Direct products and monomial characters

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All groups in this paper are finite. We assume known some notions in the representation theory of groups, such as primitivity of characters and of its utmost opposite, viz. monomiality of characters. The notation is that of Huppert's book [3] and Isaacs' book [4].

§ 1

At first we introduce certain classes $\mathcal{L}$ and $\mathcal{K}$ of groups and then we state our problem area to be treated here.

Definition 1. Let $\mathcal{L}$ be the class of groups consisting entirely of those groups G containing some $N \trianglelefteq G$ with G/N nilpotent and with the Sylow p-subgroups of N abelian for all primes p.

Definition 2. Let $\mathcal{K}_1$ be the class of groups consisting entirely of those groups $G \neq \{1\}$ which have all their maximal subgroups contained in $\mathcal{L}$. Put $\mathcal{K} = \{1\} \cup \mathcal{K}_1$; whence $\mathcal{L} \subseteqq \mathcal{K}$.

Examples.

The alternating group A_5 is a member of $\mathcal{L}$.
All nilpotent groups are members of $\mathcal{L}$.
All Schmidt groups T (i.e. T is not nilpotent but it has all of its proper subgroups nilpotent) are members of $\mathcal{K}$.
The symmetric group S_4 is a member of $\mathcal{L}$.
The group $SL(2, 3)$ is a member of $\mathcal{K}$ but not of $\mathcal{L}$. □

The central topics in the representation theory (and character theory) of groups are undoubtedly primitivity, induction, restriction, monomiality in connection with Clifford's theorem. See [4], Chapters 5 and 6.

In this paper we are given a group G which is an (internal) direct product of the subgroups N and H. Hence $G = NH$, $N \cap H = \{1\}$, $N \trianglelefteq G$, $H \trianglelefteq G$. In his paper [1], N. S. Hekster proved that $\chi \in \mathrm{Irr}(G)$ is a primitive character if and only if the irreducible constituents of χ_N and χ_H are primitive characters. One may conjecture that $\chi \in \mathrm{Irr}(G)$ is monomial if and only if the irreducible constituents of χ_N and χ_H are monomial. We give in § 3 an affirmative answer to the conjecture when $N \in \mathcal{K}$. As the reader will observe, the following ingredients are already used in that case, viz.

1) groups of central type are solvable ([2], Theorem (7.3)),
2) the theory of character triples ([5], Theorem (8.2)),
3) a theorem of A. Parks: Let T be a group, $M \trianglelefteq T$, $M \in \mathscr{L}$. Suppose $\chi \in \operatorname{Irr}(T)$ is monomial and let ζ be an irreducible constituent of χ_M. Then the so-called Clifford correspondent of χ over ζ, χ_ζ, is monomial ([6], Theorem 3.1).

There is also an affirmative answer to the conjecture when $(\mu(1), \nu(1))=1$, where μ is the (unique) irreducible constituent of χ_N, and ν that of χ_H.

Now, in trying to reduce the general case of the conjecture via a hypothetical counterexample G of smallest order, I was able to derive that the subgroups N and H had to be solvable, having all their abelian normal subgroups cyclic and central, while N and H were also of central type.

§ 2

After these hopeful events the progress in the argumentation stopped. No wonder, because the obtained configuration leads very quickly to the existence of a group T, being an internal direct product of two isomorphic copies of a group S, say $T=S_1 S_2$, an irreducible monomial character $\chi \in \operatorname{Irr}(T)$, $\chi_{S_i}=e\varphi_i$ $(i=1, 2)$, $e=\varphi_1(1)=\varphi_2(1)=|S/Z(S)|^{1/2}$, $\varphi_i \in \operatorname{Irr}(S_i)$ $(i=1, 2)$ (whence S is of central type), the φ_i being non-monomial characters. The final touch of this phenomenon is due to E. C. Dade, written in a letter to the author dated August 25, 1985. More precisely we have

