Some theorems on wreath products

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1. *Discussion**)

The Embedding Theorem constructs, for each group G and each subgroup H of index n in G, embeddings φ of G in the (unrestricted, permutational) wreath product $H \operatorname{Wr} S_n$ of H by the relevant symmetric group. Such wreath products have a functorial property which gives for each homomorphism $\alpha\colon H \to A$ a homomorphism $\alpha \operatorname{Wr} S_n\colon H \operatorname{Wr} S_n \to A \operatorname{Wr} S_n$. The composites $\alpha\uparrow$ of φ and $\alpha \operatorname{Wr} S_n$ are of fundamental importance. For example, if n is finite and A is a general linear group, GL_k say, so α is a linear representation of H, then $\alpha\uparrow$ (composed with the obvious inclusion of $GL_k \operatorname{Wr} S_n$ in GL_{kn}) is the induced representation of G. In this sense at least, the Embedding Theorem goes back all the way to Frobenius. (For recent expositions, see § 5 in COSSEY, KEGEL, KOVÁCS [1] and § 4 in ROBINSON, WILSON [4].)

The first question considered here is: how does one recognize whether a homomorphism $G \to H \operatorname{Wr} S_n$ is one of the embeddings given by that Theorem? What distinguishes these embeddings from others?

Towards an answer we must emphasize first that the Theorem gives not just one embedding but a whole lot: one for each of the $|H|^n$ transversals of H in G. Second, the symmetric group which really occurs in the Theorem is that acting on the set of all cosets of G modulo H, while the functorial view demands that we think of S_n as the symmetric group on some set given without reference to G or H: so we have to choose one of the $n!$ possible identifications of these two sets. All told, we have $n!|H|^n$ options. It is not hard to see that the resulting embeddings differ precisely by inner automorphisms of the wreath product: if we let $\operatorname{Inn}(H \operatorname{Wr} S_n)$ act on $\operatorname{Hom}(G, H \operatorname{Wr} S_n)$ by composition, they form a single complete orbit of this action. (In general, there are some coincidences so we get fewer than $n!|H|^n$ distinct embeddings: we shall return to this point later.)

More notation is needed before we can proceed. It will *not* be assumed that n is finite. Throughout, I shall denote a fixed set of cardinality n, and for emphasis we shall often write S_I rather than S_n. The wreath product $A \operatorname{Wr} S_I$ is the semidirect product of S_I and the group A^I of all functions $I \to A$. [Permutations and

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functions will be written on the left and composed accordingly. The action of S_I on A^I is defined in terms of this composition but written exponentially: so

$$f^p(i)=f(pi) \quad \text{whenever} \quad f\in A^I, \quad p\in S_I, \quad i\in I.]$$

The natural projection of $A \operatorname{Wr} S_I$ onto S_I will be denoted π (or π^A when a distinction appears necessary). Given $W=A \operatorname{Wr} S_I$ and an i in I, the elements pf of W such that $pi=i$ form a subgroup, W_i say, which has an obvious direct decomposition $A\times(A \operatorname{Wr} S_{I\setminus\{i\}})$: the corresponding projection $W_i\to A$, $pf\mapsto f(i)$ will be called π_i (or π_i^A when appropriate). [Homomorphisms will be written on the right and composed accordingly.] The answer to the recognition problem above may now be expressed as follows.

Theorem 1. *A homomorphism $\varphi\colon G\to W=H \operatorname{Wr} S_I$ is one of the embeddings given by the Embedding Theorem if and only if*

(a) $G\varphi\pi$ is transitive (as subgroup of S_I), and
(b) there is an element 0 in I such that
(b1) the stabilizer of 0 in G with respect to the permutation representation $\varphi\pi$ is H, and
(b2) the restriction $\varphi\downarrow\colon H\to W_0$ followed by π_0 is an inner automorphism of H.

