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On the cyclic and dihedral cohomology of Banach spaces

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Abstract. Given a functor $\mathcal{F} : \mathfrak{K} \longrightarrow \mathcal{B}$, where $\mathfrak{K}$ is a small category and $\mathcal{B}$ is a category of Banach $\underline{\mathfrak{K}}$-modules, where $\underline{\mathfrak{K}}$ is a Banach algebra associated with the category $\mathfrak{K}$, the cyclic and dihedral cohomology of $\mathcal{F}$ will be studied. We prove that the cyclic cohomology is isomorphic to the direct sum of dihedral cohomologies.

0. Introduction

In [2] Connes introduced the concept of cyclic cohomology. He had shown that the cyclic cohomology can be considered as special functor Ext, in the category of cyclic linear space. In [3] Helemckii studied the analogies Banach cyclic (co)homology as Banach derived functor. The cyclic cohomology of some classes of C^*-algebra has been studied by Wodzicki [9], Christensen and Sinclair [1]. The dihedral (co)homology for discrete involutive algebra has been studied in [4], [5] and [6]. One of the main basic theorems, when formulated the dihedral (co)homology, says that the cyclic (co)homology splits into the direct sum of dihedral (co)homologies. In the peresent paper, we study the Banach simiplicial cyclic and dihedral cohomology, as derived functor, in a Banach category and prove that the cyclic cohomology is isomorphic to the direct sum of the dihedral cohomologies.

1. Notation and definitions

In this part we recall the main definitions and results of [3]. Let $\mathfrak{K}$ be a small category with the countable set $\{0, 1, 2, \dots\}$ as objects. The set of all morphisms in the category $\mathfrak{K}$ is denoted by $[\,.\,,\,.\,]_{\mathfrak{K}}$; the set of morphisms from $\underline{m}$ to $\underline{n}$ by $[\underline{m}, \underline{n}]_{\mathfrak{K}}$; the set of morphisms from $\underline{m}$ to somewhere by $[\underline{m}, .]_{\mathfrak{K}}$, and the set of morphisms from somewhere to $\underline{n}$ by $[\,.\,, \underline{n}]_{\mathfrak{K}}$. The identity morphism from $\underline{n}$ into $\underline{n}$ is given by the map $1_n : \underline{n} \longrightarrow \underline{n}$. We denote by $\ell_1[m, n]_{\mathfrak{K}}$, $\ell_1[m, .]_{\mathfrak{K}}$, $\ell_1[\,.\,, n]_{\mathfrak{K}}$, $\ell_1[\,.\,, .]_{\mathfrak{K}}$ the ℓ_1-Banach spaces constructed on the corresponding set morphisms. Obviously the space $\ell_1[\,.\,, .]_{\mathfrak{K}}$ is a Banach algebra under the following rule: the product of any two morphisms is equal to their composite, if this exists; otherwise, it is equal to zero.

Definition 1.1. The Banach algebra $\ell_1[\,.\,, .]_{\mathfrak{K}}$ is called the Banach algebra associated with $\mathfrak{K}$ (or $\mathfrak{K}$-Banach algebra), and denoted by $\underline{\mathfrak{K}}$. We denote by $\underline{\mathfrak{K}}$-mod the category of left Banach module over $\underline{\mathfrak{K}}$, and by $\mathfrak{h}_{\mathfrak{K}}(X, Y)$ the set of all morphisms between $X, Y \in \underline{\mathfrak{K}}$-mod. For any $X \in \underline{\mathfrak{K}}$-mod we denote by X^n the closed subspace $X^n = \{1_n \cdot x; x \in X\} = \{x; 1_n \cdot x = x\}$. Obviously $X^m \cap X^n = \phi$ for $m \neq n$.

Definition 1.2 [8]. The module $X \in \underline{\mathfrak{K}}$-mod is called geometrically essential if $X = \sum_{n=0}^{\infty} X^n$ and it satisfies an inequality:

$$\|a \cdot x\| \leq \|a\| \, \|x\|, \quad a \in \mathfrak{K}, \ x \in X.$$

The symbol $\sum$ is the ℓ_1-sum of Banach spaces. Note that the algebra $\underline{\mathfrak{K}}$ is an example of a geometrically essential module.

Definition 1.3. The $\mathfrak{K}$-categorial Banach space or $\mathfrak{K}$-Banach space is a covariant functor $\mathcal{F} : \mathfrak{K} \longrightarrow \mathcal{B}$, where $\mathcal{B}$ is the category of Banach spaces and continuous operators.

For given functor $\mathcal{F}$, suppose $E(\mathcal{F}) = E^n(\mathcal{F})$, $E^n(\mathcal{F}) = \mathcal{F}(n)$.

Lemma 1.4. *$E(\mathcal{F})$ has the structure of left Banach $\underline{\mathfrak{K}}$-module (associated with $\mathcal{F}$), which uniquely determined for $\xi \in [m, .]$ and $x \in E^n(\mathcal{F})$ by the equalities:*

$$\xi \cdot x = \begin{cases} [\mathcal{F}(\xi)]x, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

This module is geometrically essential.

Let $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ be a natural transformation of $\mathfrak{K}$-spaces (as functor), then its components $\alpha_n : \mathcal{F}(\underline{n}) \longrightarrow \mathcal{G}(\underline{n})$ generate the operators $\underline{\alpha} : E(\mathcal{F}) \longrightarrow E(\mathcal{G})$. These morphisms are elements in $\underline{\mathfrak{K}}$-mod. We shall denote by $[\mathfrak{K}, \mathcal{B}]$, the category of $\mathfrak{K}$-space with natural transformations as morphisms.

Lemma 1.5. *The assignment $\mathcal{F} \longrightarrow E(\mathcal{F})$; $\alpha \longrightarrow \underline{\alpha}$ is an isomorphism of the categories $[\mathfrak{K}, \mathcal{B}]$ and $\mathfrak{K}$-mod.*

Proof (see [3]).