Theorem 1. *Let* $T=S_1 S_2$, $S_1 \trianglelefteq T$, $S_2 \trianglelefteq T$, $S_1 \cap S_2=\{1\}$, $S_1 \cong S_2$. *Let α be an isomorphism of S_1 onto S_2. Let $Z(S_1)$ be cyclic,* $\neq \{1\}$. *Let λ_1 be a faithful linear character of $Z(S_1)$. Assume there exists* $\varphi_1 \in \operatorname{Irr}(S_1)$ *with* $\varphi_1|_{Z(S_1)}=|S_1/Z(S_1)|^{1/2}\lambda_1$ *(whence also* $\lambda_1^{S_1}=|S_1/Z(S_1)|^{1/2}\varphi_1$*). Define* $\varphi_2(\alpha(s))=\overline{\varphi_1(s)}$ *(= the complex conjugate of* $\varphi_1(s)$*), for all* $s \in S_1$. *Then* $\varphi_2 \in \operatorname{Irr}(S_2)$ *and* $\varphi_2|_{Z(S_2)}=|S_2/Z(S_2)|^{1/2}\lambda_2$, $\lambda_2^{S_2}=|S_2/Z(S_2)|^{1/2}\varphi_2$, *where λ_2 is the faithful linear character of $Z(S_2)$ determined by* $\lambda_2(\alpha(z))=\overline{\lambda_1(z)}$, *for all* $z \in Z(S_1)$. *Then there exists a unique* $\chi \in \operatorname{Irr}(T)$ *such that* $\chi_{S_1}=\varphi_2(1)\varphi_1$, $\chi_{S_2}=\varphi_1(1)\varphi_2$. *This χ is a monomial character.*

Proof. We have $Z(T)=\{uv \mid u \in Z(S_1),\ v \in Z(S_2)\}$. Further $\chi_{Z(T)}=|S_1/Z(S_1)|\lambda=|T/Z(T)|^{1/2}\lambda$, where λ is the linear character of $Z(T)$ defined by $\lambda(uv)=\lambda_1(u)\cdot\overline{\lambda_1(\alpha^{-1}(v))}$, $u \in Z(S_1)$, $v \in Z(S_2)$. In particular $\chi(1)=|T/Z(T)|^{1/2}$ and so $\lambda^T=|T/Z(T)|^{1/2}\chi=|S_1/Z(S_1)|\chi$. We have here that $\chi(t)=\varphi_1(a)\varphi_2(b)$ with $t=ab$, $a \in S_1$, $b \in S_2$. The set S, defined by $S=\{s\alpha(s) \mid s \in S_1\}$ is a subgroup of T with $S_1 \cong S_2 \cong S$. So, as φ_1 is zero outside $Z(S_1)$ and as φ_2 is zero outside $Z(S_2)$, it follows that $\operatorname{Ker}\chi=\operatorname{Ker}\lambda=S \cap Z(T)$ just by the faithfulness of λ_1. Moreover $S \cap Z(T)=Z(S)$. Hence there exists a linear character μ of $SZ(T)$ such that $\mu_S=1_S$ and $\mu_{Z(T)}=\lambda$. We calculate μ^T. It follows that any irreducible constituent of μ^T is equal to some irreducible constituent of $\lambda^T=(\lambda^{SZ(T)})^T=(\mu+\dots)^T=\mu^T+\dots$. Therefore immediately $\mu^T=f\chi$, for some positive integer f. Counting degrees we see that

$$\mu^T(1)=|T\colon SZ(T)|=|T/Z(T)|/|SZ(T)/Z(T)|=|T/Z(T)|/|S/Z(S)|=$$
$$=|S/Z(S)|=\chi(1).$$

Hence $\chi=\mu^T$ and so χ is a monomial character. □

From the theorem just proved, it follows that, in order to disprove the conjecture, we only need to construct a group S satisfying

1) S has cyclic center $Z(S)\neq\{1\}$,
2) $\lambda\in\mathrm{Irr}\,(Z(S))$ is faithful,
3) $\lambda^S=|S/Z(S)|^{1/2}\Lambda$, $\Lambda\in\mathrm{Irr}\,(S)$, $\Lambda_{Z(S)}=|S/Z(S)|^{1/2}\lambda$,
4) Λ is not monomial.

Now put E_t=extra special t-group of exponent t, t odd prime, $|E_t|=t^3$.