It must never be forgotten that here W is the group concretely constructed above, with a distinguished copy (the "top group") of S_I and a distinguished copy (the "base group") of H^I as semidirect factors, and equipped with π and the π_i. Changing to a different wreath decomposition of this group may easily spoil the result. For example, let G be a nonabelian group of order 6 and H a subgroup of index 3 in G. Then the base group has two conjugacy classes of complements in W, one being the class containing the top group; it is easy to verify that the relevant embeddings are precisely those whose whose images fall into the other class. This illustrates the sensitivity of Theorem 1 to the slightest change in the wreath decomposition: one cannot even replace the top group by another (nonconjugate) complement of the base group, without upsetting the conclusions.

This recognition problem has an obvious variant: given a homomorphism $\gamma\colon G\to W=A \operatorname{Wr} S_I$, how can one tell whether $\gamma=\alpha\uparrow$ for some suitable α?

Theorem 1′. *Let $\gamma\colon G\to W=A \operatorname{Wr} S_I$ be any homomorphism. There is a subgroup H in G (of index equal to the cardinality of I) and a homomorphism $\alpha\colon H\to A$ such that $\alpha\uparrow=\gamma$ (for a suitable identification of I with the set of the left cosets of G modulo H, and for a suitable transversal of H in G), if and only if*

(a) $G\gamma\pi$ is transitive (as subgroup of S_I), and
(b) there is an element 0 in I such that

$$G\gamma\leqq(G\gamma\cap W_0)\pi_0 \operatorname{Wr} S_I.$$

Of course here $(G\gamma\cap W_0)\pi_0 \operatorname{Wr} S_I$ is thought of as a subgroup of $A \operatorname{Wr} S_I$ [embedded via $\beta \operatorname{Wr} S_I$ where β is the inclusion of $(G\gamma\cap W_0)\pi_0$ in A]. Note that (a), (b) do not involve γ directly, only its image $G\gamma$. Also, once (a) is assumed, the inclusion in (b) holds either for all elements of I or for none at all.

The second question of this paper also comes in two versions. One, what is the cardinality of the set of all embeddings φ constructed by the Embedding Theorem for given G and H? The discussion above leads to the conclusion that it is the index in $H \operatorname{Wr} S_I$ of the centralizer $\mathbf{C}_W(G\varphi)$ of the image of any one of these embeddings, so the real question is to determine $\mathbf{C}_W(G\varphi)$.

Theorem 2. *Let H be a subgroup of index n in a group G, and $\varphi\colon G\to W=H \operatorname{Wr} S_n$ any one of the embeddings given by the Embedding Theorem. Then $\mathbf{C}_w(G\varphi)\cong\mathbf{C}_G(H)$. If G is finite, the number of distinct φ of this kind is therefore*

$$(n-1)!\,|H|^{n-1}|G:\mathbf{C}_G(H)|.$$

The second version asks: given G, H, and $\alpha\colon H\to A$, what is the cardinality of the set of all homomorphisms $\alpha\uparrow\colon G\to A \operatorname{Wr} S_n$ "induced" by this α? An argument similar to the discussion above yields that it is the index in W of any one $\mathbf{C}_W(G(\alpha\uparrow))$, except that W must be taken as $(H\alpha) \operatorname{Wr} S_n$, *not* as $A \operatorname{Wr} S_n$. In place of $\mathbf{C}_G(H)$, the answer will involve the subgroup $\mathbf{C}_G(H/\ker\alpha)$ defined as the set of those elements g of G for which the mutual commutator $[H,g]$ is contained in $\ker\alpha$: that is, those g which normalize both H and $\ker\alpha$, and whose (conjugation) action on $H/\ker\alpha$ is trivial. Of course when $A=H$ and α is the identity map, this is just $\mathbf{C}_G(H)$, and the $\alpha\uparrow$ are just the φ of Theorem 2. That result is therefore a special case of the following.

Theorem 2′. *Let H be a subgroup of index n in a group G, let $\alpha\colon H\to A$ be a homomorphism, and $\alpha\uparrow\colon G\to A \operatorname{Wr} S_n$ any one of the homomorphisms induced by α. Set $W=(H\alpha) \operatorname{Wr} S_n$: then $\mathbf{C}_W(G(\alpha\uparrow))\cong\mathbf{C}_G(H/\ker\alpha)/\ker\alpha$. If G is finite, the number of distinct $\alpha\uparrow$ induced by the given α is*

$$(n-1)!\,|H\alpha|^{n-1}|G:\mathbf{C}_G(H/\ker\alpha)|.$$

It may be worth noting that the proofs of Theorems 2 and 2′ yield explicit isomorphisms, not just the existence of isomorphisms.

