# An integral operator generating solutions of partial differential equations of higher order

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*Abstract*

In this paper we shall introduce an integral operator for representing solutions of the $n$-th order partial differential equation

$$w_{z_1 z_2 \dots z_n} + C(z_1, \dots, z_n) w = 0$$

n $n$ independent complex variables.

## Introduction

Integral operators of various types have been used for a long time in the mathematical literature. Solutions of second order elliptic and hyperbolic differential equations can be represented by the integral operators of B. RIEMANN, I. N. VEKUA and S. BERGMAN to mention the most popular.

In the present paper we shall introduce a new class of integral operators for $n$-th order linear partial differential equations with $n$ independent variables. These integral operators can be seen as a generalisation of the Bergman operators for solutions of second order differential equations with $n=2$ variables.

We shall consider two problems, proceeding as follows. Section 1 is devoted to representations of solutions for equations of the form

$$(1) \qquad Lw = w_{z_1 z_2 \dots z_n} + Cw = 0, \quad w = w(z_1, \dots, z_n), \quad n \geqq 2,$$

where $C$ is a holomorphic function of $z_1, z_2, \dots, z_n$. In order to obtain independent particular solutions of (1), we represent them in the form

$$(2) \quad w(z_1, \dots, z_n) = \int_{\mathcal{L}_j} E_j(z_1, \dots, z_n, t) f_j\left[\frac{z_j}{2}(1-t^2)\right] dt/\sqrt{(1-t^2)}, \quad 1 \leqq j \leqq n.$$

$f_j(z_j)$ is an arbitrary holomorphic function of $z_j$. In Theorem 1 is shown that under certain restrictions on $E_j$, which is a particular solution of a certain partial differential equation which depends only on $L$, (2) is a solution of (1).

Section 2 contains the corresponding proof for the existence of kernels $E_j$ as solutions of a differential equation in the form of a power series in the variable of

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integration with coefficients depending on the independent variables $z_1, z_2, \ldots, z_n$. The convergence of the infinite series is proved by a slight modification of CAUCHY's classical method of dominants. Clearly, this process generally yields local representations of kernels and, hence, of solutions. This is said in Theorem 2.

In the last section we give examples. We shall see that operators (2) are applicable explicitely to many equations (1) for which no representations of solutions have been known previously. This will simultaneously motivate the new representation of solutions to be constructed in this paper.

## 1. Integral operators for the differential equation (1)

For $1 \leqq k \leqq n$, $n \geqq 2$ let $G_k$ be simply connected domains in the complex plane. We consider partial differential equations of the form $Lw=0$ where $z_1, \ldots, z_n$ are independent complex variables and $C$ is holomorphic in the poly domain $G=G_1 \times \ldots \times G_n \subset \mathbf{C}^n$ containing the origin. Let $\mathcal{L}_j$, $j=1, 2, \ldots, n$, oriented rectifiable arcs joining $t=-1$ to $t=1$ and lying in the disk $\mathcal{D}=\{t/|t| \leqq 1\} \subset \mathbf{C}$. Then taking any function $E_j(z_1, \ldots, z_n, t)$, holomorphic in $G \times \mathcal{D}$, and arbitrary functions $f_j(z_j)$ of $F_j=\{f_j/\text{holomorphic in } G_j \ni 0\}$, we can define

$$(3) \quad (P_j f_j)(z_1, \ldots, z_n) = \int_{\mathcal{L}_j} E_j(z_1, \ldots, z_n, t) f_j\left[\frac{z_j}{2}(1-t^2)\right] \frac{dt}{\sqrt{1-t^2}}, \quad j = 1, \ldots, n.$$

Call $P_j$ an integral operator for $Lw=0$ if for all $f_j \in F_j$

$$Lw = LP_j f_j = 0, \quad P_j \neq 0,$$

on $G$. $E_j(z_1, \ldots, z_n, t)$ is denoted as the generating function and $f_j(z_j)$ as the associated function of this operator. $f_j$ is independent of the coefficient $C$ of (1). The operators $P_j$ transform holomorphic functions into (complex) particular solutions of the given partial differential equation (1).