Now we shall define the special $\mathfrak{K}$-space denoted by $\mathbb{Y}$, that assign to each element $\underline{n}$ the complex space $\mathbb{C}$ and to each morphism $\xi \in [\,.\,,\,.\,]_{\mathfrak{K}}$ the identity operator in $\mathbb{C}$. The associated $\underline{\mathfrak{K}}$-mod $\underline{\mathbb{Y}}$ is obviously ℓ_1-Banach space with the external multiplication: $\zeta \cdot P^k = \delta_m^k P^n$, where $\xi \in [m,n]_{\mathfrak{K}}$ and P^k is the k-th unit vector $(0,\dots,0,1,0,\dots) \in \ell_1$ and

$$\delta_m^k = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{if } k \neq m. \end{cases}$$

Definition 1.7 [3]. Let $\mathcal{F}$ be the $\mathfrak{K}$-Banach space, then the n-dimensional cohomology $\mathcal{H}^n(\mathcal{F})$ of $\mathcal{F}$, is complete normed space $\operatorname{Ext}^n_{\mathfrak{K}}(\underline{\mathbb{Y}}, E(\mathcal{F}))$.

We now consider a certain class of $\underline{\mathfrak{K}}$-modules that plays an important role in the study of $\mathfrak{K}$-cohomology for certain special Banach categories. Clearly the Banach space $\ell_1[n,\,.\,]_{\mathfrak{K}}$ is a closed left ideal in $\mathfrak{K}$ generated by the idempotent map 1_n. We shall denote it by $\underline{\mathfrak{K}}_n$. Obviously that $\underline{\mathfrak{K}}_n$ is a geometrically essential $\underline{\mathfrak{K}}$-module, and the $\mathfrak{K}$-space related with this ideal is a functor sends an object $\underline{m}$ into $\ell_1[n,m]$ and the morphism $\xi \in [q,m]$ into the operator $\zeta : \ell_1[n,q] \longrightarrow \ell_1[n,m] : \eta \longrightarrow \xi \circ \eta$.

The idea of the main result of this work depends on the following assertions, that proved in [3].

Lemma 1.8. *For any $X \in \underline{\mathfrak{K}}$-mod, there exist a topological Banach space isomorphism $\Phi^n : \mathcal{H}_{\mathfrak{K}}(\underline{\mathfrak{K}}_n, X) \cong X$ taking φ into $\varphi(1_n)$. The inverse $(\Phi^n)^{-1}$ takes $x \in X^n$ into the morphism $\underline{\mathfrak{K}}_n \longrightarrow X : \xi \longrightarrow \xi \cdot x$.*

Lemma 1.9. *The idal $\underline{\mathfrak{K}}_n$ is a projective Banach $\underline{\mathfrak{K}}_n$-module.*

2. The simplicial, cyclic, reflexive, and dihedral Banach category

In this part, we consider some special small categories play an important role in the sequel. They have the objects n, $n = 0,1,2,\dots$, and differ in the family of morphisms.

The simpliest one is called a simplicial category and denoted by Δ. It is generated by the set of morphisms δ_n^i and $\sigma_n^j \in [m,n]_\Delta$, such that;

$\delta_n^i \in [\underline{n}, \underline{n}-1]$, $\sigma_n^j \in [\underline{n}+1, \underline{n}]$, $0 \leq i \leq n$, $0 \leq j \leq n$. The morphisms δ_n^i and σ_n^j satisfy the following conditions:

$$
\begin{aligned}
&\delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{j-1}, && i < j,\\
&\sigma_n^j \sigma_{n-1}^i = \sigma_n^i \sigma_{n+1}^{j-1}, && i \leq j,\\
&\delta_n^i \sigma_{n-1}^j = \begin{cases} \sigma_{n-2}^{i-1} \delta_{n-1}^j, & i < j,\\ \mathrm{Id}, & i = j,\ j+1,\\ \sigma_{n-2}^j \delta_{n-1}^{i-1}, & i > j. \end{cases}
\end{aligned}
\tag{2-1}
$$

The second important category is called cyclic category is introduced by Connes in [2] and denoted by $\mathfrak{C}$. It has the same object $\underline{n}$ and morphisms δ_n^i, σ_n^j and τ_n, where $\tau_n \in [n,n]_{\mathfrak{C}}$ is a Connes's cyclic operator. The relation between morphisms δ_n^i, σ_n^j and τ_n are given in (2-1) and the following relations:

$$
\begin{aligned}
&\tau_n \delta_n^i = \delta_n^{i-1} \tau_{n-1}, && 1 \leq i < n,\\
&\tau_n \sigma_n^j = \sigma_n^{j-1} \tau_{n+1}, && 1 \leq j \leq n,\\
&(\tau_n)^{n+1} = 1_n.
\end{aligned}
\tag{2-2}
$$

The dihedral category, introduced and studied in [4], [5] and denoted by Ξ, has the same object $\boldsymbol{n}$ and morphisms δ_n^i, σ_n^j, τ_n and ρ_n. These morphisms satisfy the relations (2-1), (2-2) and also

$$
\begin{aligned}
&\rho_n \delta_n^i = \delta_n^{n-i} \rho_{n-1}, && 0 \leq i \leq n,\\
&\rho_n \sigma_n^j = \sigma_n^{n-j} \rho_{n-1}, && 0 \leq j \leq n,\\
&\rho_n^2 = 1_n, && \tau_n \rho_n = \rho_n \tau_n^{-1}.
\end{aligned}
\tag{2-3}
$$

Note that, if we take only the morphisms δ_n^i, σ_n^j and ρ_n, we get a reflexive category which denoted by Ω (see [4]).

Definition 2.1. The $\mathfrak{K}$-Banach space is called cosimplicial if $\mathfrak{K} = \Delta$, cocyclic if $\mathfrak{K} = \mathfrak{C}$, coreflexive if $\mathfrak{K} = \Omega$ and codihedral if $\mathfrak{K} = \Xi$. For codihedral Banach space $\mathcal{F}$ we put $\mathcal{F}(n) = E^n$, $\mathfrak{d}_n^i = \mathcal{F}(\delta_n^i)$, $\mathfrak{s}_n^j = \mathcal{F}(\sigma_n^j)$, $\mathfrak{t}_n = \mathcal{F}(\tau_n)$, $\mathfrak{r}_n = \mathcal{F}(\rho_n)$. These morphisms satisfy the relations (2-1), (2-2) and (2-3). We shall represent the Banach spaces Δ, $\mathfrak{C}$, Ω and Ξ

