Nonparametric identification of Uryson and Volterra nonlinear systems with autoregressive input processes

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1. *Introduction*

In our earlier papers [3] we dealt in details with the nonparametric statistical identification of the discrete and continuous Zadeh systems.

This paper discusses the discrete and continuous first order Zadeh (so called Uryson) systems furthermore the nonlinear systems represented by Volterra functional series with autoregressive processes as input signals. Although the nonparametric identification of Volterra models is well-known see for example Schetzen's [5] general spectral-factorization method in frequency domain, we introduce a very simple procedure for the estimation of the Wiener Kernels given in time domain by explicit formulae.

It is also known see for example Anderson [6] that one can get a good approximation for the ARMA processes by autoregressive processes. Therefore the obtained results have practical values in case of ARMA input as well.

The last argument is of course valid for the identification of the Uryson models too. Furthermore the obtained results for the determination of the first order Rajbman kernels (see [1]) of the Uryson system can partly be used for the estimation of the higher order Rajbman kernels of the Zadeh nonlinear systems. Thus in the first part of this paper we are dealing with the nonparametric identification of continuous and discrete Uryson systems, mostly with the first and second order autoregressive input processes. Furthermore in the second part we present the identification of continuous and discrete Volterra models with autoregressive input of pth order.

Finally we point out the relationship between the Volterra and the Zadeh model identifications when the input process is an autoregressive one. A general identification of the Zadeh model using autoregressive and ARMA input process will be discussed in a forthcoming paper.

2. *Identification of continuous Uryson models with AR* (1) *input process*

It is well known that the identification of first order nonlinear dynamic systems can be frequently described by Uryson models (which is a first order Zadeh one). Its most important special cases are the simple and cascade Hammerstein systems both in the scalar and multiple variable cases.

The usual form of discrete stationary Uryson system is

$$y(t) = \sum_{k=0}^{\infty} u[x(t-k), k] + \zeta(t)$$

Here u is an arbitrary Borel measurable function and ζ is a zero mean noise process, which is independent of the input. In the case of active identification the input signal is chosen as discrete white noise with zero mean and unit variance. This case was dealt with in details in [3]. In the continuous time case, instead of the general form

$$y(t) = \int_{0}^{\infty} u[x(t-s), s]\, ds + \zeta(t)$$

the following (analytical) Uryson model seems to be useful, if the input $x(t)$ is an arbitrary Gaussian process

$$y(t) = \int_{0}^{\infty} u[x(t-s), s]\, ds + \zeta(t) = \int_{0}^{\infty} \sum_{n=1}^{\infty} a_n(s) H_n[x(t-s)]\, ds + \zeta(t)$$

where the input is independent of $\zeta(t)$ and

$$H_n(x) = \frac{(-1)^n}{2^{n/2}\sqrt{n!}} \exp\left[-\frac{x^2}{2}\right] \frac{d^n}{dx^n} \exp\left[-\frac{x^2}{2}\right]$$

are the standard Hermite polynomials and $a_n(s)$ are continuous functions and $a_n(s) \in L^2(\infty, \infty)$ for all n

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} a_n^2(s)\, ds < \infty$$

For the sake of simplicity in handling of the formulae it is assumed that the process $x(t)$ is of zero mean and unit variance.

Let's consider now the determination and estimation of the normed dispersion function (correlation ratio) by a finite sample. For Uryson models the dispersion function has the following simple formula

$$K_{yx}^2(\tau) = \sum_{n=1}^{N} \hat{R}_{yZ_n}^2(\tau) \quad \text{and} \quad Z_n(t-\tau) = H_n[x(t-\tau)]$$

where

$$\hat{R}_{yZ_n}(\tau) = \frac{1}{T} \sum_{t} y(t+\tau) H_n[x(t)]$$

are the cross correlations between the output and the Hermite polynomial variables. For ergodic continuous time series the estimators are

$$K_{yx}^2(\tau) = \sum_{n=1}^{N} \hat{R}_{yZ_n}^2(\tau)$$

where

$$\hat{R}_{yZ_n}(\tau) = \frac{1}{T}\int_0^T y(t+\tau)H_n[x(t)]\,dt, \quad n = 1, \ldots, N.$$

It is easy to see that the estimator of $R_{yZ_n}(\tau)$ is unbiassed, but the estimator of the dispersion function is biassed because of the squaring and the finite upper limits of summation. However, this estimation method is simple and also the bias is expectedly smaller than the well known estimator of the dispersion function computed by the formula in [1]. In the linear case it is clear that

$$K^2_{yx}(\tau) = R^2_{yZ_1}(\tau)+0+\ldots = R^2_{yx}(\tau),$$

i.e. all the cross correlations vanish except the one of order one

$$R_{yZ_n}(\tau) = 0, \quad n > 1$$

Since the generation of exact continuous white noise process for the input is almost impossible, it is useful to choose a zero mean and unit variance stationary first order autoregressive Gaussian process, the so called coloured noise [7] with autocorrelation function $e^{-a|\tau|}$ and spectral density,

$$F_{xx}(\omega) = \frac{2a}{a^2+\omega^2}$$

If we use the above specification of the input Gaussian process, we can get the kernels of the Uryson systems from the following simple relationships

$$a_n(s) = naR_{yH_n}(s) - \frac{1}{na}\ddot{R}_{yH_n}(s), \quad n = 1, 2, \ldots$$

$$s\in[0, \tau] \quad \text{or} \quad [0, \infty)$$

or by the expression of differential operators

$$a_n(s) = \frac{1}{na}\left[(na)^2 - \frac{d^2}{ds^2}\right]R_{yH_n}(s) = \frac{1}{na}\left(na-\frac{d}{ds}\right)\left(na+\frac{d}{ds}\right)R_{yH_n}(s) \quad n = 1, 2, \ldots.$$

For the proof of the above equation we start from the spectral density function of the continuous first order autoregressive process. On the basis of the relationships

$$F_{xx}(\omega) = F^+_{xx}(j\omega)F^-_{x}(j\omega) = \frac{\sqrt{2a}}{a+j\omega}\,\frac{\sqrt{2a}}{a-j\omega}$$

and similarly for the processes $H_n[x(t)]$

$$F_{H_n}(\omega) = F^+_{H_n}(j\omega)F^-_{H_n}(j\omega) = \frac{2na}{(na)^2+\omega^2} = \frac{\sqrt{2na}}{na+j\omega}\,\frac{\sqrt{2na}}{na-j\omega}$$

it is not necessary to use the spectral factorization procedure but it is satisfactory to analyse the equation

$$R_{yH_n}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_n(j\omega) \frac{2na}{(na)^2+\omega^2} e^{-j\omega s}\, d\omega \tag{1}$$

naturally only for positive values of time delay "s" (where F is the symbol of the Fourier transformation)

$$G_n(j\omega) = F[a_n(s)]$$

and

$$F(R_{yH_n}(s)) = G_n(j\omega) \frac{(na)^2}{(na)^2+\omega^2}$$

Thus by (1) differentiating $R_{yH_n}(s)$ twice we get that

$$\frac{d^2}{ds^2} R_{yH_n}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_n(j\omega) \frac{2na\omega^2}{(na)^2+\omega^2} e^{j\omega s}\, d\omega \tag{2}$$

and so from (1) and (2)

$$naR_{yH_n}(s) - \frac{d^2}{ds^2} R_{yH_n}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_n(j\omega) e^{j\omega s}\, d\omega = a_n(s)$$

Theorem 1. *The estimation of kernels $a_n(s)$ can be carried out by the formula*

$$\hat{a}_n(s) = \frac{1}{na}\left[(na)^2 - \left(\frac{d}{ds}\right)^2\right] \hat{R}_{yH_n}(s)$$

where

$$\hat{R}_{yH_n}(s) = \frac{1}{T-s} \int_{S}^{T} y(t) H_n[x(t-s)]\, dt$$

It can be proved that the estimate of $\hat{a}_n(s)$ is umbiassed and consistent one.

