Directed, topological and transitive relators

By Á. SZÁZ (Debrecen)

Introduction

Relators are simply nonvoid families of reflexive relations on sets. They are straightforward generalizations of the various uniformities [27], and are essentially identical to the generalized uniformities of KONISHI [14] and KRISHNAN [16] and to the connector systems of NAKANO—NAKANO [24].

Relators were proposed in our former paper [33] as the most suitable basic terms which topology and analysis should be based on. In [33], we have mainly studied limits, closures and closed sets in relator spaces and uniform, proximal and topological continuities of functions from a relator space into another.

In the present paper, we aim to provide a primary classification for relator spaces which is necessary to formulate and prove generalized forms of many of the important theorems of topology and analysis. To this end, the following fundamental properties of a relator $\mathscr{R}$ on a set X will be introduced and investigated.

Uniform directedness: $T\subset R\cap S$ for some $T\in\mathscr{R}$ whenever $R, S\in\mathscr{R}$.

Proximal directedness: $T(A)\subset R(A)\cap S(A)$ for some $T\in\mathscr{R}$ whenever $A\subset X$ and $R, S\in\mathscr{R}$.

Topological directedness: $T(x)\subset R(x)\cap S(x)$ for some $T\in\mathscr{R}$ whenever $x\in X$ and $R, S\in\mathscr{R}$.

Strong topologicalness: $R(x)^{\circ}=R(x)$ for all $x\in X$ and $R\in\mathscr{R}$.

Topologicalness: $x\in R(x)^{\circ\circ}$ for all $x\in X$ and $R\in\mathscr{R}$.

Weak topologicalness: $\overline{\overline{\{x\}}}=\overline{\{x\}}$ for all $x\in X$.

Strong transitiveness: $R\circ R=R$ for all $R\in\mathscr{R}$.

Uniform transitiveness: $T\circ S\subset R$ for some $S, T\in\mathscr{R}$ whenever $R\in\mathscr{R}$.

Proximal transitiveness: $T\big(S(A)\big)\subset R(A)$ for some $S, T\in\mathscr{R}$ whenever $A\subset X$ and $R\in\mathscr{R}$.

Topological transitiveness: $T(S(x))\subset R(x)$ for some $S, T\in\mathscr{R}$ whenever $x\in X$ and $R\in\mathscr{R}$.

Weak transitiveness: $(\bigcap\mathscr{R})\circ(\bigcap\mathscr{R})=\bigcap\mathscr{R}$.

The results obtained mainly answer the following important questions: 1. How can these properties be expressed in terms of the induced limits, closures or closed sets? 2. To what extent are these properties preserved under uniform, proximal or topological equivalence of relators? 3. When can a given relator with some of these properties be replaced by a better equivalent one?

In particular, some of the results of ČECH [2], DAVIS [5], EFRĔMOVIČ—ŠVARC [8], HUŠEK [11], KELLEY [13], KONISHI [14], LEVINE [17], MORDKOVIČ [19], MURDESHWAR—

NAIMPALLY [21], NAKANO—NAKANO [24], NIEMYTSKI [26], PERVIN [29] and WILLIAMS [36] are generalized, improved or reformulated. For the details, see the "Notes and comments" section.

The only prerequisites for reading this paper are a basic knowledge of relations and nets [13] and a familiarity with a few things from our former paper [33] which will be briefly laid out in the next preparatory section.

Notation and terminology

By a relator space we mean an ordered pair $X(\mathcal{R})=(X, \mathcal{R})$ consisting of a set X and a nonvoid family $\mathcal{R}$ of reflexive relations on X which we call a relator on X.

Denoting by $\mathcal{N}=\mathcal{N}(X)$ and $\mathcal{P}=\mathcal{P}(X)$ the classes of all nets and sets in X, respectively, we introduce the next basic tools in $X(\mathcal{R})$:

$$\mathrm{Lim}_{\mathcal{R}},\ \mathrm{Adh}_{\mathcal{R}} \subset \mathcal{N}\times\mathcal{N},\quad \mathrm{lim}_{\mathcal{R}},\ \mathrm{adh}_{\mathcal{R}} \subset \mathcal{N}\times X,$$

$$\mathrm{Cl}_{\mathcal{R}},\ \mathrm{Int}_{\mathcal{R}} \subset \mathcal{P}\times\mathcal{P},\quad \mathrm{cl}_{\mathcal{R}},\ \mathrm{int}_{\mathcal{R}} \subset \mathcal{P}\times X$$

and $\mathcal{F}_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}\subset\mathcal{P}$ such that for any $(x_\alpha), (y_\alpha)\in\mathcal{N}$, $A, B\in\mathcal{P}$ and $x\in X$ we have

(i) $(y_\alpha)\in\mathrm{Lim}_{\mathcal{R}}((x_\alpha))$ $\big((y_\alpha)\in\mathrm{Adh}_{\mathcal{R}}((x_\alpha))\big)$ iff $((y_\alpha, x_\alpha))$

is eventually (frequently) in each $R\in\mathcal{R}$;

(ii) $x\in\mathrm{lim}_{\mathcal{R}}((x_\alpha))$ $\big(x\in\mathrm{adh}_{\mathcal{R}}((x_\alpha))\big)$ iff $(x)\in\mathrm{Lim}_{\mathcal{R}}((x_\alpha))$ $\big((x)\in\mathrm{Adh}_{\mathcal{R}}((x_\alpha))\big)$;

(iii) $B\in\mathrm{Cl}_{\mathcal{R}}(A)$ $\big(B\in\mathrm{Int}_{\mathcal{R}}(A)\big)$ iff $A\cap R(B)\neq\emptyset$ $\big(R(B)\subset A\big)$

for all (some) $R\in\mathcal{R}$;

(iv) $x\in\mathrm{cl}_{\mathcal{R}}(A)$ $\big(x\in\mathrm{int}_{\mathcal{R}}(A)\big)$ iff $\{x\}\in\mathrm{Cl}_{\mathcal{R}}(A)$ $\big(\{x\}\in\mathrm{Int}_{\mathcal{R}}(A)\big)$;

(v) $A\in\mathcal{F}_{\mathcal{R}}$ $(A\in\mathcal{T}_{\mathcal{R}})$ iff $\mathrm{cl}_{\mathcal{R}}(A)=A$ $\big(\mathrm{int}_{\mathcal{R}}(A)=A\big)$.

Trusting the reader's good sense to avoid confusion, we shall always use the simplied notations $y_\alpha\in\underset{\alpha}{\mathrm{Lim}}_{\mathcal{R}}\, x_\alpha$, $y_\alpha\in\underset{\alpha}{\mathrm{Adh}}_{\mathcal{R}}\, x_\alpha$, $x\in\underset{\alpha}{\mathrm{lim}}_{\mathcal{R}}\, x_\alpha$ and $x\in\underset{\alpha}{\mathrm{adh}}_{\mathcal{R}}\, x_\alpha$. Moreover, when it is convenient, we shall simply write $\bar{A}$ and $\mathring{A}$ instead of $\mathrm{cl}_{\mathcal{R}}(A)$ and $\mathrm{int}_{\mathcal{R}}(A)$, respectively.

We define a function (or a relation) f from a relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$ to be $(\mathcal{R}, \mathcal{S})$ — continuous if $f^{-1}\circ\mathcal{S}\circ f\subset\mathcal{R}$, i.e., $f^{-1}\circ S\circ f\in\mathcal{R}$, for all $S\in\mathcal{S}$.

To obtain the most important continuity properties of f as particular cases of the above definition, we introduce the following basic refinements of a relator $\mathcal{R}$ on X:

$$\mathcal{R}^* = \{S\subset X\times X\colon\ \exists R\in\mathcal{R}\colon\ R\subset S\},$$

$$\mathcal{R}^\# = \{S\subset X\times X\colon\ \forall A\subset X\colon\ \exists R\in\mathcal{R}\colon\ R(A)\subset S(A)\},$$

$$\hat{\mathcal{R}} = \{S\subset X\times X\colon\ \forall x\in X\colon\ \exists R\in\mathcal{R}\colon\ R(x)\subset S(x)\}.$$

After this, f may be called uniformly, proximally, resp. topologically continuous if it is $(\mathcal{R}^*, \mathcal{S})$-, $(\mathcal{R}^\#, \mathcal{S})$-, resp. $(\hat{\mathcal{R}}, \mathcal{S})$-continuous.