respectively, by: $\mathcal{F}(E,\mathfrak{d},\mathfrak{s})$, $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t})$, $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{r})$, and $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t},\mathfrak{r})$, where $E=E^n$, $\mathfrak{d}=\mathfrak{d}_n^i$, $\mathfrak{s}=\mathfrak{s}_n^i$, $\mathfrak{t}=\mathfrak{t}_n$, $\mathfrak{r}=\mathfrak{r}_n$. Consider for the Banach space $\mathcal{F}(E,\mathfrak{d},\mathfrak{s})$ the following complex

$$C(\mathcal{F}): \quad 0\longrightarrow E^0 \xrightarrow{\mathfrak{b}^0} E^1 \longrightarrow \ldots \longrightarrow E^{n-1} \xrightarrow{\mathfrak{b}^{n-1}} E^n \longrightarrow \ldots\,,$$

where $\mathfrak{b}^{n-1}=\sum_{k=0}^{n}(-1)^k\mathfrak{d}_n^k$, $n=1,2,\ldots$ Clearly the complex $(C(\mathcal{F}),\mathfrak{b}^n)$ is a chain complex, and from relations between the operators d, it is a complex in $\mathcal{B}$. For a given $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t})$, $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{r})$ and $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t},\mathfrak{r})$, it is easily seen that there exists in $C(\mathcal{F})$ the subcomplexes $CC(\mathcal{F})$, $C\mathcal{R}(\mathcal{F})$ and $C\mathcal{D}(\mathcal{F})$:

$$CC(\mathcal{F}):0 \longrightarrow EC^0 \xrightarrow{\mathfrak{bc}^0} EC^1 \longrightarrow\ldots\longrightarrow EC^{n-1} \xrightarrow{\mathfrak{bc}^{n-1}} EC^n \longrightarrow \ldots\,,$$

$$C\mathcal{R}(\mathcal{F}):0 \longrightarrow {}^{\alpha}E\mathcal{R}^0 \xrightarrow{\mathfrak{b}r^0} {}^{\alpha}E\mathcal{R}^1 \longrightarrow\ldots\longrightarrow {}^{\alpha}E\mathcal{R}^{n-1} \xrightarrow{\mathfrak{b}r^{n-1}} {}^{\alpha}E\mathcal{R}^n \longrightarrow \ldots\,,$$

$$C\mathcal{D}(\mathcal{F}):0 \longrightarrow {}^{\alpha}E\mathcal{D}^0 \xrightarrow{\mathfrak{bd}^0} {}^{\alpha}E\mathcal{D}^1 \longrightarrow\ldots\longrightarrow {}^{\alpha}E\mathcal{D}^{n-1} \xrightarrow{\mathfrak{bd}^{n-1}} {}^{\alpha}E\mathcal{D}^n \longrightarrow \ldots\,,$$

where $EC^n=\{x\in E^n;\ x=(-1)^n\,\mathfrak{t}_n(x)\}$,

$$ {}^{\alpha}E\mathcal{R}^n=\Big\{x\in E^n;\ x=(-1)^{\frac{n(n+1)}{2}}\,\mathfrak{r}_n(x)\Big\},$$

$$ {}^{\alpha}E\mathcal{D}^n=\Big\{x\in E^n;\ x=(-1)^n\,\mathfrak{t}_n(x),\ x=(-1)^{\frac{n(n+1)}{2}}\,\mathfrak{r}_n(x)\Big\},\ \alpha=\pm 1.$$

$$\Big({}^{-}E\mathcal{D}^n=\Big\{x\in E^n;\ x=(-1)^n\,\mathfrak{t}_n(x),\ x=\alpha(-1)^{\frac{n(n+1)}{2}}\,\mathfrak{r}_n(x),\alpha=-1\Big\}\Big).$$

Definition 2.3. (i) The $n^{\underline{\text{th}}}$dimendsional cohomology of the complex $C(\mathcal{F})$ is called the n-dimensional simplicial cohomolgy of cosimplicial space $\mathcal{F}(E,\mathfrak{d},\mathfrak{s})$ and denoted by $\mathcal{H}^n(\mathcal{F})$.

(ii) The $n^{\underline{\text{th}}}$dimensional cohomology of the complex $CC(\mathcal{F})$ is called the $n^{\underline{th}}$dimensional cyclic cohomology of $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t})$ and denoted by $\mathcal{H}C^n(\mathcal{F})$.

(iii) The $n^{\underline{\text{th}}}$dimensional cohomology of the complex $C\mathcal{D}(\mathcal{F})$ $(C\mathcal{R}(\mathcal{F}))$ is called the $n^{\underline{\text{th}}}$dimensional dihedral (reflexive) cohomology of $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t},\mathfrak{r})$ $(\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{r}))$, and denoted by ${}^{\alpha}\mathcal{H}\mathcal{D}^n(\mathcal{F})$ $({}^{\alpha}\mathcal{H}\mathcal{R}^n(\mathcal{F}))$, $\alpha=\pm 1$.

3. The relation between the cyclic and dihedral cohomologies

In this part, we study the concepts of the $\underline{\mathfrak{K}}$-modules $\underline{\Delta}_n$, $\underline{\mathfrak{C}}_n$, $\underline{\Omega}_n$ and $\underline{\Xi}_n$ in the category $\mathcal{B}$, express the cyclic and dihedral cohomology in terms of Ext functor and prove that the dihedral cohomology is a direct summand of cyclic cohomology.

In what follows let $\Theta_n = (-1)^n \tau_n$ and $R_n = (-1)^{\frac{n(n+1)}{2}} \rho_n$; obviously $\Theta^{n+1} = R^2 = 1$. Consider the following diagram in the category $\mathcal{B}$:

$$
\begin{array}{ll}
 & 0 \longrightarrow \underline{\mathfrak{C}}_n \underset{P_1'}{\overset{N_1}{\rightleftarrows}} \underline{\Xi}_n \underset{L_1}{\overset{M_1}{\rightleftarrows}} \underline{\Xi}_n \underset{N_1'}{\overset{P_1}{\rightleftarrows}} \underline{\mathfrak{C}}_n \longrightarrow 0 \\
(3\text{-}1) & \\
 & 0 \longrightarrow \underline{\Omega}_n \underset{P_2'}{\overset{N_2}{\rightleftarrows}} \underline{\Xi}_n \underset{L_2}{\overset{M_2}{\rightleftarrows}} \underline{\Xi}_n \underset{N_2'}{\overset{P_2}{\rightleftarrows}} \underline{\Omega}_n \longrightarrow 0
\end{array}
$$