3. *Estimation of weighting coefficients for discrete Uryson models using autoregressive input processes*

Let us consider the following Uryson model

$$y(n) = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} g_l(i) H_l[Z(n-i)]$$

We shall estimate the weighting coefficients $g_l(i)$ when $Z(n)$ is a first order discrete autoregressive random process, i.e.

$$Z(n) = \lambda Z(n-1) + \varepsilon_n$$

Here ε_n is a discrete Gaussian white noise series with zero expectation and unit variance as well as $|\lambda|<1$. Then the autocorrelation function of $Z(n)$ is

$$c(|i-j|) = \frac{\lambda^{|i-j|}}{1-\lambda^2}$$

Let consider the expectation

$$Ey(k)H_l[Z(k-j)] = \sum_{i=0}^{\infty} g_l(i)\left(\frac{\lambda^{|i-j|}}{1-\lambda^2}\right)^l$$

from where we can get an estimation for

$$g_l(i) \quad i = 0, 1, 2, \dots \quad \text{if} \quad |g_l(i)| \leqq A^{-c_l i}, \quad \text{i.e.}$$

the weighting coefficients have exponential characters, the system of equations

$$EY(k)H_l[Z(k-j)] = \sum_{l=0}^{N} g_l(i)\left(\frac{\lambda^{|i-j|}}{1-\lambda^2}\right)^l \quad j = 0, 1, 2, \dots, N$$

Now we introduce an another more simple estimator for $g_l(i)$ using explicit formula and so a direct computation procedure. Let us take the generator function of series $g_l(i)$ by

$$G_l(v) = \sum_{i=0}^{\infty} g_l(i)v^i$$

using $G_l(v)$ we can get the generator function for the cross correlation function between $H_l[z(n-j)]$ and $y(n)$ i.e.

$$K_l(v) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} g_l(i)\left(\frac{\lambda^{|i-j|}}{1-\lambda^2}\right)^l v^j = \frac{1}{(1-\lambda^2)^l}\sum_{i=0}^{\infty} g_l(i)\sum_{i=0}^{\infty}\lambda^{|i-j|l}v^j =$$

$$= \frac{1}{(1-\lambda^2)^l}\sum_{i=0}^{\infty} g_l(i)\left[\lambda^{il}\sum_{j=0}^{i-1}\left(\frac{v}{\lambda^l}\right)^j + v^i\sum_{j=0}^{\infty}(\lambda_v^l)^{j-i}\right] =$$

$$= \frac{1}{(1-\lambda^2)^l}\sum_{i=0}^{\infty} g_l(i)\left[\lambda^{il}\frac{1-\left(\frac{v}{\lambda^l}\right)^i}{1-\frac{v}{\lambda^l}} + \frac{v^i}{1-\lambda^l v}\right] =$$

$$= \frac{1}{(1-\lambda^2)^l}\sum_{i=0}^{\infty} g_l(i)\left[\lambda^{li}\frac{\lambda^l}{\lambda^l - v} + v^i\frac{v(\lambda^{2l}-1)}{(\lambda^l - v)(1-\lambda^l v)}\right] =$$

$$= \frac{1}{(1-\lambda^2)^l}\left[\frac{\lambda^l}{\lambda^l - v}G_l(\lambda^l) + \frac{v(\lambda^{2l}-1)}{(\lambda^l - v)(1-\lambda^l v)}G_l(v)\right]$$

From where

$$G_l(v)=\left[(1-\lambda^2)^l K_l(v)-\frac{\lambda^l}{\lambda^l-v}\,G_l(\lambda^l)\right]\frac{(\lambda^l-v)(1-\lambda^l v)}{v(\lambda^{2l}-1)}=$$

$$=\left[(1-\lambda^2)^l(\lambda^l-v)K_l(v)-\lambda^l G_l(\lambda^l)\right]\frac{1}{\lambda^{2l}-1}(1/v-\lambda^l)$$

Because of

$$EY(n)H_i(x(n))=K_l(o)=\frac{1}{(1-\lambda^2)^l}\,G_l(\lambda^l)$$

we get, that

$$G_l(v)=\left[(1-\lambda^2)^l\cdot\lambda^l\sum_{j=1}^{\infty}EY(n)H_l(x(n-j))v^j-(1-\lambda^2)^l vK_l(v)\right]\frac{1}{\lambda^{2l}-1}\left(\frac{1}{v}-\lambda^l\right)$$

Theorem 2. *And so it follows*

$$g_l(o)=\frac{(1-\lambda^2)^l}{\lambda^{2l}-1}\left(\lambda^l EY(n)H_l(x(n-1))-EY(n)H_l(x(n))\right)$$

$$g_l(1)=\frac{(1-\lambda^2)^l}{\lambda^{2l}-1}\left(\lambda^l EY(n)H_l(x(n-2))-EY(n)H_l(x(n-1))(1+\lambda^{2l})+\right.$$
$$\left.+\lambda^l EY(n)H_l(x(n))\right)$$

$$\vdots$$

$$g_l(j)=\frac{(1-\lambda^2)^l}{\lambda^{2l}-1}\left[\lambda^l EY(n)H_l(x(n-j-1))-EY(n)H_l(x(n-j))(1+\lambda^{2l})+\right.$$
$$\left.+\lambda^l EY(n)H_l(x(n-j+i))\right]$$

We get the estimates by the empirical moments instead of the theoretical ones

$$\hat{g}_l(o)=\frac{(1-\lambda^2)^l}{1-\lambda^{2l}}\left[\overline{Y(n)H_l^0(x(n))}-\lambda^l\overline{Y(n)H_l^0(x(n-1))}\right]$$

$$\hat{g}_l(j)=\frac{(1-\lambda^2)^l}{1-\lambda^{2l}}\left[(1+\lambda^{2l})\overline{Y(n)H_l(x(n-j))}-\lambda^l\overline{(Y(n)H_l(x(n-j-1))}+\right.$$
$$\left.+\overline{Y(n)H_l(x(n-j+1))}\right].$$

It can be proved easily that these estimates are unbiassed and consistent. We show that there is another interesting method for the determination of $g_l(i)$ using a relationship between the spectral density and autocorrelation functions of stationary input series.

If $\sigma(0), \sigma(1), \ldots$ are an autocorrelation series of a stationary time series for which

$$\sigma(o)+2\sum_{n=1}^{\infty}|\sigma(n)|=\sum_{n=-\infty}^{\infty}|\sigma(n)|<\infty$$

then its spectral density function is

$$f(\lambda) = \frac{1}{2\pi}\sigma(o) + \frac{1}{\pi}\sum_{n=1}^{\infty}\sigma(n)\cos(\lambda n) =$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sigma(n)\cos(-\lambda n) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sigma(n)e^{-i\lambda n}$$

From the spectral density function we get the autocorrelation function by

$$\sigma(n) = \int_{-\pi}^{\pi} e^{i\lambda n} f(\lambda)\, d\lambda$$

These formula are valid properly for the Fourier transform of nonnegative definit absolutely summeable series [6].