Let us introduce the differential operator

$$D_j := \prod_{\substack{i=1 \\ i \neq j}}^{n} \partial/\partial z_i, \quad 1 \leqq j \leqq n.$$

As a first result, we state the following theorem.

**Theorem 1.** *Let* $E_j(z_1, \ldots, z_n, t)$ *be a solution of*

$$(4) \qquad (1-t^2) D_j(\partial E_j/\partial t) - \frac{1}{t} D_j E_j + 2 z_j t (L E_j) = 0,$$

*such that, for* $z_j \neq 0$,

$$(5) \qquad \frac{\sqrt{1-t^2}}{z_j t} D_j E_j(z_1, \ldots, z_n, t)$$

*is continuous for* $t=0$, *and tends to zero for each* $(z_1, \ldots, z_n)$ *as* $t$ *approaches to* $\pm 1$

*Then, if $f_j$ is any arbitrary holomorphic function, the function $w$ defined by*

$$w = \alpha_1 w_1 + \ldots + \alpha_n w_n \quad \text{where} \quad w_j = P_j f_j \quad (\alpha_j \in \mathbf{C})$$

*is a solution of $Lw=0$ in $G$.*

PROOF. Let $z_j \neq 0$, $1 \leqq j \leqq n$, $j$ arbitrary fixed. Writing for brevity, $f_j$ instead of $f_j\left(\frac{1}{2} z_j(1-t^2)\right)$, we have, upon differentiating (3) and substituting into (1)

$$Lw_j = \int_{\mathscr{L}_j} \{(LE_j)f_j + D_j(E_j)\,\partial f_j/\partial z_j\}\; dt/\sqrt{1-t^2} = 0.$$

Since $\partial f_j/\partial z_j = -(\partial f_j/\partial t)(1-t^2)/2z_j t$, we obtain by integrating by parts

$$Lw_j = \int_{\mathscr{L}_j} \left\{ \frac{LE_j}{\sqrt{1-t^2}} + \left[ \frac{\sqrt{1-t^2}}{2z_j t} D_j E_j \right]_t \right\} f_j\, dt - \left[ \frac{\sqrt{1-t^2}}{2z_j\, t} D_j\,(E_j) f_j \right]\Bigg|_{-1}^{+1} = 0.$$

The desired conclusion now follows from the hypothesis made about (5) and the fact that $f_j$ is an arbitrary holomorphic function of one complex variable in $G_j$ including the origin. Q.E.D.

$w_j = P_j f_j$ is continuously differentiating $n$ times in $z_j = 0$ and so satisfying equation $Lw_j = 0$ also for $z_j = 0, j = 1, 2, \ldots, n$.

Without loss of any generality $E_j$ can be considered as an even function of $t$ as can be seen from (3). So $E_j(z_1, \ldots, z_n, t)$ may be replaced by its even part $[E_j(z_1, \ldots, z_n, t) + E_j(z_1, \ldots, z_n, -t)]/2$ in (3). Hence, instead of (2) resp. (3), consider

$$w_j = 2 \int_{\substack{0 \\ \mathscr{L}_j^*}}^{1} E_j(z_1, \ldots, z_n, t) f_j \left[ \frac{z_j}{2} (1-t^2) \right] dt/\sqrt{1-t^2}.$$

Then, for $z_j \neq 0$, the transformation $\sigma_j = z_j(1-t^2)$, i.e. $t = \sqrt{(z_j - \sigma_j)/z_j}$ carries equation (3) over into

$$w_j = \int_0^{z_j} E_j \left( z_1, \ldots, z_n, \sqrt{\frac{z_j - \sigma_j}{z_j}} \right) f_j(\sigma_j/2)\, d\sigma_j/\sqrt{\sigma_j(z_j - \sigma_j)}.$$

Now, easily can be shown that

$$W_j(z_1, \ldots, z_n, \sigma_j) = E_j \left( z_1, \ldots, z_n, \sqrt{\frac{z_j - \sigma_j}{z_j}} \right) \Big/ \sqrt{\frac{\sigma_j}{z_j - \sigma_j}}$$

is a solution of (1) containing a parameter $\sigma_j$.