where

$$
\begin{aligned}
&N_1 : a \longrightarrow a\left(1_n + \Theta_n + \ldots + \Theta_n^n\right), \\
&M_1 : a \longrightarrow a(1_n - \Theta_n), \\
&P_1 : \xi\Theta^k \longrightarrow \xi, \text{ where } \xi \in [n,\, .\,]_\Omega,\ 0 \le k \le n, \\
&P_1' = \frac{1}{n+1} P_1, \qquad N_1' = \frac{1}{n+1} N_1, \\
&L_1 : a \longrightarrow \frac{a}{n+1} \sum_{k=0}^{n-1} \left(1_n + \Theta_n + \ldots + \Theta_n^k\right), \\
&N_2 : a \longrightarrow a\left(1_n + R_n\right), \qquad M_2 : a \longrightarrow a\left(1_n - R_n\right), \\
&P_2 : \xi R_n^k \longrightarrow \xi, \text{ where } \xi \in [n,\, .\,]_{\mathfrak{C}},\ 0 \le k \le n, \\
&P_2' = \frac{1}{2} P_2,\ N_2' = \frac{1}{2} N_2,\ L_2 = \frac{1}{2} 1_\Xi .
\end{aligned}
$$

Note that (see [4]), any morphism $\xi \in [n,\, .\,]_\Xi$ can uniquely, determined by $\xi = \eta \circ \mu$ where $\mu \in [n, m]_\Delta$ and η is an isomorhism from n into n, generated by the operators τ_n and ρ_n. It is easy to prove that the rows in the diagrams (3-1) are complexes in $\mathcal{B}$. These complexes, by [3], split in $\mathcal{B}$.

Lemma 3.1. *In the diagram (3-1), the following holds:*
(i) *The relations*

$$
\begin{aligned}
a \cdot x &= P_1'(a \cdot N_1(x)) = P_1(a \cdot N_1'(x)), \\
a \cdot y &= P_2'(a \cdot N_2(y)) = P_2(a \cdot N_2'(y)),
\end{aligned}
\tag{3-2}
$$

give the structure of left Banach $\underline{\Xi}$-module on $\underline{\Omega}_n$ and $\underline{\mathfrak{C}}_n$.

(ii) *The families $\{P_1', L_1, N_1'\}$ and $\{P_2', L_2, N_2'\}$ of morphisms are contracting homotopies.*

Proof. (i) Taking the second relation of the complex (3-1), since M_2 is a $\underline{\Xi}$-module morphism and the complex (3-1) is split in $\mathcal{B}$, then the morphism N_2 induces a linear homomorphism between $\underline{\Omega}_n$ and a submodule $\operatorname{Im} N_2 = \ker M_2$ in $\underline{\Xi}$, and the morphism P_2 induces a linear homomorphisms between the module $\underline{\Omega}_n$ and the quotient module $\Xi_n/\operatorname{Im} M_2 = \Xi_n/\ker P_2$. Furthermore $P_2'(N_2(y)) = P_2(N_2'(y)) = y$, for any $y \in \underline{\Omega}_n$. Hence $a \cdot y = P_2'(a \cdot N_2(y)) = P_2(a \cdot N_2'(y))$.

Similarly we can show that $a \cdot x = P_1'(a \cdot N_1(x)) = P_1(a \cdot N_1'(x))$. To prove that (3-2) are Banach $\underline{\Xi}$-module it is sufficient to prove that N_2 and P_2 are morphisms in the category $\underline{\Xi}$-mod. Since the composition $N_2 \circ P_2'$ on $\operatorname{Im} N_2 \in \underline{\Xi}$, then $N_2(a \cdot y) = N_2(P_2'(a \cdot N_2(y)) = aN_2(y)$. Let $y \in \underline{\Xi}_n$, clearly that $a \cdot y = N_2P_2'(a \cdot y) + L_2M_2(a \cdot y) = N_2P_2'(a \cdot y) + a \cdot L_2M_2(y)$, then $N_2P_2'(a \cdot y) = a \cdot (y - L_2M_2(y)) = a \cdot N_2P_2'(y)$, then $P_2'(a \cdot y) = P_2'N_2(P_2'(a \cdot y) = P_2'(N_2P_2'(a \cdot y)) = P_2'(a \cdot N_2P_2'(y)) = P_2'(N_2(a \cdot P_2'(y))) = aP_2'(y)$. Hence it is easy to check that N_2 and P_2 are $\underline{\Xi}$-module. Similarly N_1 and P_1 are $\underline{\Xi}$-module morphisms.

(ii) Since all morphisms in the diagram (3-1) are $\underline{\Xi}$-module morphisms and the complex (3-1) is split in the category Ξ-mod, then the families $\{P_2', L_2, N_2'\}$ and $\{P_2', L_2, N_2'\}$ are contracting homotopies.

Lemma 3.2. *The modules $\underline{\mathfrak{C}}_n$ and $\underline{\Omega}_n$ are projective Ξ-modules.*

Proof. These follow by Lemmas 3.1, 1.9 and (Proposition 15) in [3], since the modules $\underline{\mathfrak{C}}_n$ and $\underline{\Omega}_n$ are retract of the projective $\underline{\Xi}_n$-module $\underline{\mathfrak{C}}_n$ and $\underline{\Omega}_n$ and the retract projective Banach module is projective.

Consider now the following diagram:

$$
\begin{array}{lccccccccc}
 & 0 & \longrightarrow & \Delta_n & \xrightarrow{N} & \Omega_n \oplus \mathfrak{C}_n & \xrightarrow{M} & \Omega_n \oplus \Xi_n \oplus \mathfrak{C}_n & \longrightarrow & \\
\text{(3-3)} & & & \downarrow & & {\scriptstyle P}\downarrow & & {\scriptstyle L}\downarrow & & \\
 & & & 0 & \longrightarrow & \Delta_n & \xrightarrow[N]{} & \Omega_n \oplus \mathfrak{C}_n & \xrightarrow[M]{} & \Omega_n \oplus \Xi_n \oplus \mathfrak{C}_n \longrightarrow
\end{array}
$$

$N : a \longrightarrow (a(1_n + R_n); -a(1_n + \Theta_n, \dots, \Theta_n^n))$,
$M : (a, b) \longrightarrow (a(1_n - R_n); a\,(1_n + \Theta_n, \dots, \Theta_n^n) + b(1_n + R_n); b(1 - \Theta_n)$,