We remark that if the spectral representation of two stationary time series are

$$v_n = \int_{-\pi}^{\pi} e^{in\lambda} g_1(\lambda)\, d\mu$$

and

$$w_n = \int_{-\pi}^{\pi} e^{in\lambda} g_2(\lambda)\, d\mu$$

where the stochastic spectral measure μ is same for both cases then their cross correlation function is

$$c_{vw}(n-m) = \int_{-\pi}^{\pi} e^{i(n-m)\lambda} g_1(\lambda)\overline{g_2(\lambda)}\, E|d\mu|^2$$

Now let us consider the spectral representation of the original input series i.e.

$$Z_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{in\omega} \frac{1}{1-\lambda e^{-i\omega}}\, d\mu_2 \quad \left(EZ_n Z_{n+k} = \frac{\lambda^{|k|}}{1-\lambda^2}\right)$$

where $E|d\mu_2|^2 = d\omega$ and the spectral representation of stationary time series $H_l[Z(n)]$ is

$$H_l[Z(n)] = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{in\omega} \sqrt{\frac{1-\lambda^{2l}}{(1-\lambda^2)^l}}\, \frac{1}{1-\lambda^l e^{-i\omega}}\, d\mu$$

where

$$E|d\mu|^2 = d\omega.$$

We get for the spectral representation of the process

$$y_l(n) = \sum_{j=0}^{\infty} g_l(j) H_l[Z(n-j)]$$

the following result

$$y_l(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sum_{j=n}^{\infty} e^{i(n-j)\omega} g_l(j) \sqrt{\frac{1-\lambda^{2l}}{(1-\lambda^2)^l}} \frac{1}{1-\lambda^l e^{-i\omega}} d\mu =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{in\omega} \sum_{j=0}^{\infty} e^{-ij\omega} g_l(j) \sqrt{\frac{1-\lambda^{2l}}{(1-\lambda^2)^l}} \frac{1}{1-\lambda^l e^{-i\omega}} d\mu =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{in\omega} G_l(e^{-i\omega}) S_l(e^{-i\omega}) \, d\mu,$$

where

$$G_l(z) = \sum_{j=0}^{\infty} g_l(j) Z^j$$

$$S_l(z) = \sqrt{\frac{1-\lambda^{2l}}{(1-\lambda^2)^l}} \frac{1}{1-\lambda^l z}$$

Because of the orthogonality of polynomials H_l it follows that

$$EY(n) H_l[Z(n-j)] = Ey_l(n) H_l[Z(n-j)]$$

The spectral density function of this cross correlation function is

$$K_l(\omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} Ey_l(n) H_l[Z(n-k)] e^{-i\omega n+1}$$

The formula

$$Ey_l(n) H_l[Z(n-k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\omega} G_l(e^{-i\omega}) S_l(e^{-i\omega}) \, d\omega$$

holds for any integer number k so

$$K_l(\omega) = \frac{1}{2\pi} G_l(e^{-i\omega}) S_l(e^{-i\omega}) S_l(e^{i\omega})$$

from where

$$\frac{1}{2\pi} G_l(e^{-i\omega}) = \frac{1}{|S_l(e^{i\omega})|^2} - K_l(\omega) = \frac{(1-\lambda^2)^l}{1-\lambda^{2l}} (1-\lambda^l e^{i\omega})(e^{i\omega}-\lambda^l) e^{-i\omega} K_l(\omega)$$

$$\frac{1}{2\pi} G_l(e^{-i\omega}) = \frac{1}{2\pi} \frac{(1-\lambda^2)^l}{1-\lambda^{2l}} [-\lambda^l e^{2i\omega} + (1+\lambda^{2l}) e^{i\omega} - \lambda^l] K_l(\omega) e^{-i\omega} =$$

$$= \frac{(1-\lambda^2)^l}{1-\lambda^{2l}} [-\lambda^l e^{i\omega} + (1+\lambda^{2l}) - \lambda^l e^{-i\omega}] K_l(\omega)$$

and so

$$g_l(o) = \frac{(1-\lambda^2)^l}{1-\lambda^{2l}}\big[(1+\lambda^{2l})Ey(n)H_l[Z(n)] -$$

$$-\lambda^l Ey(n)H_l[Z(n-1)] - \lambda^l Ey(n)H_l[Z(n+1)]\big].$$

Theorem 3.

$$g_l(o) = \frac{(1-\lambda^2)^l}{1-\lambda^{2l}}\big[Ey(n)H_l[Z(n)] - \lambda^l Ey(n)H_l[Z(n-1)]\big],$$

$$g_l(j) = \frac{(1-\lambda^2)^l}{1-\lambda^{2l}}\big[(1+\lambda^{2l})Ey(n)H_l[Z(n-j)] -$$

$$-\lambda^l[Ey(n)H_l[Z(n-j+1)] + Ey(n)H_l[Z(n-j-1)]\big].$$

We approximate the variance of obtained estimator

$$\hat{g}_l(j) = \frac{(1-\lambda^2)^l}{1-\lambda^{2l}}\left[(1+\lambda^{2l})\,\overline{y(n)}\,\overline{H_l(Z_{n-j})} - \lambda^l\left[\overline{y_n H_l(Z_{n-j+1})} + \overline{y_n H_l(Z_n)}\right]\right]$$

by the following

$$\hat{\sigma}_{g'_l(j)} = \frac{(1-\lambda^2)^l}{1-\lambda^{2l}}\left[(1+\lambda^{2l})\sqrt{\frac{1}{T-j}\sum_{k=j}^{T}[y_k H_l(Z_{k-j}) - \overline{y_n H_l(Z_{n-j})}]^2} +\right]$$

$$+\lambda^l\left[\sqrt{\frac{1}{T-j+1}\sum_{k=j+1}^{T}[y_k H_l(Z_{k-j+1}) - \overline{y_n H_l(Z_{n-j+1})}]^2} +\right.$$

$$\left. + \sqrt{\frac{1}{T-j-1}\sum_{k=j+1}^{T}[y_n H_l(Z_{k-j-1}) - \overline{y_n H_l(Z_{n-j-1})}]^2}\right].$$

4. *Identification of the continuous and discrete Uryson systems using $AR(2)$ input processes*

In this case for the identification of continuous Uryson model the input is a second order autoregressive process with equation

$$\ddot{Z}(t) = \alpha\dot{Z}(t) + \beta Z(t) + e(t)$$

where $e(t)$ is a white noise, i.e. its autocorrelation function is

$$R_{zz}(s) = \frac{1}{\lambda_2 - \lambda_1}[\lambda_2 e^{-\lambda_1|s|} - \lambda_1 e^{-\lambda_2|s|}]$$

where λ_1, λ_2 are two different roots of the equation $x^2 - \alpha x - \beta = 0$. Let us denote its Fourier transform by

$$\varphi(\omega) = \frac{1}{\lambda_1 - \lambda_2}\left[\lambda_1 \frac{\lambda_2}{\omega^2 + \lambda_2} - \lambda_2 \frac{\lambda_1}{\omega^2 + \lambda_1^2}\right] = \frac{\alpha\beta}{\omega^4 + (\alpha^2 - 2\beta)\omega^2 + \beta^2}$$

and consider for example the Fourier transform of the autocorrelation function ϱ of H_2

$$\varrho_2(\tau) = cov\left[H_2[\tau(t)], H_2[2(t)]\right] = 2R_{xx}^2(\tau)$$

which has the next form

$$\varphi_2(\omega) = 2\frac{-2\alpha\beta\omega^2 - 8\alpha\beta(\beta+\alpha^2)}{\omega^6 + (5\alpha^2 - 8\beta)\omega^4 + 4(\alpha^4 - 2\alpha^2\beta + 4\beta^2)\omega^2 + 16\alpha^2\beta^2}$$