Example (E. C. Dade). Let p, q, r be three distinct odd primes with $q\equiv r\equiv -1 \pmod p$. Consider the internal direct product E_qE_r of E_q and E_r. Then there exists a semi-direct product $S=(E_qE_r)E_p$ such that $E_p=C_{E_p}(E_q)C_{E_p}(E_r)$ and $Z(E_p)=C_{E_p}(E_q)\cap C_{E_p}(E_r)$, and $E_q/Z(E_q)$ and $E_r/Z(E_r)$ are irreducible E_p-groups. The center $Z(S)=Z(E_p)Z(E_q)Z(E_r)$ is cyclic of order pqr. Thus there exists at least one faithful linear character λ of $Z(S)$. There is a unique irreducible character ψ of E_qE_r lying over $\lambda_{Z(E_q)Z(E_r)}$ with $\psi(1)=qr$. So ψ is inert in S. As $(|E_p|, |E_qE_r|)=1$, it then follows that ψ has an extension $\hat{\psi}$ to $\mathrm{Irr}\,(S)$. There exists also $\xi\in\mathrm{Irr}\,(S)$ with $\xi(1)=p$ such that $\xi_{Z(E_p)}=\lambda_{Z(E_p)}$ and $\mathrm{Ker}\,\xi=E_qE_r$. From corollary (6.17) of [4] we conclude that $\hat{\psi}\xi\in\mathrm{Irr}\,(S)$. Write $\Lambda=\hat{\psi}\xi$. Hence $\Lambda_{Z(S)}=pqr\lambda=|S/Z(S)|^{1/2}\lambda$. This implies that $\lambda^S=|S/Z(S)|^{1/2}\Lambda$. The character Λ is not monomial for there is no subgroup M of S with $Z(S)\subset M\subset S$ for which $|S:M|=pqr$. This follows from the choice of the primes p, q, r in connection with the well-defined action of E_p on E_qE_r inside S by conjugation. □

§ 3

In this section the group G is the (internal) direct product of the groups N and H, whence $N\trianglelefteq G$, $H\trianglelefteq G$, $NH=G$, $N\cap H=\{1\}$. Note that $nh=hn$ for all $h\in H$, $n\in N$. Let $\chi\in\mathrm{Irr}\,(G)$. Assume that ν is an irreducible constituent of χ_N and that η is an irreducible constituent of χ_H. It follows from Clifford's theorem that χ_N and χ_H are homogeneous, hence that $\chi_H=\nu(1)\eta$ and $\chi_N=\eta(1)\nu$. We prove now a sort of a rehabilitation to the (negative) result of the second section.

Theorem 2. *Using the notations and hypotheses just given and assuming in addition that $N\in\mathcal{K}$, it holds that χ is a monomial character if and only if η and ν are both monomial characters.*

Proof. If η and ν are both monomial then it follows that χ is monomial. Indeed, $\chi=\hat{\nu}\otimes\hat{\eta}$, where $\hat{\eta}\in\mathrm{Irr}\,(G)$ is the character of G, well-defined by $\hat{\eta}(nh)=\eta(h)$, $(n\in N, h\in H)$ and analogously $\hat{\nu}(nh)=\nu(n)$ defines $\hat{\nu}\in\mathrm{Irr}\,(G)$. As $\hat{\nu}$ and $\hat{\eta}$ are both monomial, χ is monomial too.

So conversely we assume from now on that χ is monomial. Hence there exists a subgroup T of G and a linear character λ of T such that $\lambda^G=\chi$. Assume $T\neq G$. (Otherwise η and ν are both one-dimensional monomial characters). Let M be a proper maximal subgroup of G containing T. Then write $\varphi=\lambda^M$ and so φ is monomial and $\varphi^G=\chi$. There are three cases to be considered.

α) $N\subseteq M\subset G$,
β) $H\subseteq M\subset G$,
γ) $MN=G=MH$.