2. *Proofs*

Theorems 1 and 1′ depend on the answer to a related question: how can one recognize whether two homomorphisms $\gamma,\gamma'\colon G\to W=A \operatorname{Wr} S_I$ are the same up to composition with an inner automorphism of W? In turn, this is an extension of the familiar question: how can one recognize whether $\gamma\pi$ and $\gamma'\pi$ are equivalent as permutation representations $G\to S_I$? The answer to that is of course classical, the essential case being that of transitive representations. Accordingly, let us narrow down our question: after correction by an inner automorphism of W induced by an element of the top group S_I, we assume that $\gamma\pi$ and $\gamma'\pi$ are equal and transitive, and ask whether γ and γ' differ only by an inner automorphism of W induced by some element of the base group A^I. The answer is: if and only if $(\gamma\downarrow)\pi_0$ and $(\gamma'\downarrow)\pi_0$ differ only by an inner automorphism of A. Here 0 is any element of I, and $\gamma\downarrow$, $\gamma'\downarrow$ are the restrictions of γ, γ', respectively, to $H\to W_0$ where H is the stabilizer of 0 with respect to $\gamma\pi$. This is contained in the Uniqueness Theorem of [2], which may be conveniently paraphrased as follows.

Uniqueness Theorem. *Let γ and γ' be homomorphisms of a group G into a wreath product $A \operatorname{Wr} S_I$, such that $\gamma\pi=\gamma'\pi$ and $G\gamma\pi$ is transitive as subgroup of S_I. Consider*

$$F = \{f\in A^I \mid \gamma' = \gamma(\operatorname{inn} f)\}$$
$$= \{f\in A^I \mid g\gamma' = f^{-1}(g\gamma)f \text{ for all } g \text{ in } G\},$$
$$B = \{b\in A \mid (\gamma'\downarrow)\pi_0 = (\gamma\downarrow)\pi_0 (\operatorname{inn} b)\}$$
$$= \{b\in A \mid h\gamma'\pi_0 = b^{-1}(h\gamma\pi_0)b \text{ for all } h \text{ in } H\}.$$

Then π_0 maps F one-to-one onto B.

Addendum. The inverse of this bijection may be described in terms of a transversal of H in G but is of course independent of that. To each i in I choose a t_i in G such that $(t_i\gamma\pi)0=i$ [equivalently, $(t_i\gamma'\pi)0=i$]. Write $t_i\gamma$ as $p_i f_i$ with p_i from the top group S_I and f_i from the base group A^I; similarly, set $t_i\gamma'=p_i f_i'$. The inverse bijection maps an element b of B to the element f of F defined by

$$f(i) = f_i(0)bf_i'(0)^{-1} \quad \text{for all } i \text{ in } I.$$

Proof of Theorem 1. The "only if" claim comes straight from the proof of the Embedding Theorem and we shall not spell it out: the reader can easily elaborate details from the sketch given on p. 216 of [1]. Take 0 as the element of I identified with the trivial coset of H in G; the inner automorphism of H in question is induced by the representative of this coset in the transversal chosen.

For the "if" part, suppose (a) and (b) hold; let t_0 be an element of H which induces the inner automorphism $(\varphi\downarrow)\pi_0$. For each i in I other than this 0, choose a t_i in G such that $(t_i\varphi\pi)0=i$: this gives a transversal of H in G. Identify I with the set of the cosets of G modulo H by matching each i to t_iH. Let φ' be the embedding constructed with this choice of transversal and identification. It is obvious that $\varphi\pi=\varphi'\pi$ and that $(\varphi\downarrow)\pi_0=\operatorname{inn} t_0=(\varphi'\downarrow)\pi_0$. Invoke the Uniqueness Theorem with φ, φ', H in place of γ, γ', A, noting that now $1\in B$: hence F is also nonempty. Take any f in F: then $\varphi=\varphi'(\operatorname{inn} f^{-1})$, and of course $\varphi'(\operatorname{inn} f^{-1})$ is just an embedding constructed from a different transversal [namely, from that with $t_i f(i)^{-1}$ in place of t_i]. This completes the proof of Theorem 1.