$P:\big(\xi R_n^k,\eta\Theta_n^\ell\big)\rightarrow\frac{1}{2}\left(\frac{1}{2}\xi-\frac{1}{n+1}\eta\right)$, where $\xi,\eta\in\Delta[n,\,.]$,

$$L\ :\ \big(a,\xi R_n^k\Theta_n^\ell,\ell\big)\ \rightarrow\ \left(\frac{1}{2}a+\frac{1}{2}\xi R_n^k,\frac{2}{n+1}\sum_{k=0}^{n-1}\Big(1_n+\Theta_n,\dots,\Theta_n^k\Big)+\xi\Theta_n^d\right),$$

where

$$d=\left[\begin{array}{ll}\ell, & \text{if } k=0,\\ n+1-\ell, & \text{if } k=1 \quad\text{and}\quad \xi\in[n,\,.\,]_\Delta.\end{array}\right.$$

The direct calculation shows that $NP+LM=1_{\underline{\Omega}_n\oplus\underline{\mathfrak{C}}_n}$, $PN=1_{\underline{\Delta}_n}$. Let us prove now that M and L in the diagram (3-3) are $\underline{\Xi}$-module morphisms. To do this consider the following diagrams, related to the diagram (3-1);

$$(i)\quad \begin{tikzcd} \underline{\mathfrak{C}}_n \arrow[r] \arrow[d, "N_1"'] & \underline{\mathfrak{C}}_n \arrow[d, shift right, "N_1"'] \\ \underline{\Xi}_n \arrow[r, "M_1"'] & \underline{\Xi}_n \arrow[u, shift right, "P_1'"'] \end{tikzcd} \qquad (ii)\quad \begin{tikzcd} \underline{\mathfrak{C}}_n \arrow[d, "N_1"'] \\ \Xi_n \\ \underline{\Omega}_n \arrow[u, "N_2"] \end{tikzcd} \qquad (iii)\quad \begin{tikzcd} \underline{\Omega}_n \arrow[r] \arrow[d, "N_2"'] & \underline{\Omega}_n \arrow[d, shift right, "N_2"'] \\ \underline{\Xi}_n \arrow[r, "M_2"'] & \underline{\Xi}_n \arrow[u, shift right, "P_2'"'] \end{tikzcd}$$

From diagram (3-3), clearly the diagrams (i), (ii) and (iii) respectively, describe the image of the morphism M as follows: $(\underline{\Omega}_n,0)\rightarrow(\underline{\Omega}_n,0,0)$, $(\underline{\Omega}_n,\underline{\mathfrak{C}}_n)\rightarrow(0,\underline{\Xi}_n,0)$, $(0,\underline{\mathfrak{C}}_n)\rightarrow(0,0,\underline{\mathfrak{C}}_n)$. This means that the action of morphism M can be described, either a composition of morphisms in $\underline{\Xi}$-mod as (i), (iii), or a sum of morphisms in $\underline{\Xi}$-mod as (ii). Thus M is $\underline{\Xi}$-module morphism. Similarly we can show that L is $\underline{\Xi}$-module morphism. By Lemmas 3.1 and 3.2 the following is easily established:

Lemma 3.3. (i) *The relation $a\cdot x=P(a\cdot N(x))$ gives $\underline{\Delta}_n$ the structure of left Banach Ξ-module.*

(ii) *The morphisms N and P are $\underline{\Xi}$-module morphisms.*

(iii) *The module $\underline{\Delta}_n$ is a projection $\underline{\Xi}$-module.*

Note that if we replace the maps R_n by $-R_n$ in the precending discussion we get the $\underline{\Xi}$-module $\underline{\Delta}_n^-$ instead of $\underline{\Delta}_n^+$. This module is also projective $\underline{\Xi}$-module.

We now induce from diagram (3-3) a direct sum modules diagram and prove that it is in $\underline{\Xi}$-mod.

Lemma 3.4. *Suppose that the diagram:*

$$\underline{\Delta}_n^+ \underset{P_u}{\overset{u}{\rightleftarrows}} \underline{\Omega}_n \underset{P_w}{\overset{w}{\leftrightarrows}} \underline{\Delta}_n^- . \tag{3-4}$$

where $u : a \longrightarrow a(1_n - R_n), \quad w : a \longrightarrow a(1_n - R_n),$
$P_u : \xi R_n^\alpha \longrightarrow \frac{1}{2}\xi, \quad P_w : \xi R_n^\alpha \longrightarrow (-1)^\alpha \frac{1}{2}\xi, \quad \alpha = \pm 1,$
is in $\mathcal{B}$. *Then it is also a diagram in* $\underline{\Xi}$*-mod.*

Proof. It is sufficient to prove that all morphisms in (3-4) are $\underline{\Xi}$-module morphisms. Consider the diagram (3-3), if we take the image of the first part of the direct sum $\Omega_n \oplus \mathfrak{C}_n$. Clearly the morphisms N and P coincide with the morphisms u, P_u. Since the morphisms N, P are $\underline{\Xi}$-module morphisms the u, P are $\underline{\Xi}$-module morphisms. Similarly the morphisms w and P_w are $\underline{\Xi}$-module morphisms.

We now consider for all $n = 0, 1, 2, \ldots,\ k = 0, 1, \ldots, n+1$ the operators: $D_n^k : \underline{\Delta}_{n+1} \longrightarrow \Delta_n : \zeta \longrightarrow \zeta\delta_{n+1}^k$ and put $D_n = \sum\limits_{k=0}^{n+1}(-1)^k D_n^k$. Similarly define the operator $\widetilde{D}_n^k : \underline{\Omega}_{n+1} \longrightarrow \underline{\Omega}_n : \zeta \longrightarrow \zeta\delta_{n+1}^k$. In addition, define $\widetilde{D}_n = \sum\limits_{k=0}^{n+1}(-1)^k \widetilde{D}_n^k$. Consider the following diagram:

$$\begin{tikzcd}
0 & \underline{\Delta}_0^+ \arrow[l] \arrow[d, shift left, "u_0"] & \underline{\Delta}_1^+ \arrow[l, "D_0"'] \arrow[d, shift left, "u_1"] & \ldots \arrow[l] & \underline{\Delta}_n^+ \arrow[l] \arrow[d, shift left, "u_n"] & \underline{\Delta}_{n+1}^+ \arrow[l, "D_n"'] \arrow[d, shift left, "u_{n+1}"] & \ldots \arrow[l] \\
0 & \underline{\Omega}_0 \arrow[l] \arrow[u, shift left, "P_{u_0}"] \arrow[d, shift right, "P_{w_0}"'] & \underline{\Omega}_1 \arrow[l, "\widetilde{D}_0"'] \arrow[u, shift left, "P_{u_1}"] \arrow[d, shift right, "P_{w_1}"'] & \ldots \arrow[l] & \underline{\Omega}_n \arrow[l] \arrow[u, shift left, "P_{u_n}"] \arrow[d, shift right, "P_{w_n}"'] & \underline{\Omega}_{n+1} \arrow[l, "\widetilde{D}_n"'] \arrow[u, shift left, "P_{u_{n+1}}"] \arrow[d, shift right, "P_{w_{n+1}}"'] & \ldots (\mathfrak{L}) \arrow[l] \\
0 & \underline{\Delta}_0^- \arrow[l] \arrow[u, shift right, "w_0"'] & \underline{\Delta}_1^- \arrow[l, "D_0"'] \arrow[u, shift right, "w_1"'] & \ldots \arrow[l] & \underline{\Delta}_n^- \arrow[l] \arrow[u, shift right, "w_n"'] & \underline{\Delta}_{n+1}^- \arrow[l, "D_n"'] \arrow[u, shift right, "w_{n+1}"'] & \ldots \arrow[l]
\end{tikzcd}$$

Lemma 3.5. *The diagram* $(\mathfrak{L})$ *is commutative.*

Proof. It is sufficient to prove that $u_n D_n = \widetilde{D}_n u_{n+1}$, $\widetilde{D}_n w_{n+1} = W_n D_n$, $P_{u_n}\widetilde{D}_n = D_n P_{u_{n+1}}$ and $\widetilde{D}_n w_{n+1} = W_n D_n$. Since

$$u_n D_n : \xi \rightarrow \sum_{k=0}^{n+1}(-1)^k \xi\delta_{n+1}^k \rightarrow \sum_{k=0}^{n+1}(-1)^k \xi\delta_{n+1}^k(1_n + R_n),$$

$$\widetilde{D}_n u_{n+1} : \xi \longrightarrow \xi(1_{n+1} + R_{n+1}) \longrightarrow \sum_{k=0}^{n+1} (-1)^k \xi(1_{n+1} + R_{n+1})\delta_{n+1}^n,$$

and

$$\begin{aligned}(-1)^k \xi R_{n+1}\delta_{n+1}^k &= (-1)^k(-1)^{\frac{(n+1)(n+2)}{2}} R_{n+1}\delta_{n+1}^k \\ &= (-1)^{k+\frac{(n+1)(n+2)}{2}} \xi\delta_{n+1}^{n+1-k} R_n \\ &= (-1)^{k+\frac{(n+1)(n+2)}{2}-\frac{n(n+1-k)}{2}} \xi\delta_{n+1}^{n+1-k} R_n \\ &= (-1)^{k+n+1}\xi\delta_{n+1}^{n+1-k} R_n,\end{aligned}$$

then $\sum_{k=0}^{n+1}(-1)^k\xi\delta_{n+1}^n R_n = \sum_{k=0}^{n+1}(-1)^k\xi R_{n+1}\delta_{n+1}^k$. So $u_n D_n = \widetilde{D}_n u_{n+1}$. Similarly we can prove that $\widetilde{D}_n W_{n+1} = W_n D_n$. Now we shall show that $P_{u_n}\widetilde{D}_n = D_n P_{u_{n+1}}$. Let $\eta = \xi R_{n+1}^{\alpha}$, where $\xi \in [n+1]_\Delta$, then $D_n P_{u_{n+1}}(\eta) = \frac{1}{2}\sum_{k=0}^{n+1}(-1)^k\xi\delta_{n+1}^k$,

$$\begin{aligned}P_{u_n}\widetilde{D}_n(\eta) &= P_{u_n}\left[\sum_{k=0}^{n+1}(-1)^k\xi R_{n+1}\delta_{n+1}^k\right] \\ &= P_{u_n}\left[\sum_{k=0}^{n+1}(-1)^k\xi\delta_{n+1}^k R_n^{\alpha}\right] = \frac{1}{2}\sum_{k=0}^{n+1}(-1)^k\xi\delta_{n+1}^k.\end{aligned}$$

Similarly we obtain $\widetilde{D}_n w_{n+1} = W_n D_n$.

Lemma 3.6. *The diagram $(\mathfrak{L})$ is in the category $\underline{\Xi}$-mod.*

Proof. It is enough to prove that D_n and $\widetilde{D}_n$ are $\underline{\Xi}$-module morphisms. To do this, consider the following diagram:

$$\begin{array}{ccc} \underline{\Omega}_n & \xrightarrow{\widetilde{D}_n} & \underline{\Omega}_{n+1} \\ {\scriptstyle N_1}\downarrow & & {\scriptstyle N_1}\downarrow \\ \underline{\Xi}_n & \xrightarrow{\widehat{D}_n} & \underline{\Xi}_{n+1} \end{array} \tag{3-5}$$

Where $\widehat{D}_n : \xi \to \sum_{k=0}^{n+1}(-1)^k\xi\delta_{n+1}^k$. Clearly the morphism $\widehat{D}_n$ is $\underline{\Xi}$-module morphisms (by definition) and the diagram (3-5) is commutative, since

$N_1\widetilde{D}_n : \xi \longrightarrow \sum_{k=0}^{n+1}(-1)^k\xi\delta_{n+1}^k \longrightarrow \sum_{\ell=0}^{n}\sum_{k=0}^{n+1}(-1)^k\xi\delta_{n+1}^k\Theta^\ell$ and $\widehat{D}_nN_1 : \xi \longrightarrow \sum_{k=0}^{n+1}\xi\Theta^\ell = \sum_{k=0}^{n+1}\sum_{\ell=0}^{n}(-1)^k\xi\Theta^\ell\delta_{n+1}^k$.