At first glance it might appear that the problem of representing solutions of the form $w_j = P_j f_j$ has been made more difficult because we must solve now a partial differential equation in $n+1$ variables instead of $n$. But this is misleading since only one non-vanishing partial solution of (4), which is satisfying (5), is needed to construct infinitely solutions $w_j = P_j f_j$.

## 2. A constructive existence proof for the generating function $E_j$

It had been shown in Section 1, that (3) represents a solution of $Lw_j=0$ if a solution $E_j$ exists to (4).

For obtaining solutions of the generating function equation (4) we construct solutions in the form of infinite series. Their convergence we show by using a modification of CAUCHY's classical method of dominants. In this way we shall obtain local solutions of $Lw=0$, in general.

We seek a solution $E_j$ which possesses the form

$$E_j(z_1, \dots, z_n, t) = 1 + \sum_{k\geqq 1} p_{j;2k}(z_1, \dots, z_n)\, t^{2k}, \quad 1 \leqq j \leqq n \tag{6}$$

for $(z_1, \dots, z_n)\in\hat{G} := \{(z_1, \dots, z_n)/|z_i|\leqq a, a>0, 1\leqq i\leqq n\}\subset G$. The $\{p_{j;2k}\}_{k\in\mathbf{N}}$, $j=1, \dots, n$, are sequences of holomorphic functions in $G$ defined by the recurrence formulas

$$D_j p_{j;2} = -2z_j C$$

$$\frac{2k+1}{2} D_j p_{j;2k+2} = (kD_j - z_j L) p_{j;2k}, \quad j = 1, 2, \dots, n, \; k = 1, 2, \dots. \tag{7}$$

By solving (7) we can choose the arbitr. integration functions to zero. The recurrence formulas are obtained by substituting the series (6) into (4).

*Remark.* If $E_j$ possesses the form (6) the function (5) is continuous for $t=0$. Then $\mathscr{L}_j$ can pass though the origin of the $t$-plane.

**Theorem 2.** *Suppose that the coefficient* $C(z_1, \dots, z_n)$ *of the equation* (1) *is a holomorphic function in the poly domain* $\hat{G}\ni(0, \dots, 0)$. *Then exists a generating function* $E_j(z_1, \dots, z_n, t)$, $j=1, \dots, n, t$, $n\geqq 2$, *according to* (6) *which is holomorphic in the neighbourhood of the origin.*

PROOF. The existence of a kernel $E_j$ which we can calculate by a constructive method is realised if we can show the holomorphy of series (6).

Let us introduce the operators

$$J_j := \prod_{\substack{i=1\\ i\neq j}}^{n} J_i^*, \quad (J_i^* F)(z_1, \dots, z_n) := \int_0^{z_j} F(z_1, \dots, \hat{z}_i, \dots, z_n)\, d\hat{z}_i, \quad 1 \leqq j \leqq n.$$

In addition, we set

$$D_j p_{j;2k}(z_1, \dots, z_n) = z_j^k P_{j;2k}(z_1, \dots, z_n). \tag{8}$$

Then (7) implies

$$P_{j;2} = -2C \tag{9}$$

$$\frac{2k+1}{2} P_{j;2k+2} = \frac{-\partial}{\partial z_j} P_{j;2k} - C\cdot J_j(P_{j;2k}), \quad 1 \leqq j \leqq n, \; k \geqq 1.$$

The arbitrary integration functions we defined by zero. Finally we can start to show the uniformly convergence of series (6).