Re α) *Let* $N \subseteq M \subset G$. Notice $MH=G$ and consider $M \cap H$. As $N \cap (M \cap H)=\{1\}$ we see by counting orders that $M=N(M \cap H)$. Since $[M \cap H, N]=\{1\}$, we have $M \cap H \trianglelefteq M$. It follows from the character theory of direct products that $\varphi_N=\varrho(1)\sigma$ and $\varphi_{M \cap H}=\sigma(1)\varrho$ for some $\varrho \in \mathrm{Irr}(M \cap H)$ and $\sigma \in \mathrm{Irr}(N)$. But χ_N, being homogeneous, has ν as its unique irreducible constituent, whereas $(\chi_N, \sigma)=(\chi, \sigma^G)=(\chi, (\sigma^M)^G) \geqq (\chi, \varphi^G)=(\chi, \chi)=1$. Hence $\sigma=\nu$. By induction on M, seen here as direct product of N and $M \cap H$, we can require that the monomiality of φ implies that ν and ϱ are both monomial. Now Mackey's theorem can be used ([3], V.16.9) and this yields $\varphi^G|_H=\nu(1)\eta=(\varphi_{M \cap H})^H=\sigma(1)\varrho^H=\nu(1)\varrho^H$. Hence $\eta=\varrho^H$ and thus η is monomial.

Re β) *Let* $H \subseteq M \subset G$. Notice $MN=G$ and consider $M \cap N$. Therefore, as $H \cap (M \cap N)=\{1\}$, we see by counting orders that $M=H(M \cap N)$. As $[M \cap N, H]=\{1\}$, we have $M \cap N \trianglelefteq M$. It follows again that $\varphi_H=\gamma(1)\delta$ and $\varphi_{M \cap N}=\delta(1)\gamma$ for some $\gamma \in \mathrm{Irr}(M \cap N)$ and $\delta \in \mathrm{Irr}(H)$. But χ_H has η as its unique irreducible constituent, whereas $(\chi_H, \delta)=(\chi, \delta^G)=(\chi, (\delta^M)^G) \geqq (\chi, \eta^G)=(\chi, \chi)=1$. Hence $\delta=\eta$. Now M is the direct product of the groups $M \cap N$ and H. Observe that here $M \cap N \in \mathscr{L} \subseteq \mathscr{K}$. Hence by induction $\delta=\eta$ is monomial. Since $\varphi^G|_N=\eta(1)\nu=(\varphi_{M \cap N})^N=\delta(1)\gamma^N=\eta(1)\gamma^N$, it follows that $\nu=\gamma^N$ and so ν is monomial.

Re γ) *Let* $MN=G=MH$. Since H centralizes N it follows here that $M \cap N \trianglelefteq MH=G$. Let ψ be an irreducible constituent of $\varphi_{M \cap N}$. We will argue that it will be sufficient to assume that $\varphi_{M \cap N}$ is homogeneous, i.e. that ψ is inert in M. For let $U=I_M(\psi)$, the inertia group of ψ in M, and let $\alpha \in \mathrm{Irr}(U)$ be the unique character such that $\alpha^M=\varphi$ and $(\alpha_{M \cap N}, \psi) \geqq 1$. (The knowledge of the existence of such a Clifford correspondent $\varphi_\psi=\alpha \in \mathrm{Irr}(U)$ can be found in [4], (6.11)). Hence $\chi=\varphi^G=(\alpha^M)^G=\alpha^G$. Now α is monomial as φ is, just by Parks' theorem 3.1 in [6], applied on the group $M \cap N \in \mathscr{L}$ as normal subgroup of M. Then put $S=UN$ and $\zeta=\alpha^S$, whence ζ is monomial and $\zeta^G=(\alpha^S)^G=(\alpha^M)^G=\chi$. Now, if S is a proper subgroup of G, we can find a maximal subgroup $\bar{M}$ of G containing S, and we can find the monomial character $\zeta^{\bar{M}}$ with $(\zeta^{\bar{M}})^G=\chi$. So then we have reduced the problem to the case α). Thus we may assume that $S=G$. Hence $|U/(M \cap N)|=|U/(U \cap N)|=|UN/N|=|G/N|=|MN/N|=|M/(M \cap N)|$. So $U=M$ and ψ is inert in M. In that case ψ is inert in $MH=G$, as H centralizes $M \cap N$. Therefore we may assume from now on that $\varphi_{M \cap N}=e\psi$, some integer $e \geqq 1$, and that ψ is inert in N.