We can determine analogously the Fourier transform of autocorrelation function

$$\varrho_3(\tau) = 6R_{ZZ}^3(\tau).$$

If the Uryson model has the next form

$$Y(t) = \sum_{k=1}^{\infty}\int_0^{\infty} g_n(s)H_n[Z(t-s)]\,ds + \zeta(t)$$

then the first order cross correlation function is

$$EY(t)H_1[Z(t-\tau)] = \int_0^{\infty} g_1(s)k(\tau-s)\,ds = R_{YH_1}(\tau)$$

and its Fourier transform is

$$F_{YH_1}(\omega) = \hat{g}_1(\omega)\varphi(\omega)$$

From here

$$g_1(s) = \frac{1}{\varphi(D)}R_{YH_1}(s)$$

where D denotes the $\frac{d}{ds}$ operator and hence

$$g_1(s) = \frac{1}{\alpha\beta}\left(R_{yH_1}^{(i)}(s) + (\alpha^2 - 2\beta)R_{yH_1}^{(2)}(s) + \beta^2 R_{yH_1}(s)\right)$$

We cannot get explicit expression unfortunately, for the determination of the weighting function $g_2(s)$. We have to solve the following differential equation

$$-2\alpha\beta g_2^{(2)}(s) - 8\alpha\beta(\beta+\alpha^2)g_2(s) =$$
$$= R_{yH_1}^{(6)}(s) + (5\alpha^2 - 8\beta)R_{yH_1}^{(4)}(s) + 4(\alpha^4 - 2\alpha^2\beta + 4\beta^2)R_{yH_1}^{(2)}(s) + 16\alpha^2\beta^2 R_{yH_1}(s)$$

Therefore for this case and for higher kernels as well we may apply the spectral factorization on the basis of the orthogonality of Hermite polynomials.

Thus if we compute the Fourier transform of

$$R_{ZZ}^n(s) \quad \text{i.e.} \quad F\left(\varrho^n(s)\right) = F_n(\omega)$$

then with the factorization $F_n(\omega)$ we get that

$$F_n(\omega) = F_n^+(j\omega)F_n^-(j\omega)$$

Using this result and calculating the cross spectral density function

$$F[R_{yH_n}(s)] = F_{yH_n}(j\omega)$$

for fixed n we can use the linear spectral factorization procedure [5] that leads to the determination of the transferfunction of Uryson model $F[g_n(s)] = G_n(j\omega)$.

In the case of the discrete Uryson system we can the above computation in entirely analogous way using the spectral factorization method through the discrete Laplace or Z transformation.

Now let us try by the $AR[2]$ the identification of the discrete Uryson system y_t analogously to the identification of Uryson system with $AR[1]$ input where

$$y_t = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g_n(k) H_n(Z_{t-k}) + \zeta(t)$$

Here the difference equation for the $AR[2]$ input process is

$$Z_t = \beta_1 Z_{t-1} + \beta_2 Z_{t-2} + u_t$$

where $u_t \in N[0, 1]$ is a discrete white noise series. Let the roots of the equation

$$w^2 - \beta_1 w - \beta_2 = 0$$

be different and real numbers w_1, w_2. Then

$$cov(Z_t, Z_s) = \Gamma_{|t-s|} = \frac{1}{(w_1 - w_2)(1 - w_1 w_2)} \left(\frac{w_1^{|t-s|+1}}{1 - w_1^2} - \frac{w_2^{|t-s|+1}}{1 - w_2^2} \right)$$

and so

$$cov[H_l(Z_t) H_l(Z_s)] = \Gamma^l_{|t-s|} = \frac{1}{(w_1 - w_2)^l (1 - w_1 w_2)^l} \left(\frac{w_1^{|t-s|+1}}{1 - w_1^2} - \frac{w_2^{|t-s|+1}}{1 - w_2^2} \right)$$

In this case the Fourier transformations of the crosscorrelation function

$$E y_t H_l(Z_{t-k}) = \frac{1}{(w_1 - w_2)^l (1 - w_1 w_2)^l} \sum_{s=0}^{l} \binom{l}{s} \frac{1}{(1 - w_1^2)^l (1 - w_2^2)^{l-s}}$$

$$\sum_{j=0}^{\infty} g_l(j) w_1^{(|j-k|+1)s} (-w_2)^{(|j-k|+1)(l-s)}$$

is given by relationship

$$k_l(\omega) = \frac{1}{2\pi} \frac{1}{(w_1 - w_2)^l (1 - w_1 w_2)^l} \sum_{s=0}^{l} \binom{l}{s} \frac{w_1^s w_2^{l-s}}{(1 - w_1^2)^s (1 - w_2^2)^{l-s}} \times$$

$$\times \sum_{j=0}^{\infty} g_l(j) \sum_{k=-\infty}^{\infty} e^{-i\omega k} (w_1^s w_2^{l-s})^{|k-j|} =$$

$$= \frac{1}{2\pi} \frac{1}{(w_1 - w_2)^l (1 - w_1 w_2)^l} \sum_{s=0}^{l} \binom{l}{s} \frac{w_1^s w_2^{l-s}}{(1 - w_1^2)^s (1 - w_2^2)^{l-s}} \frac{1}{|1 - w_1^s w_2^{l-s} e^{-i\omega}|^2} G(\omega)$$

It is clear that from here we can get result for weighting function $g_n(j)$ in a rather complicated way.

Therefore to avoid the "difficult" spectralfactorization procedure and the above complicated computation the authors constructed a so called semiparametric identification method of identification for not only the Uryson but even for the nonlinear system represented by Zadeh functional series using autoregressive input process (3).

The first few Wiener G-functionals are

$$G_0[g_0;\ x(t)] = g_0,$$

$$G_1[g_1;\ x(t)] = \int_0^\infty g_1(s)x(t-s)\,ds,$$

$$G_2[g_2;\ x(t)] = \iint_0^\infty g_2(s_1, s_2)x(t-s_1)x(t-s_2)ds_1 ds_2 - \int_0^\infty g_2(s, s)\,ds,$$

$$G_3[g_3;\ x(t)] = \int_0^\infty \int_0^\infty \int_0^\infty g_3(s_1, s_2, s_3)x(t-s_1)x(t-s_2)x(t-s_3)\,ds_1 ds_2 ds_3 - \\ -3\int_0^\infty \int_0^\infty g_3(s_1, s_2, s_3)x(t-s_1)\,ds_1 ds.$$

Now let us first consider the n-variable Appel polynomial system (8) for the identification of the continuous Volterra-model:

1. $$A_0 = 1,$$

2. $$A_n(x_1, x_2, \ldots, x_n)$$

is an n-variable symmetric polynomial of degree n in the variables $x_1, \ldots, x_n$;

3. $$\frac{\partial}{\partial x_n} A_n(x_1, x_2, \ldots, x_n) = A_{n-1}(x_1, x_2, \ldots, x_{n-1}),$$

4. $$EA_n(x_1, x_2, \ldots, x_n) = 0,$$

where $\underline{x}=(x_1, x_2, \ldots, x_n)$ is a vector-valued random variable with Gaussian distribution and $E\underline{x}=\underline{0}$.