The corresponding comparisons of relators $\mathscr{R}$ and $\mathscr{S}$ on the same set are to be defined accordingly. For instance, $\mathscr{R}$ may be called proximally finer than (equivalent to) $\mathscr{S}$ if $\mathscr{S}\subset\mathscr{R}^{\#}$ ($\mathscr{S}^{\#}=\mathscr{R}^{\#}$). Similarly, we can also say that $\mathscr{R}$ is proximally fine if $\mathscr{R}^{\#}=\mathscr{R}$.

It is not hard to see that the above basic refinements are closely related to the induced basic tools. For instance, a function f from $X(\mathscr{R})$ into $Y(\mathscr{S})$ is $(\mathscr{R}^{\#}, \mathscr{S})$-continuous iff $B\in \mathrm{Cl}_{\mathscr{R}}(A)$ implies $f(B)\in\mathrm{Cl}_{\mathscr{S}}(f(A))$, or equivalently $B\in\mathrm{Int}_{\mathscr{S}}(A)$ implies $f^{-1}(B)\in\mathrm{Int}_{\mathscr{R}}(f^{-1}(A))$. Hence, by letting f be the identity function of X, one can easily derive that $\mathscr{R}^{\#}$ is the largest relator on X such that $\mathrm{Cl}_{\mathscr{R}^{\#}}=\mathrm{Cl}_{\mathscr{R}}$ ($\mathrm{Int}_{\mathscr{R}^{\#}}=\mathrm{Int}_{\mathscr{R}}$).

If $\mathscr{R}_\alpha$ is a relator on X_α for all α in a nonvoid set Γ and π_α is the projection of $X=\bigtimes_{\alpha\in\Gamma} X_\alpha$ onto X_α for all $\alpha\in\Gamma$, then the relator

$$\mathscr{R}=\bigcup_{\alpha\in\Gamma}\pi_\alpha^{-1}\circ\mathscr{R}_\alpha\circ\pi_\alpha$$

on X is called the projectively generated product of the relators $\mathscr{R}_\alpha$.

Clearly, $\mathscr{R}$ is the smallest relator on X such that each π_α is $(\mathscr{R}, \mathscr{R}_\alpha)$-continuous. However, the relator $\mathscr{R}$ is usually too small for several purposes. For instance, if $\Gamma=\{1, 2\}$ and $A\subset X$, then we only have

$$\mathrm{cl}_{\mathscr{R}}(A)=\bigcap\mathscr{R}_2^{-1}\circ\{A\circ X^{-1}\circ A\}\circ\mathscr{R}_1$$

as a striking analogue of [21, Theorem 1.37].

Therefore, we have also to consider some larger relators on the product set X. An immediate candidate for this is the relator

$$\mathscr{R}'=\{\bigcap\mathscr{A}\colon\ \emptyset\neq\mathscr{A}\subset\mathscr{R}\text{ is finite}\}.$$

However, sometimes a direct product of the relators $\mathscr{R}_\alpha$ is more appropriate than $\mathscr{R}'$.

If for each $(R_\alpha)_{\alpha\in\Gamma}\in\bigtimes_{\alpha\in\Gamma}\mathscr{R}_\alpha$, $\bigotimes_{\alpha\in\Gamma}R_\alpha$ is the relation on X such that

$$\Big(\bigotimes_{\alpha\in\Gamma}R_\alpha\Big)(x)=\bigtimes_{\alpha\in\Gamma}R_\alpha(x_\alpha)$$

for all $x=(x_\alpha)_{\alpha\in\Gamma}\in X$, then the relator

$$\bigotimes_{\alpha\in\Gamma}\mathscr{R}_\alpha=\Big\{\bigotimes_{\alpha\in\Gamma}R_\alpha\colon\ (R_\alpha)_{\alpha\in\Gamma}\in\bigtimes_{\alpha\in\Gamma}\mathscr{R}_\alpha\Big\}$$

on X will be called the direct product of the relators $\mathscr{R}_\alpha$.

It is clear that $\mathscr{R}\subset\bigotimes_{\alpha\in\Gamma}\mathscr{R}_\alpha$ if $X_\alpha\times X_\alpha\in\mathscr{R}_\alpha$ for all $\alpha\in\Gamma$, and $\bigotimes_{\alpha\in\Gamma}\mathscr{R}_\alpha\subset\mathscr{R}'$ if Γ is finite. Moreover, one can also easily check that if $\Gamma=\{1, 2\}$ and $A\subset X$, then we have

$$\mathrm{cl}_{\mathscr{R}_1\otimes\mathscr{R}_2}(A)=\bigcap\mathscr{R}_2^{-1}\circ A\circ\mathscr{R}_1$$

as a certain improvement of [21, Theorem 1.37].

1. Directed relators

Definition 1.1. A relator $\mathscr{R}$ on X, or a relator space $X(\mathscr{R})$, will be called

(i) uniformly directed if for each $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T \subset R \cap S$;

(ii) proximally directed if for each $A \subset X$ and $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T(A) \subset R(A) \cap S(A)$;

(iii) topologically directed if for each $x \in X$ and $R, S \in \mathscr{R}$ there exists $T \in \mathscr{R}$ such that $T(x) \subset R(x) \cap S(x)$.

Remark 1.2. Clearly "uniformly directed" implies "proximally directed", and "proximally directed" implies "topologically directed".

On the other hand, the next useful example shows that the converse implications do not, in general, hold.

Example 1.3. Let X be a set and $\mathscr{A}$ be a nonvoid family of subsets of X. For each $A \in \mathscr{A}$, define the relation R_A on X such that $R_A(x) = A$ if $x \in A$ and $R_A(x) = X$ if $x \in X \setminus A$. Then

$$\mathscr{R}_{\mathscr{A}} = \{R_A:\ A \in \mathscr{A}\}$$

is a relator on X such that

(i) $\mathscr{R}_{\mathscr{A}}$ is uniformly diercted if and only if $\mathscr{A} \subset \{\emptyset, A, X\}$ for some $A \subset X$;

(ii) $\mathscr{R}_{\mathscr{A}}$ is proximally directed if and only if $A \cap B \in \mathscr{A}$ for all $A, B \in \mathscr{A}$ with $\emptyset \neq A \cap B \neq X$;

(iii) $\mathscr{R}_{\mathscr{A}}$ is topologically directed if and only if for each $A, B \in \mathscr{A}$ and $x \in A \cup B$ there exists $C \in \mathscr{A}$ such that $x \in C \subset A \cap B$.

The proof is quite straightforward, but rather lengthy. For instance, we shall only prove here the "only if part" of (ii). For this, let $\mathscr{R}$ be proximally directed, and assume on the contrary that there exist $A, B \in \mathscr{A}$ with $\emptyset \neq A \cap B \neq X$ such that $A \cap B \notin \mathscr{A}$. Then, because of the proximal directedness of $\mathscr{R}$, there exists $C \in \mathscr{A}$ such that

$$R_C(A \cap B) \subset R_A(A \cap B) \cap R_B(A \cap B).$$

Hence, since $A \cap B \neq \emptyset$, it follows that

$$R_C(A \cap B) \subset A \cap B.$$

From this latter inclusion, using $A \cap B \neq X$, we can infer that $A \cap B \setminus C = \emptyset$, i.e., $A \cap B \subset C$. Now, taking into account $\emptyset \neq A \cap B \subset C$, from the above inclusion we can also infer that $C \subset A \cap B$. Consequently, we have $C = A \cap B$, which is a contradiction.

Remark 1.4. Note that the condition given in (iii) can be briefly expressed by saying that $\mathscr{A}$ is a base for a topology on $\bigcup \mathscr{A}$.

The importance of uniformly directed relators lies mainly in the fact that they do not need nondirected nets.

This is well shown by the next three basic theorems which rest upon the same arguments as [33, Theorems 1.11, 3.1 and 5.2].

Theorem 1.5. *If (x_α) and (y_α) are directed nets in a uniformly directed relator space $X(\mathscr{R})$, then the following assertions are equivalent:*

(*i*) $y_\alpha \in \operatorname{Adh}_{\mathscr{R}} x_\alpha$;

(*ii*) $w_\beta \in \operatorname{Lim}_{\beta} {}_{\mathscr{R}}\, z_\beta$ *for some directed subnet* $((z_\beta, w_\beta))$ *of* $((x_\alpha, y_\alpha))$.