Proof of Theorem 1'. For the "only if" part, we have to show that (a) and (b) hold when $\gamma=\alpha\uparrow$. Let $\alpha\uparrow=\varphi(\alpha \operatorname{Wr} S_I)$ with a $\varphi\colon G\to H \operatorname{Wr} S_I$ given by the Embedding Theorem, and 0 an element of I such that (b1) and (b2) of Theorem 1 hold. The proof depends on the fact that π and π_0 are "natural". To express this we now distinguish π^H from π^A and π_0^H from π_0^A, but simply keep W and W_0 for the domains of π^A and π_0^A, leaving the domains of π^H and π_0^H unnamed. The naturality of π means that $(\alpha \operatorname{Wr} S_I)\pi^A=\pi^H$; this yields that $G(\alpha\uparrow)\pi^A=G\varphi\pi^H$, so $G(\alpha\uparrow)\pi^A$ is transitive by (a) of Theorem 1. The naturality of π_0 means that $((\alpha \operatorname{Wr} S_I)\downarrow)\pi_0^A= =\pi_0^A\alpha$ for the relevant restriction $(\alpha \operatorname{Wr} S_I)\downarrow$: this yields that

$$(\alpha\uparrow\downarrow)\pi_0^A = (\varphi\downarrow)\big((\alpha Wr S_I)\downarrow\big)\pi_0^A = (\varphi\downarrow)\pi_0^H\alpha.$$

As $H(\varphi\downarrow)\pi_0^H=H$ by (b2) of Theorem 1, we have $H(\alpha\uparrow\downarrow)\pi_0^A=H(\varphi\downarrow)\pi_0^H\alpha=H\alpha$. Of course $H(\alpha\uparrow\downarrow)=H(\alpha\uparrow)$, while $H(\alpha\uparrow)\pi^A=H\varphi\pi^H$ and (b1) of Theorem 1 give that $H(\alpha\uparrow)\leqq G(\alpha\uparrow)\cap W_0$: hence by the conlusion of the previous sentence $H\alpha\leqq$

$\leqq (G(\alpha\uparrow)\cap W_0)\pi_0^A$. In view of $G(\alpha\uparrow)\leqq(H\,\mathrm{Wr}\,S_I)(\alpha\,\mathrm{Wr}\,S_I)=(H\alpha)\,\mathrm{Wr}\,S_I$, this proves the inclusion claimed in (b).

The proof of the "if" claim depends on the Addendum to the Uniqueness Theorem: so assume (a), (b), and define H as the stabilizer (with respect to the permutation representation $\gamma\pi^A$) of the 0 of (b), so $H\gamma=G\gamma\cap W_0$. Define $\alpha\colon H\to A$ as $(\gamma\downarrow)\pi_0^A$; the inclusion in (b) may then be written as $G\gamma\leqq(H\alpha)\,\mathrm{Wr}\,S_I$. By (a), to each i in I one may choose a t_i in G such that $(t_i\gamma\pi^A)0=i$, and these form a transversal of H in G. Define γ' as $\alpha\uparrow$ formed with respect to such a transversal and the matching identification of i with t_iH, for each i in I. Elaborating this definition of γ' shows that $(g\gamma'\pi^A)i=j$ means $gt_iH=t_jH$; by the definition of H, this is equivalent to $((gt_i)\gamma\pi^A)0=j$. It follows that $\gamma\pi^A=\gamma'\pi^A$. Define f_i and f_i' as in the Addendum. We have seen that $G\gamma\leqq(H\alpha)\,\mathrm{Wr}\,S_I$: hence $f_i\in(H\alpha)^I$. Similarly, $f_i'\in(H\alpha)^I$ because by its definition γ' factors through $\alpha\,\mathrm{Wr}\,S_I$. Let φ be the embedding $G\to H\,\mathrm{Wr}\,S_I$ used in forming $\alpha\uparrow$: we know from the proof of Theorem 1 that $(\varphi\downarrow)\pi_0^H=\mathrm{inn}\,t_0$. As π_0 is natural,

$$(\gamma'\downarrow)\pi_0^A=(\varphi\downarrow)\big((\alpha Wr S_I)\downarrow\big)\pi_0^A=(\varphi\downarrow)\pi_0^H\alpha=(\mathrm{inn}\,t_0)\alpha=\alpha(\mathrm{inn}\,t_0\alpha)=(\gamma\downarrow)\pi_0^A\,(\mathrm{inn}\,t_0\,\alpha).$$