Any normal subgroup of N is already a normal subgroup of G, by $H \subseteq C_G(N)$. This fact combined with $G=MN$ implies that $N/(N \cap M)$ is a simple group $\neq \{1\}$. The theorem of Mackey yields $\chi_N=\varphi^G|_N=\eta(1)\nu=(\varphi_{M \cap N})^N=e\psi^N$. Since $\eta(1)=(\eta(1)\nu, \nu)=e(\psi^N, \nu) \geqq e$, it follows that e divides $\eta(1)$. Therefore $\frac{\eta(1)}{e}\,\nu=\psi^N$. We saw above that we work with the assumption that ψ is inert in N. This means that $\nu_{M \cap N}$ is homogeneous as $M \cap N \trianglelefteq G$ and $(\nu_{M \cap N}, \psi)=(\nu, \psi^N)=\frac{\eta(1)}{e} \geqq 1$. Hence ν is fully ramified over $M \cap N$. In particular $|N/(N \cap M)|=(\eta(1)/e)^2$. The triple $(N, N \cap M, \psi)$ is then a so-called fully ramified character triple. It follows from Isaacs' theory of these triples ([5], (8.2)) in connection with the fact that groups of central type are solvable, that $N/(N \cap M)$ is a solvable group. See here [2] for details on groups of central type. Thus the simple group $N/(N \cap M)$ is solvable and so cyclic of prime order. This contradicts $|N/(N\cap M)|$ being a square and so the proof of Theorem 2 is complete. $\square$

Next we prove

Theorem 3. *Let G be a group with $G=NH$, $N\cap H=\{1\}$, $N\trianglelefteq G$, $H\trianglelefteq G$. Let $\chi\in\mathrm{Irr}(G)$, $\nu\in\mathrm{Irr}(N)$, $\eta\in\mathrm{Irr}(H)$ for which $\chi_N=\eta(1)\nu$ and $\chi_H=\nu(1)\eta$. Assume that $(\nu(1),\eta(1))=1$. Then χ is monomial if and only if ν and η are both monomial.*

PROOF. Just as before, if ν and η are both monomial, then χ is monomial. Now, in about the same way as in the proof of Theorem 2, we may assume that, if χ is monomial, $TN=G=TH$ holds for any subgroup T of G for which there exists a linear character λ with $\lambda^G=\chi$. Of course, in the reduction process the condition $(\alpha(1),\beta(1))=1$ has to be taken into account for appropriate irreducible characters α and β of certain subgroups of G. Now we reduce further. Note that we can assume that $N\neq\{1\}$.

Take such a T and linear $\lambda\in\mathrm{Irr}(T)$ with $\lambda^G=\chi$. Let $X=\mathrm{Ker}\,\nu$. As $\chi_N=\eta(1)\nu$, we have $X\trianglelefteq G$ and $X\subseteq\mathrm{Ker}\,\chi$. Then G/X is the internal direct product of the groups N/X and HX/X. Applying induction it follows that we may assume that

(1) ν is a faithful irreducible character of N.

Next we see that

(2) $T\cap N=Z(N)$.

Indeed, $T\cap N\trianglelefteq TH=G$, whence $T\cap N\trianglelefteq N$ and then $\lambda_{T\cap N}$ is inert in $TH=G$. Hence $\nu_{T\cap N}$ is a multiple of $\lambda_{T\cap N}$. Since by (1) ν is faithful, this implies $T\cap N\subseteq Z(N)$. Now, if $T\cap N\neq Z(N)$, then $(\lambda_{T\cap N})^N$ is not the multiple of a single irreducible character of N. On the other hand $\lambda^G|_N=\chi_N=\eta(1)\nu$ and $\lambda^G|_N=(\lambda_{T\cap N})^N$. Hence $T\cap N=Z(N)$. In fact we just showed that $N/Z(N)$ is a fully ramified section and so, as $(\lambda_{Z(N)})^N=(\lambda_{T\cap N})^N=\lambda^{TN}|_N=\chi_N=\eta(1)\nu$ and $\nu_{Z(N)}=\nu(1)\lambda_{Z(N)}$, it holds that

(3) $|N/Z(N)|=(\nu(1))^2=(\eta(1))^2$.

Hence, as $(\nu(1),\eta(1))=1$, this leads to $\chi(1)=\eta(1)=\nu(1)=1$ and it is clear now that we have proved Theorem 3. $\square$

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