From this commutativity and injectivity of morphism N_1, the morphism $\widetilde{D}_n$ is a $\underline{\Xi}$-module morphism. Consequently D_n is a $\underline{\Xi}$-module morphism. Consider the dihedral and the cyclic Banach spaces $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t},\mathfrak{r})$, $\mathcal{F}(E,\mathfrak{d},\mathfrak{s},\mathfrak{t})$ and their associated Banach $\underline{\Xi}$-module $E(\mathcal{F})$. Consider also, in diagram $(\mathfrak{L})$, the additive functor $\mathcal{H}_{\underline{\Xi}}(\,.\,,E(\mathcal{F}))$. We get the diagram of a direct sum of normed spaces isomorphic to the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 \longrightarrow & {}^{+}E\mathcal{D}^0 & \xrightarrow{\mathfrak{bd}^0} & {}^{+}E\mathcal{D}^1 & \longrightarrow \dots \longrightarrow & {}^{+}E\mathcal{D}^{n-1} & \xrightarrow{\mathfrak{bd}^{n-1}} & {}^{+}E\mathcal{D}^n & \longrightarrow & \dots {}^{+}C\mathcal{D}(\mathcal{F}) & \\
& i\Big\uparrow\Big\uparrow\mu & & i\Big\uparrow\Big\uparrow\mu & & i\Big\uparrow\Big\uparrow\mu & & i\Big\uparrow\Big\uparrow\mu & & & \\
0 \longrightarrow & EC^0 & \xrightarrow{\mathfrak{bc}^0} & EC^1 & \longrightarrow \dots \longrightarrow & EC^{n-1} & \xrightarrow{\mathfrak{bc}^{n-1}} & EC^n & \longrightarrow & \dots CC(\mathcal{F}) \dots & (\mathfrak{Z}) \\
& i\Big\downarrow\Big\downarrow\gamma & & i\Big\downarrow\Big\downarrow\gamma & & i\Big\downarrow\Big\downarrow\gamma & & i\Big\downarrow\Big\downarrow\gamma & & & \\
0 \longrightarrow & {}^{-}E\mathcal{D}^0 & \xrightarrow{\mathfrak{bd}^0} & {}^{-}E\mathcal{D}^1 & \longrightarrow \dots \longrightarrow & {}^{-}E\mathcal{D}^{n-1} & \xrightarrow{\mathfrak{bd}^{n-1}} & {}^{-}E\mathcal{D}^n & \longrightarrow & \dots {}^{-}C\mathcal{D}(\mathcal{F}) &
\end{array}$$

where i is the natural imbedding, $\mu : x \longrightarrow x + R_n \cdot x$, $\gamma : x \longrightarrow x - R_n \cdot x$, $R_n = (-1)^{\frac{n(n+1)}{2}}\rho_n$. From Lemma 1.8 by putting $\mathfrak{K} = \Xi$, $X = E(\mathcal{F})$, we obtain a Banach space isomorphism $\Phi_{\underline{\Xi}}^n : \mathfrak{h}_{\underline{\Xi}}(\underline{\Xi}_n, E(\mathcal{F})) \simeq E^n : \varphi \longrightarrow \varphi(1_n)$. Consider also the operators:

(i) $\Psi_{\underline{\Xi}}^n : \mathfrak{h}_{\underline{\Xi}}(\underline{\Omega}_n, E(\mathcal{F})) \longrightarrow EC^n : \psi \longrightarrow \psi(1_n)$,

(ii) $S_{\underline{\Xi}}^n : \mathfrak{h}_{\underline{\Xi}}(\underline{\Delta}^+, E(\mathcal{F})) \longrightarrow^{+1} ED^n(\mathcal{F}) : \varphi \longrightarrow \varphi(1_n)$,

(iii) $X_{\underline{\Xi}}^n : \mathfrak{h}_{\Xi}(\underline{\Delta}_n^-, E(\mathcal{F})) \longrightarrow^{-1} ED^n(\mathcal{F}) : \varphi \longrightarrow \varphi(1_n)$,

(iv) $T_{\underline{\Xi}}^n : \mathfrak{h}_{\Xi}(\underline{\Delta}_n^+, E(\mathcal{F}') \longrightarrow^{-1} ED^n(\mathcal{F}) : \varphi \longrightarrow \varphi(1_n)$,

where
$\Theta_n\psi(1_n) = \psi(\Theta_n \cdot 1_n) = \psi(1_n), \Theta_n \cdot 1_n$ in $\underline{\mathfrak{C}}_n$, $R_n\psi(1_n) = \psi(R_n \cdot 1_n) = \psi(1_n)$, $R_n \cdot 1_n$ in $\underline{\Omega}_n$.

Lemma 3.7. *The operators $\Psi_{\underline{\Xi}}^n$, $S_{\underline{\Xi}}^n$, $X_{\underline{\Xi}}^n$ and $T_{\underline{\Xi}}^n$ are topological Banach space isomorphisms.*

Proof. (i) We assign for any $x \in EC^n$ a morphism $\varphi : \underline{\Xi}_n \longrightarrow E(\mathcal{F})$, such that $\varphi(a) = a\cdot x$. From $x = (-1)^n t_n(x) = \Theta_n \cdot x$ it follows that $\psi(a) =$

0 for $a \in \operatorname{Im} M_n$. Hence φ induces a morphism $\psi : \underline{\Omega}_n = \underline{\Xi}/\operatorname{Im} M_n \longrightarrow E(\mathcal{F})$, is defined by the formula $\xi \longrightarrow \xi \cdot x$. It is easily seen that the assignment $x \longrightarrow \psi$ is a continuous operator and is the inverse of $\Psi^n_{\underline{\Xi}}$.

(ii) Since $R_n \cdot 1_n = 1_n$, $\Theta_n \cdot 1_n = 1_n$, then $\varphi(1_n) = \varphi(R_n \cdot 1_n) = R_n \cdot \varphi(1_n)$ and $\varphi(1_n) = \varphi(\Theta_n \cdot 1_n) = \Theta_n \cdot \varphi(1_n)$. This means that $\varphi(1_n) \in {}^{+1}ED^n(\mathcal{F})$. The inverse map assign for every $x \in {}^{+1}ED^n(\mathcal{F})$ a continuous map $\varphi : a \longrightarrow a \cdot x$.

(iii) Since $\varphi(1_n) = \varphi(\Theta_n \cdot 1_n) = \Theta_n \cdot \varphi(1_n)$ and $\varphi(1_n) = \varphi(R_n \cdot 1_n) = R_n \cdot (-\varphi(1_n))$, then $\varphi(1_n) \in {}^{-1}ED^n(\mathcal{F})$, and the inverse map can be defined easily, as in (i).