If we denote the covariance of the variables x_i and x_j by $R_{x_i x_j}$ then for the Appel polynomials $A_n(\underline{x})$ the following recursive formulas hold true i.e. A_n can be calculated by formula:

$$A_n(x_1, \ldots, x_n) = x_n A_{n-1}(x_1, \ldots, x_{n-1}) - \sum_{i=1}^{n-1} Rx_i x_n A_{n-2}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}).$$

For the second order moments of the Appel polynomial system (in the case of the Gaussian distribution of the variables $x_1, \ldots, x_n, z_1, \ldots, z_n$) we get

$$EA_n(x_1, x_2, \ldots, x_n)A_m(z_1, \ldots, z_m) = \delta_{nm} \sum_{n!}{}^{*} \prod_{l=1}^{n} Rx_l z_{i_l},$$

where the summation $\sum^*$ have to extend for all possible permutation $i_1, i_2, \ldots, i_n$ of numbers $1, 2, \ldots, n$ (8).

5. *Identification of continuous Volterra models using autoregressive input processes*

Now let us consider the identification of the nonlinear systems represented by the Volterra functional series in the form

$$y(t) = \sum_i \underbrace{\int_0^\infty \dots \int_0^\infty}_{i} g^*(s_1, \dots, s_i) k(t-s_1) \dots k(t-s_i)\, ds_1 \dots ds_i + \zeta(t)$$

where $\zeta(t)$ additive noise is a stationary random process independent from the input process $x(t)$. This Volterra model can be described by the help of the well-known Wiener G-functionals and using the multivariable Appel polynomial system.

In the Wiener theory the output $y(t)$, of the Volterra nonlinear system for the input $x(t)$ is expressed by the series

$$y(t) = \sum_{n=0}^{\infty} G_n[g_n, x(t)]$$

in which the Wiener G-functional (5) have the orthogonal property i.e.

$$E\{G_m[g_m;\ x(t)]\, G_n[g_n, x(t)]\} = \delta_{mn}$$

Naturally the Wiener G-functionals and the more general multivariable Appel polynomial system are close relatives. If the input is a Gaussian white noise process then the Volterra series represented by G-functional can be obtained using the Appel polynomial form too i.e.

$$y(t) = \sum_n G[g_n, x(t)] = \sum_i \int_0^\infty \dots \int_0^\infty g^*(s_1, \dots, s_i) x(t-s_1) \dots x(t-s_i)\, ds_1 \dots ds_i + \zeta(t) =$$

$$= \sum_n \int \overset{\infty}{\underset{0}{\dots}} \int g_n(s_1 \dots s_n) A_{11} \dots {}_1[x(t-s_1) \dots x(t-s_n)]\, ds_1 \dots ds_n + \zeta(t)$$

There is an analogous connection between the Appel polynomial system representation of Volterra model and the Lee—Schehen $L_n(\cdot)$ functionals when the input is a nonwhite Gaussian process. The Wiener—Hopf type multivariable integral equations for optimal estimation of Wiener—Uryson and Volterra series we get from the minimalization of

$$E[y(t) - \sum_i \int_0^\infty \dots \int_0^\infty g(s_1, \dots, s_i) A_{11-1}[x(t-s_1), \dots, x(t-xs_i)]\, ds_1 \dots ds_i]^2$$

i.e.

$$Ry_{A_{1\dots 1}}(u_1, \dots, u_i) = \int_0^\infty \dots \int_0^\infty g(s_1, \dots, s_i) R_{xx}(u_1-s_1) \dots R_{xx}(u_i-s_i)\, ds_1 \dots ds_i$$

$$i = 1, 2, \dots \quad u_1 \dots u_i \geqq 0$$

The solution of this integral equations can be performed by multivariable spectral factorization procedure given e.g. by Lee and Schelzen [9].

If the input is a continuous first order autoregressive process (a coloured noise) with autocorrelation and spectral density functions

$$R_x(s) = e^{-a|s|}, \quad F_x(\omega) = \frac{a}{a^2+\omega^2}$$

then analogously to the identification of continuous Uryson systems we can avoid the spectral factorization procedure.

We get the following estimation of Wiener—Uryson model analogously to the identification of Uryson systems see in Section 2.

$$\prod_{i=1}^{n}\left(1+\frac{1}{a}\frac{\partial}{\partial s_i}\right)\left(1-\frac{1}{a}\frac{\partial}{\partial s_1}\right) Ry_{A_{11\dots 1}}(s_1, \dots, s_n) =$$

$$= \prod_{i=1}^{n}\left(1-\frac{1}{a^2}\frac{\partial^2}{\partial s_1^2}\right) Ry_{A_{1\dots 1}}(s_1 \dots s_n) = g(s_1, \dots, s_n)$$

This result we may extend in an analogous way for the case of the any m-th order autoregressive process i.e.

$$g(s_1, \dots, s_n) = \prod_{i=1}^{n}\left(1-\sum_{l=1}^{m} c_l \frac{\partial^l}{\partial s_i^l}\right)\left(1+\sum_{l=1}^{m} c_l \frac{\partial^l}{\partial s_i^l}\right) Ry_{A_{1\dots 1}}(s_1, \dots, s_n)$$

where c_l, $l=1, 2, \dots, m$ are the appropriate parametres of $AR(m)$ process.

For example if the input is an $AR(2)$ process then for estimation of the Wiener kernels we get that

$$g(s_1, \dots, s_n) = \prod_{i=1}^{n}\left(1-\lambda_1\frac{\partial}{\partial s_i}-\lambda_1\lambda_2\frac{\partial^2}{\partial s_i^2}\right)\left(1+\lambda_1\frac{\partial}{\partial s_i}+\lambda_1\lambda_2\frac{\partial^2}{\partial s_i^2}\right)\overline{R_{yA}(s_1, \dots, s_n)}$$

6. *Estimation of Wiener kernels of discrete Volterra model with first order Gaussian autoregressive input*

Let us determine the estimation of Wiener kernels of Volterra model when the input is a discrete $AR(1)$ process. In this case the equation of the Volterra system with Wiener kernels is

$$y_n = \sum_{r=0}\sum_{k\geq 0} g(k_1, k_2, \dots, k_r) A_{11\dots 1}(z_{n-k_1}, \dots, z_{n-k_r}) + \zeta(t)$$

where $A_{11\dots 1}$, is the multivariable Appel polynomial.

Let

$$y_r^{(*)} = \sum_{k\geq 0} g(k_1, \dots, k_r) A_{11\dots 1}(z_{n-k_1}, \dots, z_{n-k_r})$$

be an output process with cross correlation function

$$R_{y_r^{(*)}A_{1\dots 1}} = \sum_{k\geq 0} g(k_1, \dots, k_r) \sum_{r!} \frac{\lambda^{\sum_{t=1}^{r}|k_t-j_{s_t}|}}{r!(1-\lambda^2)^r}$$

where $s_1, \ldots, s_r$ is a permutation of $1, 2, \ldots, r$ and so its Fourier transformation is

$$K(\omega_1, \omega_2, \ldots, \omega_r) = \left[\sum_{r!} G(\omega_{s_1}, \ldots, \omega_{s_r})\right] \frac{1}{\prod_{k=1}^{r} |1-\lambda e^{-i\omega_k}|^2}$$

where the summation $\sum_{r!}$ has to extend for all possible permutation $s_1, \ldots, s_r$ of the number $1, \ldots, r$.