We can verify that if $C(z_1, \ldots, z_n)$ is holomorphic in the poly disc $\hat{G}$ then in $G^* := \{(z_1, \ldots, z_n)/|z_i| < a,\ a > 0,\ 1 \leqq i \leqq n\}$ holds

$$\left|C(|z_1|, \ldots, |z_n|)\right| \leqq MS(|z_1|, \ldots, |z_n|), \quad S(z_1, \ldots, z_n) = \prod_{i=1}^{n} (1 - z_i/a)^{-1},$$

where $M > 0$ is a suitable choosen constant. Furthermore, we introduce dominants $\tilde{P}_{j;2k}(z_1, \ldots, z_n)$ for $P_{j;2k}(z_1, \ldots, z_n)$, i.e., $\left|P_{j;2k}(|z_1|, \ldots, |z_n|)\right| \leqq \tilde{P}_{j;2k}(|z_1|, \ldots, |z_n|)$, as follows

$$\tilde{P}_{j;2} = 2NS, \quad N \geqq M \tag{10}$$

$$\frac{2k+1}{2} \tilde{P}_{j;2k+2} = \frac{\partial}{\partial z_j} \tilde{P}_{j;2k} + NS \cdot J(\tilde{P}_{j;2k}), \quad k \geqq 1.$$

For proving the convergence of the majorant series $\tilde{E}_j$ for (6)

$$\tilde{E}_j(z_1, \ldots, z_n, t) := 1 + \sum_{k \geqq 1} z_j^k \tilde{P}_{j;2k}(z_1, \ldots, z_n) t^{2k}$$

we set[1])

$$\tilde{P}_{j;2k}(z_1, \ldots, z_n) = \frac{2^{k-1}}{1 \cdot 3 \cdot 5 \ldots (2k-1)} \left(1 - \frac{z_j}{a}\right)^{-k} Q_{j;2k}(z_1, \ldots, \hat{z}_j, \ldots, z_n), \quad k \geqq 1. \tag{11}$$

If we substitute (11) into (10), we find that

$$Q_{j;2} = 2NT_j, \quad T_j(z_1, \ldots, \hat{z}_j, \ldots, z_n) = (1 - z_j/a)\, S(z_1, \ldots, z_n) \tag{12}$$

$$Q_{j;2k+2} = \frac{k}{a} Q_{j;2k} + NT_j \cdot J_j(Q_{j;2k}), \quad k \geqq 1.$$

Furthermore, if we use

$$\left|J_j Q_{j;2k}(|z_1|, \ldots, |\hat{z}_j|, \ldots, |z_n|)\right| \leqq Q_{j;2k}(|z_1|, \ldots, |z_j|, \ldots, |z_n|), \quad k \geqq 1, \tag{13}$$

in the recurrence formula (12) we obtain

$$Q_{j;2}(|z_1|, \ldots, |\hat{z}_j|, \ldots, |z_n|) < 2N \tag{14}$$

$$Q_{j;2k+2}(|z_1|, \ldots, |\hat{z}_j|, \ldots, |z_n|) \leqq \frac{k + aN}{aT_j^{-1}} Q_{j;2k}(|z_1|, \ldots, |\hat{z}_j|, \ldots, |z_n|), \quad k \geqq 1.$$

Integrating (8) (set the integration functions to zero) and applying the inequality (13) with the integrand $P_{j;2k}$ to our integral yields

$$|p_{j;2k}(z_1, \ldots, z_n)| \leqq |z_j|^k |P_{j;2k}(|z_1|, \ldots, |z_n|)|, \quad k \geqq 1.$$

From (14) and (11) we obtain for the dominants of $P_{j;2k}$

$$\tilde{P}_{j;2}(|z_1|, \ldots, |z_n|) < 2N/(1 - |z_j|/a)$$

$$\tilde{P}_{j;2k}(|z_1|, \ldots, |z_n|) < \frac{2^k \cdot (k-1+aN)(k-2+aN) \ldots (1+aN)N}{a^{k-1}(1-|z_j|/a) S^{1-k}(|z_1|, \ldots, |z_n|)}, \quad k \geqq 2.$$

[1]) $\hat{z}_j$ means that $z_j$ does not appear in $Q_{j;2k}(z_1, \ldots, z_n)$.