(iv) Following (i) if $\varphi(1_n) \in {}^{+}ED^n(\mathcal{F}')$, then $\mathfrak{h}_{\Xi}(\underline{\Delta}_n, {}^{+}E(\mathcal{F}') \cong {}^{+1}ED^n(\mathcal{F}')$. Hence $\varphi(1_n) \in {}^{-}ED^n(\mathcal{F}')$ (by definition of $\mathcal{F}'$).

Lemma 3.8. *The operators $\Psi^n_{\underline{\Xi}}$, $X^n_{\underline{\Xi}}$ and $S^n_{\underline{\Xi}}$ determine an isomorphism between the diagrams $\mathfrak{h}_{\underline{\Xi}}(\mathfrak{L}, E(\mathcal{F}))$ and $(\mathfrak{Z})$.*

Proof. This follows from the commutativity of these diagrams and the definition of $E(\mathcal{F})$ as a $\underline{\Xi}$-module.

Consider now the upper row of diagram $(\mathfrak{L})$

$$0 \longleftarrow \underline{\Delta}^+_0 \xleftarrow{D_0} \underline{\Delta}^+_1 \longleftarrow \ldots \longleftarrow \underline{\Delta}^+_n \xleftarrow{D_n} \underline{\Delta}^+_{n+1} \longleftarrow \ldots (R_{\Delta^+}).$$

It is a comlex in $\underline{x}$-mod. Define the operator $D_{-1} : \underline{\Delta}^+_0 \longrightarrow \underline{\mathbb{Y}}$, which takes a morphism $\xi \in [0, m]$ into P^m. Clearly it is a $\underline{\Xi}$-module.

Lemma 3.9. *The complex:*

$$0 \longleftarrow \underline{\mathbb{Y}} \xleftarrow{D_{-1}} \underline{\Delta}^+_0 \xleftarrow{D_0} \underline{\Delta}^+_1 \longleftarrow \ldots \longleftarrow \underline{\Delta}^+_n \xleftarrow{D_n} \underline{\Delta}^+_{n+1} \longleftarrow \ldots \tag{3-6}$$

is a projective resolution of a module $\underline{\mathbb{Y}}$ in $\underline{\Xi}$-mod.

Proof. This follows from the fact that the complex (3-6) is admissible. This fact have proved in [3].

It should be remarked that the complex (3-6) is also a projective resolution of a module $\underline{\mathbb{Y}}$ in $\underline{\Xi}$-mod if we replace (R_{Δ^+}) by (R_{Δ^-}).

Now we can generalize the Definition 1.7, by taking the different cases of $\mathfrak{K}$-Banach space and express the simplicial, cyclic and dihedral cohomology in term of Ext functor.

Theorem 3.10. (i) *let $\mathcal{F}(E, \mathfrak{d}, \mathfrak{s})$ be a cosimplicial Banach space, then the simplicial cohomology of $\mathcal{F}$ is given by;* $\mathcal{H}^n(\mathcal{F}) \cong \mathrm{Ext}^n_{\underline{\Delta}}(\underline{\mathbb{Y}}, E(\mathcal{F}))$.

(ii) *Let $\mathcal{F}(E, \mathfrak{d}, \mathfrak{s}, \mathfrak{t})$ be a cocyclic Banach space, then the cyclic cohomology of $\mathcal{F}$ is given by;* $\mathcal{H}C^n(\mathcal{F}) \cong \mathrm{Ext}^n_{\underline{\mathfrak{C}}}(\underline{\mathbb{Y}}, E(\mathcal{F}))$.

(iii) *Let $\mathcal{F}(E, \mathfrak{d}, \mathfrak{s}, \mathfrak{t}, \mathfrak{r})$ be a codihedral Banach space, then the dihedral cohomology of $\mathcal{F}$ is given by;* ${}^{\alpha}\mathcal{HD}^n(\mathcal{F}) = \mathrm{Ext}^n(\underline{\mathbb{Y}}, E(\mathcal{F}))$, $\alpha = \pm 1$.

Proof. (i), (ii) (see [3]).

(iii) By Lemma 3.9 the space $\mathrm{Ext}_{\underline{\Xi}}(\underline{\mathbb{Y}}, E(\mathcal{F}))$ are the cohomology group of the complex $\mathfrak{h}_{\Xi}(\mathcal{R}_{\Delta}\alpha, E(\mathcal{F}))$, $\alpha = \pm 1$, by Lemma 3.8 the complexes $\mathfrak{h}_{\Xi}(\mathcal{R}_{\Delta}\alpha, E(\mathcal{F}))$ and ${}^{\alpha}ED^n(\mathcal{F})$ are isomorphic. Then (iii) follows. From all preceding discussion we state the main result of this paper.

Theorem 3.10. *The following isomorphism holds:*

$$\mathcal{H}C^n(\mathcal{F}) \cong {}^{-}\mathcal{HD}^n(\mathcal{F}) \oplus {}^{+}\mathcal{HD}^n(\mathcal{F}). \tag{3-8}$$

Poof. This follows from $(\mathfrak{Z})$ and Definition 2.3.

Example 1 [10]. Let A be a C^*-algebra without bounded traces, then the dihedral cohomology of algebra A vanishes, that is

$${}_{\alpha}\mathcal{HD}^n(A) = 0, \quad n \geq 0, \quad \alpha = \pm 1.$$

Proof. Following [1], the cyclic cohomology of C^*-algebra A without bounded traces vanishes. Then from this fact and using relation (3-8), it is clear that the dihedral cohomology ${}_{\alpha}\mathcal{HD}^n(A)$ vanishes.

Example 2 [10]. Let A be a nuclear C^*-algebra, then

$${}_{\alpha}\mathcal{HD}^{2k+1}(A) = 0, \quad {}_{\alpha}\mathcal{HD}^{2k}(A) = {}_{\alpha}A^{tr}, \quad k \geq 0,\ \alpha = (-1)^k,$$

where ${}_{\alpha}A^{tr}$ is the space of all bounded traces on A satisfies the condition that $tr(a^*) = \alpha\ tr(a)$, $\alpha = \pm 1$. The proof is directly follows from [1] and relation (3-8).

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