Theorem 1.6. *If A and B are sets in a uniformly directed relator space* $X(\mathscr{R})$, *then the following assertions are equivalent:*

(i) $B \in \mathrm{Cl}_{\mathscr{R}}(A)$;

(ii) $y_\alpha \in \operatorname{Lim}_{\alpha} {}_{\mathscr{R}}\, x_\alpha$ *for some directed net* $((x_\alpha, y_\alpha))$ *in* $A \times B$.

Theorem 1.7. *If f is a function from a uniformly directed relator space* $X(\mathscr{R})$ *into an arbitrary one* $Y(\mathscr{S})$, *then the following assertions are equivalent:*

(*i*) *f is* $(\mathscr{R}^*, \mathscr{S})$ — *continuous;*

(*ii*) $y_\alpha \in \operatorname{Lim}_{\alpha} {}_{\mathscr{R}}\, x_\alpha$ *implies* $f(y_\alpha) \in \operatorname{Lim}_{\alpha} {}_{\mathscr{S}}\, f(x_\alpha)$ *for any two directed nets* (x_α) *and* (y_α) *in X.*

Concerning uniform directedness, one can also easily prove the following useful

Theorem 1.8. *If* $\mathscr{R}$ *is a relator on X, then the following assertions are equivalent:*

(*i*) $\mathscr{R}$ *is uniformly directed;*

(*ii*) $\mathscr{R}^*$ *is uniformly directed.*

Hence, it is clear that we also have

Corollary 1.9. *If* $\mathscr{R}$ *and* $\mathscr{S}$ *are uniformly equivalent relators on X, then* $\mathscr{R}$ *is uniformly directed if and only if* $\mathscr{S}$ *is uniformly directed.*

The importance of proximally directed relators lies mainly in the next fundamental

Theorem 1.10. *If* $\mathscr{R}$ *is a relator on X, then the following assertions are equivalent:*

(*i*) $\mathscr{R}$ *is proximally directed;*

(*ii*) $\mathrm{Int}_{\mathscr{R}}(A \cap B) = \mathrm{Int}_{\mathscr{R}}(A) \cap \mathrm{Int}_{\mathscr{R}}(B)$ *for all* $A, B \subset X$.

(*iii*) $\mathrm{Cl}_{\mathscr{R}}(A \cup B) = \mathrm{Cl}_{\mathscr{R}}(A) \cup \mathrm{Cl}_{\mathscr{R}}(B)$ *for all* $A, B \subset X$.

PROOF. Suppose that (i) holds and $C \in \mathrm{Int}_{\mathscr{R}}(A) \cap \mathrm{Int}_{\mathscr{R}}(B)$. Then, by the definition of $\mathrm{Int}_{\mathscr{R}}$, there exist $R, S \in \mathscr{R}$ such that $R(C) \subset A$ and $S(C) \subset B$. Moreover, by (i), there exists $T \in \mathscr{R}$ such that $T(C) \subset R(C) \cap S(C)$. Consequently, $C \in \mathrm{Int}_{\mathscr{R}}(A \cap B)$. Hence, since the inclusion $\mathrm{Int}_{\mathscr{R}}(A \cap B) \subset \mathrm{Int}_{\mathscr{R}}(A) \cap \mathrm{Int}_{\mathscr{R}}(B)$ is always true, it is clear that (ii) also holds.

Conversely, suppose now that (ii) holds, and let $A \subset X$ and $R, S \in \mathscr{R}$. Then, again by the definition of $\mathrm{Int}_{\mathscr{R}}$, it is clear that $A \in \mathrm{Int}_{\mathscr{R}}(R(A))$ and $A \in \mathrm{Int}_{\mathscr{R}}(S(A))$. Hence, by the essential part of (ii), it follows that $A \in \mathrm{Int}_{\mathscr{R}}(R(A) \cap S(A))$, i.e., $T(A) \subset R(A) \cap S(A)$ for some $T \in \mathscr{R}$. Consequently, (i) also holds.

The equivalence of (ii) and (iii) is immediate from the fact that $\mathrm{Cl}_{\mathscr{R}}(A) = \mathscr{P}(X) \setminus \mathrm{Int}_{\mathscr{R}}(X \setminus A)$ for all $A \subset X$.

As an immediate consequence of this theorem, we have

Corollary 1.11. *If* $\mathscr{R}$ *and* $\mathscr{S}$ *are proximally equivalent relators on X, then* $\mathscr{R}$ *is proximally directed if and only if* $\mathscr{S}$ *is proximally directed.*

Concerning topological directedness, it seems convenient to start with the next striking

Theorem 1.12. *If $\mathscr{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathscr{R}$ is topologically directed;*

(ii) *$\hat{\mathscr{R}}$ is uniformly directed.*

Proof. Assume (i), and let $R, S \in \hat{\mathscr{R}}$. Then, by the definition of $\hat{\mathscr{R}}$, for each $x \in X$, there exist $R_x, S_x \in \mathscr{R}$ such that $R_x(x) \subset R(x)$ and $S_x(x) \subset S(x)$. Moreover, by (i) for each $x \in X$, there exists $T_x \in \mathscr{R}$ such that $T_x(x) \subset R_x(x) \cap S_x(x)$. Define a relation T on X by $T(x) = T_x(x)$. Then, it is clear that $T \in \hat{\mathscr{R}}$ and $T \subset R \cap S$.

Now, assume (ii), and let $x \in X$ and $R, S \in \mathscr{R}$. Then, because of $\mathscr{R} \subset \hat{\mathscr{R}}$ and (ii), there exists $T \in \hat{\mathscr{R}}$ such that $T \subset R \cap S$. Moreover, again by the definition of $\hat{\mathscr{R}}$, there exists $V \in \mathscr{R}$ such that $V(x) \subset T(x)$. Hence, it is clear that $V(x) \subset R(x) \cap S(x)$.

By immediate consequences of this theorem, we have

Corollary 1.13. *If $\mathscr{R}$ and $\mathscr{S}$ are topologically equivalent relators on X, then $\mathscr{R}$ is topologically directed if and only if $\mathscr{S}$ is topologically directed.*

Corollary 1.14. *If $\mathscr{R}$ is a relator on X, then for its topological refinement $\hat{\mathscr{R}}$, the three kinds of directedness coincide.*

The next two important theorems can be easily derived from Theorems 1.5 and 1.6 by using Theorem 1.12.

Theorem 1.15. *If x is a point and (x_α) is a directed net in a topologically directed relator space $X(\mathscr{R})$, then the following assertions are equivalent:*

(i) $x \in \operatorname{adh}_{\mathscr{R}} \underset{\alpha}{} x_\alpha$;

(ii) $x \in \lim_{\mathscr{R}} \underset{\beta}{} y_\beta$ *for some directed subnet (y_β) of (x_α).*

Theorem 1.16. *If x is a point and A is a set in a topologically directed relator space $X(\mathscr{R})$, then the following assertions are equivalent:*

(i) $x \in \operatorname{cl}_{\mathscr{R}}(A)$;

(ii) $x \in \lim_{\mathscr{R}} \underset{\alpha}{} x_\alpha$ *for some directed net (x_α) in A.*

Remark 1.17. Note that Theorems 1.15 and 1.16 improve the particular cases of Theorems 1.5 and 1.6 when $(y_\alpha) = (x)$ and $B = \{x\}$ for some $x \in X$.

In principle, the following fundamental theorem should also be derivable from Theorem 1.7 by using Theorem 1.12, but for this we should know $\operatorname{Lim}_{\hat{\mathscr{R}}}$.

Theorem 1.18. *If f is a function from a topologically directed relator space X into an arbitrary one $Y(\mathscr{S})$, then the following assertions are equivalent:*

(i) *f is $(\hat{\mathscr{R}}, \mathscr{S})$ — continuous;*

(ii) $x \in \lim_{\mathscr{R}} \underset{\alpha}{} x_\alpha$ *implies* $f(x) \in \lim_{\mathscr{S}} \underset{\alpha}{} f(x_\alpha)$ *for any point x and directed net (x_α) in X.*

Proof. Because of [33, Theorem 5.11], only the implication (ii)⇒(i) needs a separate proof. For this, it seems convenient to note now that if (ii) holds, then by Theorem 1.16, $x \in \operatorname{cl}_{\mathscr{R}}(A)$ implies $f(x) \in \operatorname{cl}_{\mathscr{S}}(f(A))$ which is equivalent to (i) by [33, Theorem 5.15].

Remark 1.19. Note that the implications (ii)⇒(i) in Theorems 1.5, 1.6, 1.15

and 1.16, and the implications (i)⇒(ii) in Theorems 1.7 and 1.18 do not need the directedness conditions on $\mathscr{R}$.