Then it is obviously, that

$$K(\omega_1, \omega_2, \ldots, \omega_r) = \sum_{j_1 \ldots j_r = -\infty}^{\infty} Ey_{(r)}^{(*)} A_{11\cdots 1}(Z_{n-j}, \ldots, Z_{n-j_r}) e^{-i \sum_{k=1}^{r} jk^{\omega}}$$

furthermore

$$G(\omega_{s_1}, \ldots, \omega_{s_r}) = \sum_{k_1 \ldots k_r \geq 0} g(k_1 \ldots k_r) \exp\left(-i \sum_{t=1}^{r} k_t \omega_{s_t}\right)$$

Let us define the shift operator by the following way

$$\mathscr{L}_i^k A(Z_{n-j_1}, \ldots, Z_{n-j_i}, \ldots, Z_{n-j_r}) = A(Z_{n-j_1} \ldots Z_{n-j_i-k}, \ldots Z_{n-j_r})$$

Assuming that the kernels satisfy the condition

$$g(k) = g(Pk)$$

where P is an arbitrary permutation we get for the Wiener kernels that

$$g(k) = \prod_{i=1}^{r} (1-\lambda\alpha_i)(1-\lambda\alpha_i^{-1}) Ey_n A_{\underbrace{1 \ldots 1}_{r}}(Z_{n-k_1}, \ldots, Z_{n-k_r})$$

Now if we assume that the system is Uryson one i.e.

$$g(k_1, k_2, \ldots, k_r) = \prod_{j=2}^{r} \delta_{k_j}^{k_1} g_r(k_1)$$

then for example for the estimation of the weighting function $g_2(k)$ the next formula holds

$$\begin{aligned} g_2(k) = {} & (1+\lambda^2)^2 Ey_n A_2(Z_{n-k}) - 2\lambda(1+\lambda^2)[Ey_n A_{11}(Z_{n-k-1}, Z_{n-k}) + \\ & + Ey_n A_{11}(Z_{n-k+1}, Z_{n-k})] + \lambda^2 [Ey_n A_2(Z_{n-k-1}) + Ey_n A_2(Z_{n-k+1}) + \\ & + 2Ey_n A_{11}(Z_{n-k-1}, Z_{n-k+1})]. \end{aligned}$$

Analogously e.g. for kernels $g_{2,1}(k, l)$ of Zadeh model (see later) the following equality holds

$$g_{2,1}(k, l) = g(k, k, l) = \prod_{i=1}^{3} (1-\lambda\mathscr{L}_i)(1-\lambda\mathscr{L}_1^{-1}) Ey_n A_{1,11}(Z_{n-k}, Z_{n-k}, Z_{n-l}).$$

The above result we shell extend for any pth order autoregressive process. If the stationary input series has an spectral density function $s(e^{i\omega})$. The cross correlation function is

$$Ey_t A_{1\dots1}(Z_{t-j}, \dots, Z_{t-j}) = \sum_{\underline{k}\geq 0} g(\underline{k}) \operatorname{cov}\left(A_{1\dots1}(\underline{k}), A_{1\dots1}(\underline{j})\right) =$$

$$= \sum_{\underline{k}\geq 0} g(\underline{k}) \sum_{r!} \frac{1}{r!} \prod_{s=1}^{r} \Gamma_{|k_{i_s}-j_s|}$$

where

$$\Gamma_{|k-j|} = \operatorname{cov}(Z_{t-k}, Z_{t-j})$$

and the cross spectral density function of the autocorrelation function is

$$K(\omega_1, \dots, \omega_r) = \frac{1}{(2\pi)^r} \sum_{-\infty}^{\infty} e^{-i(\omega_1 j_1 + \cdots \omega_r j_r)} Ey_t A_{1\dots1}(Z_{t-j_1}, \dots, Z_{t-j_r}) =$$

$$= \frac{1}{(2\pi)^r} \sum_{k\geq 0} \frac{1}{r!} g(\underline{k}) \sum_{r!} \prod_{s=1}^{r} \sum_{-\infty}^{\infty} e^{-i\omega_s j_s} \mathfrak{T}_{|k_{i_s}-j_s|} =$$

$$= \sum_{r!} \sum_{\underline{k}\geq 0} g(\underline{k}) e^{-i\sum_{s=1}^{r} \omega_{i_s} k_{i_s}} \prod_{s} \underline{s}(e^{i\omega_s})$$

from where because of the symmetricity of $g(\underline{k})$ we get that

$$G(\omega_1, \dots, \omega_r) = K(\omega_1, \dots \omega_r) \Big/ \prod_{s=1}^{r} s(e^{i\omega})$$

where

$$G(\omega_1, \dots, \omega_r) = \sum_{k\geq 0} g(k) e^{-i\sum_{1}^{r} \omega_s k_s}$$

is the multivariable transfer function of the Volterra system. This formula is very similar to the Shiryaev formula but we use the Appel polynomials instead of the input products.

The Shiryaev formula can be used only for the pure rth order Volterra model though we do not take this restriction. From this formula we may get the following special cases.

1. If the Z_t is an $AR(1)$ process than naturally

$$\operatorname{cov}(Z_t, Z_s) = \Gamma_{|t-s|} = \frac{\lambda^{|t-s|}}{1-\lambda^2}$$

$$S(e^{i\omega}) = (2\pi)^{-1} |1-\lambda e^{-i\omega}|^{-2}$$

and so the transfer function is

$$G(\omega_1, \dots, \omega_r) = (2\pi)^r \prod_{s=1}^{r} |1-\lambda e^{-i\omega_s}|^2 K(\omega_1, \dots, \omega_r)$$

and if we substitute $e^{i\omega_s}$ with shift operator L_s we get again that

$$g(k) = \prod_{s=1}^{r} (1-\lambda L_s)(1-L_s^{-1}) E y_t A_1, \dots 1(Z_{t-k_1}, \dots, Z_{t-k_r})$$

2. If Z_t is an $AR(2)$ process i.e.

$$Z_t = \beta_1 Z_{t-1} + \beta_2 Z_{t-2} + u_t$$

where $u_t \in N(0, 1)$ is discrete white noise and its spectral density function is

$$S(e^{i\omega}) = (2\pi)^{-1} |1-\beta_1 e^{-i\omega} - \beta_2 e^{-2i\omega}|^{-2}$$

$$G(\omega_1, \dots, \omega_r) = (2\pi)^r \prod_{s=1}^{r} |1-\beta_1 e^{-i\omega} - \beta_2 e^{-2i\omega}|^2 K(\omega_1 \dots \omega_r)$$

from where for the determination of appropriate kernels we get the next relatively simple expression

$$g(k) = \prod_{s=1}^{r} (1-\beta_1 L_s - \beta_2 L_s^2)(1-\beta_1 L_s^{-1} - \beta_2 L_s^{-2}) E y_t A_{1 \dots 1}(Z_{t-k_1}, \dots, Z_{t-k_r})$$

3. Z_t is a pth order Gaussian autoregressive process i.e.

$$Z_t = \sum_{l=1}^{p} \beta_l Z_{t-l} + e_t$$

where $e_t \in N(0, 1)$ discrete white noise series. Denote

$$S(e^{i\omega}) = (2\pi)^{-1} \left|1 - \sum_{l=1}^{p} \beta_l e^{-i\omega l}\right|^{-2}$$

then

$$G(\omega_1, \dots, \omega_r) = (2\pi)^r \prod_{s=1}^{r} \left|1 - \sum_{s=1}^{r} \beta_l e^{-i\omega e}\right|^2 K(\omega_1 \dots \omega_r)$$

as well as

$$g(k) = \prod_{s=1}^{r} (1 - \sum_{s=1}^{p} \beta_l l_s^l)(1 - \sum_{1}^{r} \beta_l l_s^{-e}) E y_t A_{1,\dots 1}(Z_{t-k_1}, \dots, Z_{t-k_r})$$

$$A_{1, \dots 1}(Z_{t-k_1}, \dots, Z_{t-k}).$$

7. *On the relationship between the identification of Volterra and Zadeh nonlinear models*

In our earlier paper using autoregressive input we discussed in detail the identification of nonlinear systems represented by Zadeh functional series.