The majorant series

$$\tilde{E}_j(|z_1|, \ldots, |z_n|, |t|) < 1+2N|t|^2/(1-|z_j|/a)+$$

$$+\frac{aN}{(1-|z_j|/a)}\sum_{k\geq 2}\frac{2^k(k-1+aN)(k-2+aN)\ldots(1+aN)N}{S^{1-k}(|z_1|, \ldots, |z_n|)1\cdot 3\cdot 5\ldots(2k-1)}|t|^{2k}$$

converge for $|z_i|<a$, $1\leq i\leq n$, $|t|\leq 1$. Note that $\tilde{E}_j$ does not depend on the pathes of integrations in (9).

The series (6) are therefore absolutely and uniformly convergent in $|z_i|<a$, $1\leq i\leq n$, $|t|\leq 1$. But uniformly convergent series of holomorphic functions are holomorphic; hence the series (6) represent a holomorphic function. The existence of a solution of (4) in the neighbourhood of the origin is therefore proved.

## 3. Examples

In *Example 1* we consider equation $Lw=0$ with

$$C(z_1, \ldots, z_n) = -\lambda/H^n, \quad \lambda\in\mathbf{C}, \ H = z_1+\ldots+z_n \neq 0 \text{ in } G.$$

Now we want to determine $E_j(z_1, \ldots, z_n, t)$ according (6). With (9) we can calculate the $P_{j;2k}$ successively for $k=1, 2, \ldots$ . We obtain

$$P_{j;2k} = \frac{2^k\lambda H^{-n-k+1}}{1\cdot 3\cdot 5\ldots(2k-1)}\prod_{i=0}^{k-2}\left[n+i+\frac{(-1)^{n-1}i!}{(n+i-1)!}\lambda\right], \quad k\geq 1, \quad \prod_{i=0}^{-1} := 1.$$

This relation can be proved by induction with respect to $k$. Integrating (8) yields

$$p_{j;2k}(z_1, \ldots, z_n) = z_j^k J_j(P_{j;2k}).$$

In this way we get

$$E_j(z_1, \ldots, z_n, t)=1+(-1)^{n-1}\lambda\sum_{k\geq 1}\frac{2^{2k}k!(k-1)!H^{-k}}{(2k)!(n+k-2)!}z_j^k\left[\prod_{i=0}^{k-2}\left[n+i+\frac{(-1)^{n-1}i!}{(n+i-1)!}\lambda\right]\right]t^{2k}$$

which is convergent for $|z_j|<na$.

As breaking off condition for our series $E_j$ we obtain

$$\lambda = (-1)^n\cdot(n+k-1)(n+k-2)\ldots k\,.$$

*Example 2.*

$$w_{z_1\ldots z_n}+K_1'(z_1)K_2'(z_2)\ldots K_n'(z_n)w = 0, \quad K_1'K_2'\ldots K_n' \neq 0 \text{ in } G, \quad w = w(z_1, \ldots, z_n).$$

This equation may be simplified by the variable transformation $z_j\to K_j(z_j)$, $1\leq j\leq n$, $v(K_1(z_1), \ldots, K_n(z_n))=w(z_1, \ldots, z_n)$, to the form

$$\partial^n v/\partial K_1\ldots\partial K_n+v = 0;$$

consequently, we investigate this equation.

As before we may find the kernel. We then have $e_j(K_1, \ldots, K_n, t) = E_j(z_1, \ldots, z_n, t)$ and the following generalised hypergeometric series

$$e_j(K_1, \ldots, K_n, t) = {}_0F_{n-1}\left(\frac{1}{2}, 1, 1, \ldots, 1;\ -t^2 K_1 K_2 \ldots K_n\right).$$

*"After formula*

$$e_j(K_1, \ldots, K_n, t = {}_0F_{n-1}\left(\frac{1}{2}, 1, 1, \ldots;\ -t^2 K_1 K_2 \ldots K_n\right)$$

*the following text should be printed:"*

Note that in the representation formula (2) the variable $z_j$ of the arbitrary holomorphic function $f_j(z_j)$ has to be replaced by $K_j(z_j)$. So $\Omega = \Omega_1 x \ldots x \Omega_n \ni (0, \ldots, 0)$, where $\Omega_j$, which is the range of values of $K_j$, is a simple connected domain in **C**.

## Reference

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