As an analogue of Theorem 1.10, now we can also easily prove

Theorem 1.20. *If $\mathscr{R}$ is a relator on X, then the following assertions are equivalent:*
(*i*) *$\mathscr{R}$ is topologically directed;*
(*ii*) $\operatorname{int}_{\mathscr{R}}(A\cap B)=\operatorname{int}_{\mathscr{R}}(A)\cap\operatorname{int}_{\mathscr{R}}(B)$ *for all* $A, B\subset X$;
(*iii*) $\operatorname{cl}_{\mathscr{R}}(A\cup B)=\operatorname{cl}_{\mathscr{R}}(A)\cup\operatorname{cl}_{\mathscr{R}}(B)$ *for all* $A, B\subset X$.

Proof. By [33, Theorem 6.7], we have

$$\operatorname{Int}_{\hat{\mathscr{R}}}(A) = \{B \subset X\colon B\subset \operatorname{int}_{\mathscr{R}}(A)\}$$

for all $A\subset X$. Hence, it is clear that (ii) holds if and only if

$$\operatorname{Int}_{\hat{\mathscr{R}}}(A\cap B) = \operatorname{Int}_{\hat{\mathscr{R}}}(A) \cap \operatorname{Int}_{\hat{\mathscr{R}}}(B)$$

for all A, $B\subset X$. Thus, by Theorem 1.10, (ii) is equivalent to the proximal directedness of $\hat{\mathscr{R}}$. Hence, by Corollary 1.14 and Theorem 1.12, the equivalence of (ii) and (i) is immediate.

The equivalence of (ii) and (iii) is apparent from the fact that $\operatorname{cl}_{\mathscr{R}}(A) = X\setminus\operatorname{int}_{\mathscr{R}}(X\setminus A)$ for all $A\subset X$.

Remark 1.21. Note that Theorems 1.15, 1.16, 1.18 and 1.20 can also be proved directly without using Theorem 1.12 and other former results.

2. Topological relators

Definition 2.1. A relator $\mathscr{R}$ on X, or a relator space $X(\mathscr{R})$, will be called
(i) strongly topological if $R(x)\in\mathscr{T}_{\mathscr{R}}$ for all $x\in X$ and $R\in\mathscr{R}$;
(ii) topological if $x\in R(x)^{\circ\circ}$ for all $x\in X$ and $R\in\mathscr{R}$;
(iii) weakly todological if $\overline{\{x\}}\in\mathscr{F}_{\mathscr{R}}$ for all $x\in X$.

Remark 2.2. It is clear that "strongly topological" implies "topological". The fact that "topological" also implies "weakly topological" is immediate from the next

Theorem 2.3. *If $\mathscr{R}$ is a relator on X, then the following assertions are equivalent:*
(i) *$\mathscr{R}$ is topological;*
(ii) $\mathring{A}\in\mathscr{T}_{\mathscr{R}}$ *for all* $A\subset X$;
(iii) $\bar{A}\in\mathscr{F}_{\mathscr{R}}$ *for all* $A\subset X$.

Proof. Since we have $x\in R(x)^{\circ}$ for any $x\in X$ and $R\in\mathscr{R}$, it is clear that (ii) implies (i).

To prove the converse implication, note that if $x\in\mathring{A}$, then $R(x)\subset A$ for some $R\in\mathscr{R}$. Thus, if (i) holds, then $x\in R(x)^{\circ\circ}\subset A^{\circ\circ}$.

The equivalence of (ii) and (iii) is immediate from the fact that $\bar{A}=X\setminus(X\setminus A)^{\circ}$ for all $A\subset X$.

As an immediate consequence of this theorem, we can at once state

Corollary 2.4. *If $\mathscr{R}$ and $\mathscr{S}$ are topologically equivalent relators on X, then $\mathscr{R}$ is topological if and only if $\mathscr{S}$ is topological.*

Moreover, from Theorem 2.3, we can also at once derive

Theorem 2.5. If $\mathscr{R}$ is a relator on X, then the following assertions are equivalent:
(i) $\mathscr{R}$ is topological;
(ii) $\mathring{A}=\bigcup\{G\in\mathscr{T}_{\mathscr{R}}:\ G\subset A\}$ for all $A\subset X$;
(iii) $\bar{A}=\bigcap\{F\in\mathscr{F}_{\mathscr{R}}:\ A\subset F\}$ for all $A\subset X$.

Remark 2.6. Hence, it is clear that in a topological relator space $X(\mathscr{R})$, $\operatorname{int}_{\mathscr{R}}(\operatorname{cl}_{\mathscr{R}})$ and $\mathscr{T}_{\mathscr{R}}$ $(\mathscr{F}_{\mathscr{R}})$ are also equivalent tools.

On the other hand, by using Theorem 2.3, we can also complement Theorem 1.20 with the next

Theorem 2.7. *If $\mathscr{R}$ is a topological relator on X, then the following assertions are equivalent:*
(i) *$\mathscr{R}$ is topologically directed;*
(ii) *$A, B\in\mathscr{T}_{\mathscr{R}}$ implies $A\cap B\in\mathscr{T}_{\mathscr{R}}$;*
(iii) *$A, B\in\mathscr{F}_{\mathscr{R}}$ implies $A\cup B\in\mathscr{F}_{\mathscr{R}}$.*

PROOF. The implication (i)⇒(ii) is immediate from Theorem 1.20.

To prove the converse implication, note that if $A, B\subset X$, then by Theorem 2.3, $\mathring{A}, \mathring{B}\in\mathscr{T}_{\mathscr{R}}$. Thus, if (ii) holds, we also have $\mathring{A}\cap\mathring{B}\in\mathscr{T}_{\mathscr{R}}$. Hence, it follows that

$$\mathring{A}\cap\mathring{B}\subset(\mathring{A}\cap\mathring{B})^{\circ}\subset(A\cap A)^{\circ}.$$

Now, since the converse inclusion is automatic, by Theorem 1.20, it is clear that (i) also holds.

The equivalence of (ii) and (iii) is apparent from the fact that $\mathscr{F}_{\mathscr{R}}=\{X\setminus A:\ A\in\mathscr{T}_{\mathscr{R}}\}$.

Remark 2.8. Note that the implications (i)⇒(ii)⇔(iii) do not require $\mathscr{R}$ to be topological.

The next simple example shows that the implication (ii)⇒(i) need not be true if $\mathscr{R}$ is not topological.

Example 2.9. Let $X=\{0, 1, 2\}$, and for each $i=1, 2$, define the relation R_i on X such that $R_i(0)=\{0, i\}$ and $R_i(1)=R_i(2)=\{1, 2\}$. Then $\mathscr{R}=\{R_1, R_2\}$ is a topologically nondirected relator on X such that the family $\mathscr{T}_{\mathscr{R}}=\{\emptyset, \{1, 2\}, X\}$ is still closed under intersections.

The relationship between topological and strongly topological relators will be cleared up by the following

Theorem 2.10. *Let $\mathscr{R}$ be a relator on X. For each $R\in\mathscr{R}$, define the relation R° on X by $R^{\circ}(x)=R(x)^{\circ}$. Moreover, let $\mathscr{R}^{\circ}=\{R^{\circ}:\ R\in\mathscr{R}\}$. Then $\mathscr{R}^{\circ}$ is a strongly topological relator on X such that $\mathscr{R}^{\circ}$ and $\mathscr{R}$ are topologically equivalent if and only if $\mathscr{R}$ is topological.*

PROOF. Since $x\in R(x)^{\circ}$ for all $x\in X$ and $R\in\mathscr{R}$, it is clear that $\mathscr{R}^{\circ}$ is a relator on X. Moreover, since $R^{\circ}\subset R$ for all $R\in\mathscr{R}$, we obviously have $\mathscr{R}\subset(\mathscr{R}^{\circ})^{*}\subset(\mathscr{R}^{\circ})^{\wedge}$.

On the other hand, straightforward applications of the corresponding definitions show that $\mathscr{R}$ is topological if and only if $\mathscr{R}^\circ \subset \hat{\mathscr{R}}$.

Finally, if $y \in R^\circ(x)$ for some $x \in X$ and $R \in \mathscr{R}$, then there exists $S \in \mathscr{R}$ such that $S(y) \subset R(x)$. Hence, $S^\circ(y) \subset R^\circ(x)$. Consequently, $y \in \operatorname{int}_{\mathscr{R}^\circ}(R^\circ(x))$.