In this part of this paper we analyse the relationship between the estimation of Rajbman kernels of Zadeh nonlinear system and the Wiener kernels of Volterra nonlinear model when the input is autoregressive Gaussian input process. It is well-known that the nonlinear system represented by Zadeh functional series is defined by equation

$$y(t) = \int_0^\infty u_0(s)ds + \sum_{i=1}^{n} \int_0^\infty \dots \int_0^\infty u_i[x(t-s_1), \dots, x(t-s_i), s_1, \dots, s_i] ds_1 \dots ds_i + \xi(t).$$

Here the Zadeh kernels $u_i(x_1, \dots, x_i, s_1, \dots, s_i)$ will be considered as analytical i.e.

$$u_i(x_1, \dots, x_i, s_1 \dots s_i) = \sum_{\underline{k} \geqq 1} a_{\underline{k}}(s_1, s_2, \dots, s_i) \underline{x}^k,$$

$$\text{where} \quad \underline{x}^k = \prod_{l=1}^{i} x_l^{k_l} \quad \text{moreover}$$

$$\sum_{\underline{k} \geqq 1} \Big(\prod_{l=1}^{i} k_l \Big) a_{\underline{k}}^2(s_1, \dots, s_i) < \infty$$

as well as the additive noise $\xi(t)$ is independent of input $x(t)$ and $E\xi(t)=0$.

Here the kernels $a_{\underline{k}}(s_1, \dots, s_i)$ are called Rajbman kernels. For the identification of above kernels we assume that the following equations hold true

(i) $$a_k(s_1, s_2, \dots, s_i) = a_{1_l}, (\underbrace{s_1, \dots, s_1}_{k_1}; \underbrace{s_2, \dots, s_2}_{k_2}; \dots; \underbrace{s_i, \dots, s_i}_{k_i})$$

$$\text{where} \;\; \underline{k} = (k_1, \dots, k_i), \, l = \sum_{i=1}^{i} k_j, \, D_l = (\underbrace{11 \dots 1}_{l}) \;\; \text{and} \;\; \prod_{r=p} (s_r - s_p) \neq 0,$$

(ii) $$a_k(s_1, \dots, s_i) = a_{\underline{p}k}(\underline{p}_s).$$

Here P is an arbitrary permutation of the elements.

Because of the weights $a_{\underline{k}}$ are the coefficients of $\underline{x}^{\underline{k}}$ thus the above conditions do not cause any loss of the generality.

The presented results for the Appel polynomials (in Section 4 of this paper) hold also in the case when the variables are not different. Thus, we could get the uni- and bivariables Appel polynomial systems as special cases [8]. For example, the bivariable Appel polynomial $A_{k,l}(x, y)$ $(k+l)$-variable Appel polynomial, i.e.

$$A_{k,l}(x, y) = A_{k+l}(\underbrace{x, x, \dots, x}_{k}, \underbrace{y, y, \dots, y}_{l}).$$

As a generalization of this we introduce the following symbol

$$A_{k_1, k_2, \dots, k_s}(x_1, \dots, x_s) = A_{\sum_{i=1}^{s} k_i} (\underbrace{x_1, \dots, x_1}_{k_1}, \dots, \underbrace{x_s, \dots, x_s}_{k_s}).$$

For the second order moments of Appel polynomials we get

$$EA_{k_1, k_2, \dots, k_s}(x_1, \dots, x_s) A_{l_1, \dots l_p}(y_1, \dots, y_p) =$$

$$= \delta_{k_n+\dots+k_s}^{l_1+\dots+\tau_p} \prod_{i=1}^{p} l_i! \sum_{\left\{\substack{\sum_{t=1}^{p} j_t^i = k_i \\ i=1,2,\dots,s-1}\right\}} \binom{k_s}{m_1, m_2, \dots, m_p} \prod_{i=1}^{p} C_{\alpha_s y_i}^{m_i} \prod_{i=1}^{s-1} \binom{k_i}{j_1^i, \dots, j_p^i} \prod_{i=1}^{s=1} \prod_{t=1}^{p} C_{x_i y_t}^{j_t^i}$$

$$\text{where} \quad m_t = l_t - \sum_{i=1}^{s=1} j_t^i \quad \text{and} \quad \binom{k}{l_1, \dots, l_t} = \frac{k!}{\prod_{i=1}^{t} l_i!}.$$

Using the multivariable Appel polynomials it can be seen that the equation of analytic Zadeh nonlinear system has the following equivalent form, i.e.

$$y(t) = \int_0^\infty u_0(s) ds + \sum_{i=1}^{\infty} \sum_{\underline{k} \geq 1} \int_0^\infty \dots \int_0^\infty a_k^*(s_1, \dots, s_i) \prod_{l=1}^{i} x^{k_l}(t-s_l)\, ds_1, \dots, ds_i + \xi(t) =$$

$$= \int_0^\infty u_0(s)\, ds + \sum_{i=1}^{\infty} \sum_{\underline{k} \geq 1} \underbrace{\int_0^\infty \dots \int_0^\infty}_{i} a_k(s_1, \dots, s_i) A_{\underline{k}}[x(t-s_1), \dots, x(t-s_i)] ds_1 \dots ds_i + \xi(t).$$

For the renstraction of relationship between Zadeh and Volterra nonlinear models let us consider the identification of the following fourth order Zadeh model

$$Y(t) = y^{(1)}(t) + y^{(2)}(t) + y^{(3)}(t) + y^{(4)}(t) + \xi(t)$$

where the "pure" members are

$$y^{(1)}(t) = \int_0^\infty g_1(s) A_1(Z(t-s)) ds,$$

$$y^{(2)}(t) = \iint_0^\infty g_{11}(s_1, s_2) A_{11}(Z(t-s_1), Z(t-s_2)) ds + \int_0^\infty g_2(s) A_2(Z(t-s))\, ds,$$

$$y^{(3)}(t) \cdot \iiint_0^\infty g_{111}(\underline{s}) A_{111}(Z(t-s_1), Z(t-s_2) Z(t-s_3)) d\underline{s} +$$

$$+ \iint_0^\infty g_{12}(s_1, s_2) A_{12}(Z(t-s_1), Z(t-s_2) ds_1\, ds_2 + \int_0^\infty g_3(s) A_3(Z(t-s)) d\underline{s}$$

$$y^{(4)}(t) = \iiiint_0^\infty g_{1111}(s) A_{1111}(Z(t-s_1), \dots Z(t-s_4)) d\underline{s} + \iiint_0^\infty g_{112}(\underline{s}) A_{12}(Z(t-s_1) \dots$$

$$\dots Z(t-s_3)) d\underline{s} + \iint_0^\infty g_{22}(s_1, s_2) A_{22}(Z(t-s_1), Z(t-s_2) d\underline{s} + \int_0^\infty g_4(s) A_4(Z(t-s)) ds$$

Here the input is a first order Gaussian process with zero expectation and the autocorrelation $R_{zz}(u)=e^{-\alpha u}$. Let us see first the estimation of g_1.

