)].$$

But (1) implies then

$$-c < \sup_{(x,y)\in X\times Y} \min_{(u,v)\in G} [f(x,v)-g(u,y)],$$

the existence of (x_c, y_c) in $X\times Y$ such that $0<f(x_c, v)-g(u, y_c)+c$ holds for any (u, v) in G. But this contradicts the starting assumption that $\{K^c_{u_i,v_i}\}_{i=1}^n$ hasn't a common point (as e.g. (x_c, y_c)). As a consequence a separating, nonzero vector $(\mu_1, ..., \mu_n)$ exists for $\text{co}\,\varphi(X\times Y)$ and $\text{int}\,\mathbf{R}^n_+$ in the sense that

$$\sum_{i=1}^n \mu_i[f(x, v_i)-g(u_i, y)]+c\sum_{i=1}^n \mu_i \leqq \sum_{i=1}^n \mu_i\lambda_i$$

holds for every (x, y) in $X\times Y$ and $(\lambda_1, ..., \lambda_n)\in\text{int}\,\mathbf{R}^n_+$. It then follows $\mu_i\geqq 0$ for $i=1, ..., n$, and $\sum_{i=1}^n \mu_i=1$ may then be assumed. In other words a discrete probability measure μ exists on G such that

$$\sum_{(u,v)\in G} \mu(u,v)[f(x,v)-g(u,y)] \leqq -c$$

holds for any (x, y) in $X\times Y$. Then (2) implies

$$0 \leqq \inf_{(u,v)\in X\times Y} \sup_{(x,y)\in X\times Y} [f(x,v)-g(u,y)] \leqq -c$$

a contradiction, because of positiveness of c. The proof is ended.

Theorem 2. *Let f be a real-valued function defined on $X\times Y$ and satisfying the properties of Theorem* 1 *in the case when $g=f$. Then* (MME) *holds for f on $X\times Y$.*

PROOF. As a direct consequence of Theorem 1 we have

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leqq \sup_{x\in X}\inf_{y\in Y} f(x,y)$$

for the function f. But the converse inequality is clearly holds for any function on $X\times Y$, so that the proof of (MME) is complete.

Theorem 3. *Let f be a real-valued function on $X\times Y$ satisfying the following three properties:*

(6) $$\min_{v\in G}\sum_{x\in F}\lambda(x)f(x,v) \leqq \sup_{x\in X}\min_{v\in G} f(x,v)$$

for every finite sets F in X, G in Y and a probability measure λ on F,

(7) $$\inf_{v\in Y}\sup_{x\in X} f(x,v) \leqq \sup_{x\in X}\sum_{v\in G}\mu(v)f(x,v)$$

for every finite set G in Y and a probability measure μ on G;

(8) *for any $c\in\mathbf{R}$, $c<\inf_{v\in Y}\sup_{x\in X} f(x,v)$ there exists v_c*

in Y such that if $C\subset Y$ has the property: for any x in X such that $c\leqq f(x, v_c)$ *there exists y in C with $f(x, y)<c$ then the same property holds for some finite subset of C too.*

Then f satisfies (MME).

PROOF. Take g the function on $X\times Y$ taking the only value $c^*:=\inf_{v\in Y}\sup_{x\in X} f(x, v)$ which is easily seen a real number, not the $\pm$ infinity, by (8). Theorem 1 then applies.

Corollary 4. *Let $f_1, \ldots, f_n$ be real-valued functions on Y, a nonempty set, satisfying*

(9) $$\inf_{v\in Y}\max_j f_j(v) \leqq \max_j \sum_i \mu_i f_j(v_i)$$

for any finite set (v_i) of Y and $\mu_i\geqq 0$ such that $\sum_i \mu_i=1$.

There exists then $\lambda_j\geqq 0$ $(j=1, 2, \ldots, n)$ with $\sum_j \lambda_j=1$ and such that

(10) $$\inf_{v\in Y}\max_j f_j(v) = \inf_{v\in Y}\sum_j \lambda_j f_j(v)$$

holds.

PROOF. Let X be $\{(\lambda_1, \ldots, \lambda_n)\in\mathbf{R}^n\colon \lambda_j\geqq 0\ (1\leqq j\leqq n)\ \sum_j \lambda_j=1$ and let $f(x, y)=\sum_j \lambda_j f_j(y)$ for $x=(\lambda_1, \ldots, \lambda_n)\in X$ and $y\in Y$ f is then a function on $X\times Y$, with X is compact and convex in $\mathbf{R}^n$, affine (thus concave) and continuous in the first variable. Thus Theorem 3 applies and gives our statement.

References

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