As a useful consequence of this theorem and Corollary 2.4, we have

Corollary 2.11. *A relator $\mathscr{R}$ on X is topological if and only if $\mathscr{R}$ is topologically equivalent to a strongly topological relator $\mathscr{S}$ on X.*

The next theorem offers a somewhat deeper characterization of topologicalness in temrs of nets.

Theorem 2.12. *If $\mathscr{R}$ is a relator on X, then the following assertions are equivalent:*

(i) $\mathscr{R}$ is topological;

(ii) $\lim_{\mathscr{R}}$ has the iterated limit property.

PROOF. Suppose that (i) holds. Then, because of Corollary 2.11, we may assume without loss of generality that $\mathscr{R}$ is strongly topological. To prove (ii), let $(x_{\alpha\beta})_{\beta \in \Gamma_\alpha}$ be a net in X for each α in a nonvoid preordered set Γ, and suppose that $y_\alpha \in \lim_{\mathscr{R}\,\beta} x_{\alpha\beta}$ for each $\alpha \in \Gamma$, and $z \in \lim_{\mathscr{R}\,\alpha} y_\alpha$. Then, given $R \in \mathscr{R}$, there exists $\alpha_0 \in \Gamma$ such that $y_\alpha \in R(z)$ for all $\alpha \geqq \alpha_0$. Moreover, since $R(z) \in \mathscr{T}_{\mathscr{R}}$, for each $\alpha \geqq \alpha_0$ there exists $\beta_\alpha \in \Gamma_\alpha$ such that $x_{\alpha\beta} \in R(z)$ for all $\beta \geqq \beta_\alpha$. Choose $\varphi_0 \in \bigtimes_{\alpha \in \Gamma} \Gamma_\alpha$ such that $\varphi_0(\alpha) \geqq \beta_\alpha$ for all $\alpha \geqq \alpha_0$. Then, it is clear that $x_{\alpha\varphi(\alpha)} \in R(z)$ for all $\alpha \geqq \alpha_0$ and $\varphi \geqq \varphi_0$. Consequently, we have

$$z \in \lim_{\mathscr{R}\,(\alpha,\varphi)} x_{\alpha\varphi(\alpha)}, \quad \text{where} \quad (\alpha, \varphi) \in \Gamma x \bigtimes_{\alpha \in \Gamma} \Gamma_\alpha,$$

and thus (ii) is proved.

To prove the converse implication, suppose now that (i) does not hold. Then, there exist $x \in X$ and $R \in \mathscr{R}$ such that $x \notin R(x)^{\circ\circ}$. This means that for each $S \in \mathscr{R}$ there exists $y_S \in S(x)$ such that for each $T \in \mathscr{R}$ there exists $z_{ST} \in T(y_S)$ such that $z_{ST} \notin R(x)$. Hence, by preordering $\mathscr{R}$ with the reverse set inclusion, it is not hard to check that $(z_{ST})_{T \in \mathscr{R}}$ is a net in X for each $S \in \mathscr{R}$ such that $y_S \in \lim_{\mathscr{R}\,T} z_{ST}$ for each $S \in \mathscr{R}$, and $x \in \lim_{\mathscr{R}\,S} y_S$, but

$$x \notin \lim_{\mathscr{R}\,(S,\Phi)} z_{S\Phi(S)}, \quad \text{where} \quad \Phi\colon \mathscr{R} \to \mathscr{R}.$$

And thus (ii) cannot hold too.

Remark 2.13. By using Theorem 2.3 and [33, Theorem 3.8], this latter implication can also be given a direct proof.

However, note that the above indirect proof gives a little more, namely that (i) is already implied by a particular case of (ii).

In this respect, it is also worth mentioning that using Theorem 1.12 and the same argument as above, we can also easily prove

Theorem 2.14. *If $\mathscr{R}$ is a topologically directed relator on X, then the following assertions are equivalent:*

(*i*) $\mathcal{R}$ *is topological;*

(*ii*) $\lim_{\mathcal{R}}$ *has the iterated limit property for directed nets.*

It is a striking fact that to some extent nets can also be used to describe strong topologicalness of relators. Namely, we have

Theorem 2.15. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(*i*) $\mathcal{R}$ *is strongly topological:*

(*ii*) $R^{-1}\circ\lim_{\mathcal{R}} \subset \lim_{\{R\}}$ for all $R\in\mathcal{R}$.

PROOF. Suppose that (i) holds, and let $R\in\mathcal{R}$. Moreover, let (x_α) be a net in X and assume that $y\in R^{-1}(\lim_{\alpha}{}_{\mathcal{R}} x_\alpha)$. Then, there exists $x\in\lim_{\alpha}{}_{\mathcal{R}} x_\alpha$ such that $y\in R^{-1}(x)$, i.e., $x\in R(y)$. Hence, since $R(y)\in\mathcal{T}_{\mathcal{R}}$, it is clear that (x_α) is eventually in $R(y)$, i.e., $y\in\lim_{\alpha}{}_{\{R\}} x_\alpha$. Consequently, $R^{-1}(\lim_{\alpha}{}_{\mathcal{R}} x_\alpha)\subset\lim_{\alpha}{}_{\{R\}} x_\alpha$, whence (ii) follows.

To prove the converse implication, suppose now that (i) does not hold. Then, there exist $x\in X$ and $R\in\mathcal{R}$ such that $R(x)\notin\mathcal{T}_{\mathcal{R}}$. This means that there exists $y\in R(x)$ such that for each $S\in\mathcal{R}$ there exists $y_S\in S(y)$ such that $y_S\notin R(x)$. Hence, by preordering $\mathcal{R}$ with the reverse set inclusion, it is easy to check that $(y_S)_{S\in\mathcal{R}}$ is a net in X such that $y\in\lim_{S}{}_{\mathcal{R}} y_S$, but $x\notin\lim_{S}{}_{\{R\}} y_S$, and thus (ii) cannot hold.

Remark 2.16. Note that, because of [33, Theorems 3.10, 2.16 and 2.10], we always have

$$\bigcap_{R\in\mathcal{R}} R^{-1}\circ\lim_{\mathcal{R}} \subset \lim_{\mathcal{R}} = \bigcap_{R\in\mathcal{R}} \lim_{\{R\}}.$$

From Theorem 2.15 and the second part of its proof, it is clear that we can also state

Theorem 2.17. *If $\mathcal{R}$ is a uniformly directed relator on X, then the following assertions are equivalent:*

(*i*) $\mathcal{R}$ *is strongly topological;*

(*ii*) $R^{-1}(\lim_{\alpha}{}_{\mathcal{R}} x_\alpha) \subset \lim_{\alpha}{}_{\{R\}} x_\alpha$ for any $R\in\mathcal{R}$

and any directed net (x_α) in X.

Remark 2.18. Note that by using [33, Theorem 1.17 and Remark 1.18], the above mixed condition can be written in a more uniform form.

Concerning weak topologicalness, we can at once state

Theorem 2.19. *If $\mathcal{R}$ and $\mathcal{S}$ are topologically equivalent relators on X, then $\mathcal{R}$ is weakly topological if and only if $\mathcal{S}$ is weakly topological.*

Moreover, using [33, Theorem 2.10], we can also easily prove

Theorem 2.20. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(*i*) $\mathcal{R}$ is weakly topological;

(*ii*) $\bigcap(\bigcap\mathcal{R})\circ\mathcal{R} = \bigcap\mathcal{R}$.

PROOF. We clearly have

$$\overline{\{x\}} = \bigcap_{R\in\mathscr{R}} R^{-1}(x) = (\cap\mathscr{R})^{-1}(x)$$

and

$$\overline{\overline{\{x\}}} = \bigcap_{R\in\mathscr{R}} R^{-1}((\cap\mathscr{R})^{-1}(x)) = \Big(\bigcap_{R\in\mathscr{R}} (\cap\mathscr{R})\circ R\Big)^{-1}(x)$$

for all $x\in X$, whence the theorem is immediate.

Remark 2.21. By a simple application of this theorem, one can easily check that the relator $\mathscr{R}$ given in Example 2.9 is not even weakly topological.