In terms of the Uniqueness Theorem, this means that $t_0\alpha\in B$; hence by the Addendum the element f of A^I defined by

$$f(i)=f_i(0)(t_0\alpha)f_i'(0)^{-1}\quad\text{for all } i \text{ in } I$$

lies in F: that is, $\gamma=\gamma'(\mathrm{inn}\,f^{-1})$. From the foregoing we see that in fact $f(i)\in H\alpha$ for all i, so $f^{-1}\in(H\alpha)^I$. It follows that composition with $\mathrm{inn}\,f^{-1}$ merely changes γ' to an $\alpha\uparrow$ defined with reference to a different transversal. This completes the proof of Theorem 1′.

Theorems 2 and 2′ depend on the other result from [2] as strengthened in [3]. The relevant part may be paraphrased as follows.

Centralizer Theorem. *Let $\gamma\colon G\to W=A\,\mathrm{Wr}\,S_I$ be a homomorphism such that $\gamma\pi$ is a transitive permutation representation; let H be the stabilizer in G of some point, 0 say, of I; and let S denote the image of H in the (external) direct product $G\times A$ under the embedding given by $h\mapsto(h,h\gamma\pi_0)$. Then there is a homomorphism of $\mathbf{N}_{G\times A}(S)$ onto $\mathbf{C}_W(G\gamma)$ with kernel S.*

(Strictly speaking, the statement in [3] deals with the image R of $H\gamma$ in $G\gamma\times A$ under $h\gamma\mapsto(h\gamma,h\gamma\pi_0)$, and gives an explicit homomorphism ψ of $\mathbf{N}_{G\gamma\times A}(R)$ onto $\mathbf{C}_W(G\gamma)$ with kernel R. Since S contains the kernel $(\ker\gamma)\times 1$ of the homomorphism $\gamma\times 1$ of $G\times A$ onto $G\gamma\times A$ and $S(\gamma\times 1)=R$, the composite of $\gamma\times 1$ and that ψ will serve in the present version.)

We have already noted that Theorem 2 is a special case of Theorem 2′. For the proof of the latter, one may assume without loss of generality that $A=H\alpha$, and then W can be thought of as $A\,\mathrm{Wr}\,S_I$. Further, once α is given, the isomorphism type of $\mathbf{C}_W(G(\alpha\uparrow))$ is independent of the choice of $\alpha\uparrow$, so we may as well take an $\alpha\uparrow$ defined with reference to a transversal in which the trivial coset is represented by 1, and to an identification which matches that coset to 0. We know from (a) of Theorem 1′ that $(\alpha\uparrow)\pi^A$ is transitive, while the proof of the "only if" part of

Theorem 1 and the naturality of π yield that H is the stabilizer of 0. We can therefore apply the Centralizer Theorem with $\gamma=\alpha\uparrow$. By an argument used in the proof of Theorem 1′, now $(\alpha\uparrow\downarrow)\pi_0^A=\alpha$, so S is the image of $h\mapsto(h, h\alpha)$. It is easy to see that if $(g, a)\in \mathbf{N}_{G\times A}(S)$ then g must normalize both H and $\ker\alpha$, and then (using $H\alpha=A$) that $\mathbf{N}_{G\times A}(S)=(\mathbf{C}(H/\ker\alpha)\times 1)S$ with $(\mathbf{C}(H/\ker\alpha)\times 1)\cap S=(\ker\alpha)\times 1$. Consequently $\mathbf{N}_{G\times A}(S)/S\cong \mathbf{C}_G(H/\ker\alpha)/\ker\alpha$, and so the Centralizer Theorem yields Theorem 2′.

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