Let us begin with the

$$R_{y_1A_2}(u) = \int_0^\infty g_1(s)R_{A_1A_2}(u-s)ds = \int_0^\infty g_1(s)e^{-\alpha|u-s|}ds$$

the Fourier transformation of which is

$$F_{y_1A_1}(\omega) = \hat{g}_1(\omega)F_{A_1A_1}(\omega) = \hat{g}_1(\omega)\frac{1}{\pi}\frac{\alpha}{\alpha^2+\omega^2}$$

from where

$$\frac{\pi}{\alpha}(\alpha^2+\omega^2)R_{y_1A_1}(\omega) = \hat{g}_1(\omega)$$

Applying the inverse Fourier transform we get

$$g_1(\omega) = \frac{\pi}{\alpha}(\alpha^2+D^2)R_{yA_1}(u)$$

where D denotes the differential operator d/du i.e.

$$g_1(u) = \frac{\pi}{\alpha}\left(\alpha^2R_{yA_1}(u)+\ddot{R}_{yA_1}(u)\right)$$

We obtain similarly to the above case for the "pure" fourth order member the following results

$$y^{(4)}(t) = \iiiint_0^\infty A_{1111}(\underline{s})[g_{1111}(\underline{s})+\sum_{i\neq j}\delta_{s_i-s_j}g_{1111}(\underline{s})+\sum_{i\neq j\neq k}\delta_{s_i-s_j}\delta_{s_i-s_j}g_{1111}(\underline{s})+$$

$$+\delta_{s_1-s_2}\delta_{s_3-s_4}\delta_{s_1-s_3}g_{1111}(\underline{s})]\,ds = \sum_{i\neq j\neq k}\delta_{s_i-s_j}\delta_{s_k-s_l}g_{1111}(\underline{s})+\iiiint_0^\infty A_{1111}(\underline{s})\,g_{1111}(\underline{s})\,Q(\underline{s})d\underline{s}$$

On the basis of which

$$R_{YA_{1111}}(u) = \iiiint_0^\infty R_{A_{1111}A_{1111}}(\underline{u}-\underline{s})g_{1111}(\underline{s})Q(\underline{s})d\underline{s} =$$

$$= \iiiint_0^\infty \sum_{(i_1i_2i_3i_4)} R_{zz}(u_1-s_{i_1})R_{zz}(u_2-s_{i_2})R_{zz}(u_3-s_{i_3})R_{zz}(u_4-s_{i_4})\times g_{1111}(\underline{s})Q(\underline{s})d\underline{s}$$

and

$$R_{YA_{1111}}(\underline{\omega}) = 24\prod_1^4 R_{zz}(\omega_i)[g_{1111}Q](\underline{\omega}) = [P(\underline{\omega})]^{-1}[g_{1111}Q]^i(\underline{\omega})$$

where $P(\underline{\omega})$ polinom is

$$P(\underline{\omega}) = \frac{\pi^4}{24\alpha^4} \prod_{i=1}^{4} (\alpha^2+\omega_i^2) \quad \text{and} \quad D_i = \frac{\partial}{\partial u_i}.$$

In this case

$$P(D_1, D_2, D_3, D_4)\, C_{YA_{1111}}(\underline{u}) = g_{1111}(\underline{s})Q(\underline{s})$$

if $\pi(u_i - u_j) \neq 0$ as well as

$$P(D)\, C_{YA_{1111}}(u) = g_{1111}(u)$$

In the special case when considering only the g_{zz} and g_4 kernels we get, that

$$g_{zz}(u_1, u_2) = \sum_{j\neq k} \iint_0^\infty P(D) C_{YA_{1111}}(y) du\, ; du_k$$

and

$$R_{YA_4}(u) = \iint_0^\infty g_{zz}(\underline{s}) R_{A_{zz}} A_n(u-\underline{s}) d\underline{s} + \int_0^\infty g_n(s) R^4_{z,z}(u-s) ds =$$

$$= G_{z,x}(u) + \int_0^\infty g_4(s) R^4_{z,z}(u-s) ds$$

$$C_{YA_4}(\omega) = \hat{G}_{z,z}(\omega) + \hat{g}_4(\omega) P_4^{-1}(\omega)$$

where

$$P_4(\omega) = \frac{\pi}{4\alpha}(16\alpha^2 + \omega^2)$$

from where

$$G_4(u) = P_4(D) R_{YA_4}(u) - P_4(D) \iint_0^\infty \sum_{j\neq k} \iint_0^\infty P(D) R_{YA_{1111}}(\underline{u}) du,\, du_n \times$$

$$\times R_{A_{zz}A_n}(u-u_n, u-u_m) du_n\, du_m$$

We get for third order Rajbman kernels

$$g_{1,2}(u_1, u_2) = \sum_{l=i}^{3} \int_0^\infty P(D_1, D_2, D_3) C_{YA_{111}}(u_1, u_2, u_3) du,$$

and

$$g_3(u) = P_3(D) C_{YA_3}(u) - P_3(D) \iint_0^\infty \sum_{i=1}^{3} \int_0^\infty P(D_1, D_2, D_3) C_{YA_{111}}(u) du$$

$$\times c_{A_{12}A_3}(u-u_j, u-u_k) du_j\, du_k$$

Finally the second order kernels can be determine by formulas

$$g_{11}(u_1, u_2) = P(D_1, D_2)C_{YA_{11}}(u_1, u_2)$$

$$g_2(u) = P_2(D)C_{YA_2} - P_2(D) \int_0^\infty P(D_1, D_2)C_{YA_{11}}(u_1, u_2)C_{A_{11}A_2}(u-u_1, u-u_2)du.$$

In the above cases the following

$$P_3(\omega) = \frac{\pi}{3\alpha}(9\alpha^2+\omega^2)$$

$$P_2(\omega) = \frac{\pi}{2\alpha}(4\alpha^2+\omega^2)$$

$$P(\omega_1, \omega_2, \omega_3) = \frac{\pi^3}{6\alpha^3}\pi(\alpha^2+\omega_i^2)$$

$$P(\omega_1, \omega_2) = \frac{\pi^2}{4\alpha^2}\pi(\alpha+\omega_i^2)$$

denotes were used remarking that

$$R_{zz}^n(\omega) = \frac{1}{\pi}\frac{n\alpha}{(n\alpha)^2+\omega^2}$$

$$R_{A_{13}A_3}(u_1, u_2) = 6R_{zz}(u_1)R_{zz}^2(u_2)$$

$$R_{A_{22}A_4}(u_1, u_2) = 24R_{zz}^2(u_1)R_{zz}^4(u_2)$$

$$R_{A_{11}A_2}(u_1, u_2) = 2R_{zz}(u_1)R_{xx}(u_2).$$

References

[1] N. RAJBMAN, V. KAPITONENKO, P. VÁRLAKI, Dispersional System Identification, *Nauka, Moscow*, 1981. *(in Russian)*.

[2] N. WIENER, Nonlinear Problems in Random Theory, *MIT Rep., Cambridge and Wiley, New York*, 1958.

[3] P. VÁRLAKI, J. BOKOR, G. TERDIK, Estimation of basic nonlinearity characteristics for identification of second order dynamic systems, *Proc. of JASTED Symp. of Apl. Contr. and Identification Conf.* **1** 8—12—8—17. *Copenhagen* 1983.

[4] P. VÁRLAKI and S. VERES, On measures of nonlinearity for dynamic stochastic systems. *Mathemathical Modelling in science and Technology Proc. of Fourth ICMM*. Zürich (1983), 143—149.

[5] M. SCHETZEN, A theory of non-linear system identification. *Int. J. of Control* **20** (1984), 577—592.

[6] T. W. ANDERSON, The statistical analysis of time series, *J. Wiley and Sons, New York* (1971).

[7] M. ARATÓ, Linear stochastic systems with constant coefficients, *Lecture Notes in Control and Inf. Sci.* **45**, Springer-Verlag (1982).

[8] G. TERDIK, Expectation of nonlinear functions of Gaussian process, *Publ. Math. (Debrecen)* **34** (1987).

[9] Y. LEE, M. SCHETZEN, Measurement of the Wiener kernels of a nonlinear system by cross-correlation, *Int. Journ. of Control* **2** (1965), 237—254.

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