3. Transitive relators

Definition 3.1. A relator $\mathscr{R}$ on X, or a relator space $X(\mathscr{R})$, will be called

(i) strongly transitive if each $R\in\mathscr{R}$ is transitive;

(ii) uniformly transitive if for each $R\in\mathscr{R}$ there exist $S, T\in\mathscr{R}$ such that $T\circ S\subset R$;

(iii) proximally transitive if for each $A\subset X$ and $R\in\mathscr{R}$ there exist $S, T\in\mathscr{R}$ such that $T(S(A))\subset R(A)$;

(iv) topologically transitive if for each $x\in X$ and $R\in\mathscr{R}$ there exist $S, T\in\mathscr{R}$ such that $T(S(x))\subset R(x)$;

(v) weakly transitive if $\cap\mathscr{R}$ is transitive.

Remark 3.2. It is clear that any of the properties (i), (ii) and (iii) implies the subsequent one.

The fact that "topologically transitive" also implies "weakly transitive" is immediate from the next

Theorem 3.3. *Let $\mathscr{R}$ be a relator on X. Then the following assertions hold:*

(i) If $\mathscr{R}$ is weakly topological, then $\mathscr{R}$ is weakly transitive.

(ii) If $\mathscr{R}$ is topologically transitive, then $\mathscr{R}$ is topological.

(iii) If $\mathscr{R}$ is strongly transitive, then $\mathscr{R}$ is strongly topological.

Proof. If $\mathscr{R}$ is weakly topological and $R=\cap\mathscr{R}$, then, by Theorem 2.20, $R\circ R= =R\circ\cap\mathscr{R}\subset\cap R\circ\mathscr{R}=R.$

The assertions (i) and (iii) are even more obvious consequences of the corresponding definitions.

By using a particular case of the relator $\mathscr{R}_{\mathscr{A}}$ given in Example 1.3, we can complement Theorem 2.10 with the next important

Theorem 3.4. *If $\mathscr{R}$ is a relator on X, then $\mathscr{R}_{\mathscr{T}_{\mathscr{R}}}$ is a strongly transitive relator on X such that $\mathscr{R}_{\mathscr{T}_{\mathscr{R}}}$ and $\mathscr{R}$ are topologically equivalent if and only if $\mathscr{R}$ is topological*

PROOF. From the definition of $\mathscr{R}_{\mathscr{T}_{\mathscr{R}}}$, it is clear that $\mathscr{R}_{\mathscr{T}_{\mathscr{R}}}$ is strongly transitive. Thus, if $\mathscr{R}_{\mathscr{T}_{\mathscr{R}}}$ and $\mathscr{R}$ are topologically equivalent, then, by (iii) in Theorem 3.3 and Corollary 2.11, $\mathscr{R}$ is necessarily topological.

On the other hand, it is also clear that $\mathcal{R}_{\mathcal{T}_{\mathcal{R}}} \subset \hat{\mathcal{R}}$ is always true. Finally, if $\mathcal{R}$ is topological, then by Theorem 2.3, we have $R(x)^{\circ} \in \mathcal{T}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$, and thus in this case $\mathcal{R} \subset (\mathcal{R}_{\mathcal{T}_{\mathcal{R}}})^{\wedge}$ is also true.

Using this theorem, we can improve one half of Corollary 2.11 by stating

Corollary 3.5. *A relator $\mathcal{R}$ on X is topological if and only if $\mathcal{R}$ is topologically equivalent to a strongly transitive relator $\mathcal{S}$ on X.*

Hence, by noticing that $\hat{\mathcal{S}}$ is still topologically transitive, we can also state

Corollary 3.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*
(i) $\mathcal{R}$ is topological;
(ii) $\hat{\mathcal{R}}$ is topologically transitive.

Remark 3.7. Note that actually Corollary 3.5 yields a little more: if (i) holds, then for each $x \in X$ and $R \in \hat{\mathcal{R}}$ there exists $S \in \hat{\mathcal{R}}$ such that $S(S(x)) \subset R(x)$.

The importance of uniformly transitive relators lies mainly in the next

Theorem 3.8. *If $\mathcal{R}$ is a uniformly directed relator on X, then the following assertions are equivalent:*
(i) $\mathcal{R}$ is uniformly transitive;
(ii) the restriction of $\operatorname{Lim}_{\mathcal{R}}$ to directed nets is transitive.

Proof. The proof of the implication (i)$\Rightarrow$(ii) is quite straightforward.

To prove the converse implication, note that if (i) does not hold, then there exists $R \in \mathcal{R}$ such that for each $S \in \mathcal{R}$ there exist $x_S, y_S, z_S \in X$ such that $(x_S, y_S) \in S$ and $(y_S, z_S) \in S$, but $(x_S, z_S) \notin R$. Hence, by noticing that $\mathcal{R}$ is now a directed set with respect to the reverse set inclusion, it is easy to see that $(x_S)_{S \in \mathcal{R}}$, $(y_S)_{S \in \mathcal{R}}$ and $(z_S)_{S \in \mathcal{R}}$ are directed nets in X such that

$$x_S \in \operatorname{Lim}_{S}{}_{\mathcal{R}}\, y_S \quad \text{and} \quad y_S \in \operatorname{Lim}_{S}{}_{\mathcal{R}}\, z_S, \quad \text{but} \quad x_S \notin \operatorname{Lim}_{S}{}_{\mathcal{R}}\, z_S,$$

and thus (ii) cannot hold.

Remark 3.9. Note that the implication (i)$\Rightarrow$(ii) does not require $\mathcal{R}$ to be uniformly directed.

Moreover, note also that if $\mathcal{R}$ is a relator such that $\operatorname{Lim}_{\mathcal{R}}$ is transitive, then $\mathcal{R}$ is necessarily uniformly transitive.

Concerning uniformly transitive relators, one can also easily prove the next useful

Theorem 3.10. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) $\mathcal{R}$ is uniformly transitive;
(ii) $\mathcal{R}^*$ is uniformly transitive.

Hence, it is clear that we also have

Corollary 3.11. *If $\mathcal{R}$ and $\mathcal{S}$ are uniformly equivalent relators on X, then $\mathcal{R}$ is uniformly transitive if and only if $\mathcal{S}$ is uniformly transitive.*

To demonstrate the usefulness of proximally transitive relators, we shall now prove

Theorem 3.12. *If $\mathscr{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathscr{R}$ is proximally transitive;*

(ii) $\mathrm{Int}_{\mathscr{R}} \circ \mathrm{Int}_{\mathscr{R}} = \mathrm{Int}_{\mathscr{R}}$

Proof. Since the relation $\mathrm{Int}_{\mathscr{R}}$ is always transitive, we need actually show that (i) is equivalent to the inclusion $\mathrm{Int}_{\mathscr{R}} \subset \mathrm{Int}_{\mathscr{R}} \circ \mathrm{Int}_{\mathscr{R}}$.

Suppose that (i) holds and $B \in \mathrm{Int}_{\mathscr{R}}(A)$. Then $R(B) \subset A$ for some $R \in \mathscr{R}$. Moreover, there exists $S, T \in \mathscr{R}$ such that $T(S(B)) \subset R(B)$. Consequently, $S(B) \in \mathrm{Int}_{\mathscr{R}}(A)$. Hence, since $B \in \mathrm{Int}_{\mathscr{R}}(S(B))$, it is clear that $B \in \mathrm{Int}_{\mathscr{R}}(\mathrm{Int}_{\mathscr{R}}(A))$.

Suppose now that $\mathrm{Int}_{\mathscr{R}} \subset \mathrm{Int}_{\mathscr{R}} \circ \mathrm{Int}_{\mathscr{R}}$, and let $A \subset X$ and $R \in \mathscr{R}$. Then $A \in \mathrm{Int}_{\mathscr{R}}(R(A))$. Thus, by the assumption, there exists $B \subset X$ such that $A \in \mathrm{Int}_{\mathscr{R}}(B)$ and $B \in \mathrm{Int}_{\mathscr{R}}(R(A))$. Consequently, $S(A) \subset B$ and $T(B) \subset R(A)$ for some $S, T \in \mathscr{R}$ And this implies that $T(S(A)) \subset R(A)$.

Remark 3.13. In contrast to Theorem 1.10, an equivalent reformulation of (ii) yields only that $B \notin \mathrm{Cl}_{\mathscr{R}}(A)$ with $A, B \subset X$ implies the existence of $C \subset X$ such that $C \notin \mathrm{Cl}_{\mathscr{R}}(A)$ and $B \notin \mathrm{Cl}_{\mathscr{R}}(X \setminus C)$.

As an immediate consequence of Theorem 3.12, we can at once state

Corollary 3.14. *If $\mathscr{R}$ and $\mathscr{S}$ are proximally equivalent relators on X, then $\mathscr{R}$ is proximally transitive if and only if $\mathscr{S}$ is proximally transitive.*

Analogously to Theorem 3.8, now we can also easily prove

Theorem 3.15. *If $\mathscr{R}$ is a uniformly directed relator on X, then the following assertions are equivalent:*

(i) *$\mathscr{R}$ is topologically transitive;*

(ii) *$y_\alpha \in \mathrm{Lim}_{\mathscr{R}}\, x_\alpha$ implies $\lim_{\mathscr{R}} y_\alpha \subset \lim_{\mathscr{R}} x_\alpha$ for any two directed nets (x_α) and (y_α) in X.*

Proof. Suppose that (i) holds, and let (x_α) and (y_α) be directed nets in X such that $y_\alpha \in \mathrm{Lim}_{\mathscr{R}}\, x_\alpha$. Moreover, assume that $x \in \lim_{\mathscr{R}} y_\alpha$ and $R \in \mathscr{R}$. Then, there exist $S, T \in \mathscr{R}$ such that $T(S(x)) \subset R(x)$. Moreover, there exist α_1 and α_2 such that $y_\alpha \in S(x)$ for all $\alpha \geqq \alpha_1$, and $x_\alpha \in T(y_\alpha)$ for all $\alpha \geqq \alpha_2$. Hence, by choosing α_0 such that $\alpha_0 \geqq \alpha_1$ and $\alpha_0 \geqq \alpha_2$, we can state that $x_\alpha \in R(x)$ for all $\alpha \geqq \alpha_0$. Consequently, $x \in \lim_{R} x_\alpha$. And thus (ii) also holds.

To prove the converse implication, note that if (i) does not hold, then there exists $x \in X$ and $R \in \mathscr{R}$ such that for each $S \in \mathscr{R}$ there exist $x_S, y_S \in X$ such that $x_S \in S(y_S)$ and $y_S \in S(x)$, but $x_S \notin R(x)$. Hence, by noticing that $\mathscr{R}$ is now a directed set with respect to the reverse set inclusion, it is easy to see that $(x_S)_{S \in \mathscr{R}}$ and $(y_S)_{S \in \mathscr{R}}$ are directed nets in X such that

$$y_S \in \mathrm{Lim}_{\mathscr{R}}\, x_S \quad \text{and} \quad x \in \lim_{S}{}_{\mathscr{R}}\, y_S, \quad \text{but} \quad x \notin \lim_{S}{}_{\mathscr{R}}\, x_S.$$

and thus (ii) cannot hold. Consequently, the implication (ii)$\Rightarrow$(i) is also true.

Remark 3.16. Note that the implication (i)⇒(ii) does not require $\mathcal{R}$ to be uniformly directed.

Moreover, note also that if $\mathcal{R}$ is a relator on X such that $y_\alpha \in \operatorname{Lim}_{\alpha\,\mathcal{R}} x_\alpha$ implies $\lim_{\alpha\,\mathcal{R}} y_\alpha \subset \lim_{\alpha\,\mathcal{R}} x_\alpha$ for any two nets (x_α) and (y_α) in X, then $\mathcal{R}$ is necessarily topologically transitive.

From the above theorem, using Theorem 1.12 and Corollary 3.6, we can at once derive

Corollary 3.17. *If $\mathcal{R}$ is a topologically directed relator on X, then the following assertions are equivalent:*

(i) $\mathcal{R}$ is topological;

(ii) $y_\alpha \in \operatorname{Lim}_{\alpha\,\hat{\mathcal{R}}} x_\alpha$ implies $\lim_{\alpha\,\mathcal{R}} y_\alpha \subset \lim_{\alpha\,\mathcal{R}} x_\alpha$

for any two directed nets (x_α) and (y_α) in X.

Remark 3.18. Note that because of Theorem 2.14, (ii) is equivalent to a restricted iterated limit property for $\lim_{\mathcal{R}}$.

In connection with topologically transitive relators, also worth noticing is the following

Theorem 3.19. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X such that $\mathcal{S} \subset \hat{\mathcal{R}}$ and $\mathcal{R} \subset \mathcal{S}^{\#}$, then the topological transitiveness of $\mathcal{R}$ implies the topological transitiveness of $\mathcal{S}$.*

Proof. If $x \in X$ and $S \in \mathcal{S}$, then because of $\mathcal{S} \subset \hat{\mathcal{R}}$, there exists $R \in \mathcal{R}$ such that $R(x) \subset S(x)$. Moreover, if $\mathcal{R}$ is topologically transitive, then there exist $R_1, R_2 \in \mathcal{R}$ such that $R_1(R_2(x)) \subset R(x)$. On the other hand, since $\mathcal{R} \subset \mathcal{S}^{\#}$, there exist $S_1, S_2 \in \mathcal{S}$ such that $S_1(R_2(x)) \subset R_1(R_2(x))$ and $S_2(x) \subset R_2(x)$. And hence, it is clear that $S_1(S_2(x)) \subset S(x)$.

From the above theorem, it is clear that we also have

Corollary 3.20. *If $\mathcal{R}$ and $\mathcal{S}$ are proximally equivalent relators on X, then $\mathcal{R}$ is topologically transitive if and only if $\mathcal{S}$ is topologically transitive.*

Concerning weak transitiveness, as an immediate consequence of [33, Theorem 2.22], we can at once state

Theorem 3.21. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly transitive;*

(ii) *$\varrho_{\mathcal{R}}$ is transitive.*

Remark 3.22. To feel the difference between weak transitiveness and weak topologicalness note that the relator $\mathcal{R}$ given in Example 2.9 is still weakly transitive.

4. Equivalences of transitive relators

Theorem 4.1. *Let $\mathscr{R}$ be a relator on X, and define*

$$\mathscr{R}^n = \overset{1}{\mathscr{R}} \circ \overset{2}{\mathscr{R}} \circ \ldots \circ \overset{n}{\mathscr{R}}$$

for some integer $n \geqq 2$. Then $\mathscr{R}^n$ is a relator on X such that $\mathscr{R}^n$ and $\mathscr{R}$ are uniformly, proximally, resp. topologically equivalent if and only if $\mathscr{R}$ is uniformly, proximally, resp. topologically transitive.

Proof. To prove the stated equivalences of $\mathscr{R}^n$ and $\mathscr{R}$, note that because of the reflexivity of the members of $\mathscr{R}$, we always have $\mathscr{R}^n \subset \mathscr{R}^*$. Moreover, apply the properties (ii), (iii), resp. (iv) given in Definition 3.1 repeatedly sufficiently many times and use again the reflexivity of the members of $\mathscr{R}$.

Remark 4.2. If $\mathscr{R}$ is uniformly, proximally, resp. topologically transitive, then $\mathscr{R}^n$ is also uniformly, proximally, resp. topologically transitive.

The uniform, resp. proximal transitiveness of $\mathscr{R}^n$ is immediate from the above theorem by Corollary 3.11, resp. 3.14. Unfortunately, to prove the topological transitiveness of $\mathscr{R}^n$, Theorems 4.1 and 3.19 cannot be applied. Therefore, for this latter purpose again the topological transitiveness of $\mathscr{R}$ and the reflexivity of the members of $\mathscr{R}$ have to be used.

Using Theorem 4.1, we can complement our former Theorem 2.10 with the following important

Theorem 4.3. *Let $\mathscr{R}$ be a topologically (proximally, resp. uniformly) transitive relator on X and let $\mathscr{S}$ be another relator on X such that $\mathscr{R}^{-1} \subset \hat{\mathscr{S}}$. For each $R \in \mathscr{R}$ define the relation R^- on X by $R^-(x) = \mathrm{cl}_{\mathscr{S}}(R(x))$. Moreover, let $\mathscr{R}^- = \{R^- :\ R \in \mathscr{R}\}$. Then $\mathscr{R}^-$ is a topologically (proximally, resp. uniformly) transitive relator on X such that $\mathscr{R}^-$ and $\mathscr{R}$ are topologically (proximally, resp. uniformly) equivalent.*

Proof. Since $R \subset R^-$ for any $R \in \mathscr{R}$, it is clear that $\mathscr{R}^- \subset \mathscr{R}^* \subset \hat{\mathscr{R}}$. On the other hand, since $\mathscr{R}$ is topologically transitive, for each $x \in X$ and $R \in \mathscr{R}$, there exist $S, T \in \mathscr{R}$ such that $T(S(x)) \subset R(x)$. Hence, by [33, Corollary 5.16 and Theorem 2.10], it is clear that

$$S^-(x) = \mathrm{cl}_{\mathscr{S}}(S(x)) \subset \mathrm{cl}_{\mathscr{R}^{-1}}(S(x)) = \bigcap_{V \in \mathscr{R}} V(S(x)) \subset T(S(x)) \subset R(x).$$

Consequently, now we also have $\mathscr{R} \subset (\mathscr{R}^-)^\wedge$.

Next, we show that $\mathscr{R}^-$ is also topologically transitive. For this, suppose that $x \in X$ and $R \in \mathscr{R}$. Then, by Theorem 4.1, there exist $S, T, V \in \mathscr{R}$ such that $V(T(S)x))) \subset R(x)$. Hence, since $S^-(x) \subset T(S(x))$, it is clear that $V(S^-(x)) \subset R(x)$, i.e., $V(y) \subset R(x)$ for all $y \in S^-(x)$. And this implies that $V^-(y) \subset R^-(x)$ for all $y \in S^-(x)$, i.e., $V^-(S^-(x)) \subset R^-(x)$.

Finally, to prove the remaining assertions, one has to make the necessary modifications in the first part of the above proof. Note that by Corollary 4.14, resp. 3.11 the proximal, resp. uniform equivalence of $\mathscr{R}^-$ and $\mathscr{R}$ implies the proximal, resp. uniform transitiveness of $\mathscr{R}^-$.

Remark 4.4. Note that if $\mathcal{R}$ is topologically semisymmetric in the sense that $\mathcal{R}^{-1}\subset\hat{\mathcal{R}}$, then in particular we may take $\mathcal{S}=\mathcal{R}$ in the above theorem.

Using Theorem 4.1, we can also easily prove some useful analogues of Theorems 2.10 and 4.3 which use the notion of the direct product of relators.

Theorem 4.5. *Let $\mathcal{R}$ be a uniformly transitive relator on X and $\mathcal{S}$ be a relator on $X\times X$ such that $\mathcal{R}^{-1}\otimes\mathcal{R}\subset\hat{\mathcal{S}}$. For each $R\in\mathcal{R}$, define $\mathring{R}=\operatorname{int}_{\mathcal{S}}(R)$. Moreover, let $\mathring{\mathcal{R}}=\{\mathring{R}\colon R\in\mathcal{R}\}$. Then $\mathring{\mathcal{R}}$ is a uniformly transitive relator on X such that $\mathring{\mathcal{R}}$ and $\mathcal{R}$ are uniformly equivalent.*

PROOF. Since $\mathring{R}\subset R$ for any $R\in\mathcal{R}$, it is clear that $\mathcal{R}\subset(\mathring{\mathcal{R}})^*$. On the other hand, if $R\in\mathcal{R}$, then by Theorem 4.1, there exist $S, T, V\in\mathcal{R}$ such that $V\circ T\circ S\subset R$. Hence, since

$$V\circ T\circ S=\bigcup_{(x,y)\in T} S^{-1}(x)\times V(y)=\bigcup_{(x,y)\in T}(S^{-1}\otimes V)(x,y),$$

it is clear that

$$T\subset \operatorname{int}_{\mathcal{R}^{-1}\otimes\mathcal{R}}(R)\subset\operatorname{int}_{\mathcal{S}}(R)=\mathring{R}.$$

Consequently, now we also have $\mathring{\mathcal{R}}\subset\mathcal{R}^*$. Finally, from the uniform equivalence of $\mathring{\mathcal{R}}$ and $\mathcal{R}$, by Corollary 3.11, it is clear that $\mathring{\mathcal{R}}$ is also uniformly transitive.

Theorem 4.6. *Let $\mathcal{R}$ be a uniformly transitive relator on X and let $\mathcal{S}$ be a relator on $X\times X$ such that $\mathcal{R}\otimes\mathcal{R}^{-1}\subset\hat{\mathcal{S}}$. For each $R\in\mathcal{R}$, define $\bar{R}=\operatorname{cl}_{\mathcal{S}}(R)$. Moreover, let $\bar{\mathcal{R}}=\{\bar{R}\colon R\in\mathcal{R}\}$. Then $\bar{\mathcal{R}}$ is a uniformly transitive relator on X such that $\bar{\mathcal{R}}$ and $\mathcal{R}$ are uniformly equivalent.*

PROOF. Since $R\subset\bar{R}$ for any $R\in\mathcal{R}$, it is clear that $\bar{\mathcal{R}}\subset\mathcal{R}^*$. On the other hand, if $R\in\mathcal{R}$, then again by Theorem 4.1, there exist $S, T, V\in\mathcal{R}$ such that $V\circ T\circ S\subset R$. Hence, since

$$\operatorname{cl}_{\mathcal{R}\otimes\mathcal{R}^{-1}}(T)=\bigcap\{W\circ T\circ\Omega\colon \Omega, W\in\mathcal{R}\},$$

it is clear that

$$\bar{T}=\operatorname{cl}_{\mathcal{S}}(T)\subset\operatorname{cl}_{\mathcal{R}\otimes\mathcal{R}^{-1}}(T)\subset R.$$

Consequently, now we also have $\mathcal{R}\subset(\bar{\mathcal{R}})^*$. Finally, the uniform transitiveness of $\bar{\mathcal{R}}$ is immediate from the uniform equivalence of $\bar{\mathcal{R}}$ and $\mathcal{R}$ by Corollary 3.11.

Remark 4.7. Note that in particular we may take

$$\mathcal{S}=(\mathcal{R}^{-1}\times\mathcal{R})' \quad\text{and}\quad \mathcal{S}=(\mathcal{R}\times\mathcal{R}^{-1})'$$

in Theorems 4.5 and 4.6, respectively.

Notes and comments

The most complete classification scheme for generalized uniform spaces has formerly been given by NAKANO—NAKANO [24]. Unfortunately, they did not consider proximal directedness and transitiveness which were first investigated by MORDKOVIČ [19] under the name correctness. A curious directedness property lying strictly between uniform and proximal directedness was earlier utilized by ALFSEN—NJÅSTAD [1]. Accounts on proximity motivated generalized uniform spaces can be found in [23] and [20].

The relator $\mathscr{R}_{\mathscr{A}}$ given in Example 1.3 will be called the DAVIS—PERVIN relator on X generated by $\mathscr{A}$ since its important particular case when $\mathscr{A}$ is an ordinary topology on X seems to have been first introduced independently by DAVIS [5] and PERVIN [29] in the proofs of their striking generalized uniformization theorems. The relationship between $\mathscr{A}$ and $\mathscr{R}_{\mathscr{A}}$ was later more fully explored by LEVINE [17]. Note that the assertion (i) in Example 1.3 is an extension of [17, Theorem 2.9].

Theorems 1.5 and 1.15 are analogues of Theorem 2.6 of KELLEY [13]. Theorems 1.6 and 1.7 were first established for WEIL's uniform spaces by EFRĔMOVIČ—ŠVARC [8]. (See also MAMUZIČ [18, p. 120].) Extensions to more general spaces were later given by HUŠEK [11]. The origin of Theorems 1.10 and 3.12 goes back to MORDKOVIČ [19]. Theorem 1.12 is essentially equivalent to the statement (1.23) of NAKANO—NAKANO [24]. Theorem 1.20 is closely related to the first assertion in Proposition 1 of KONISHI [14].

Topologicalness of relators is an equivalent of the fourth axiom of neighbourhoods [30, p. 45]. Theorems 2.3, 2.5 and 2.10 correspond to Theorem 15 A.2 and Corollary 15 A. 3 of ČECH [2]. In connection with Theorem 2.7, see also the second assertion in Proposition 1 of KONISHI [14] and the statements (2.2.1) and (2.2.23) of CSÁSZÁR [4]. Theorems 2.12 and 2.14 are closely related to Theorem 15 B.13 of ČECH [2] and Theorems 2.4 and 2.9 of KELLEY [13]. Theorems 2.15 and 2.20 seem to have no analogues in the existing literature.

Analogously to strong topologicalness, topological, uniform and strong transiveness are also natural specializations of topologicalness. Particular cases of the assertions (i) and (ii) in Theorem 3.3 were already proved by NACHBIN [22, p. 58], MOZZOCHI—GAGRAT—NAIMPALLY [20 (2.32)] and WILLIAMS [36, Lemma 1.3]. Theorem 3.4 is patterned upon the generalized uniformization theorems of DAVIS [5] and PERVIN [29]. Corollary 3.6 partly explains the apparently very strange terminology of NAKANO—NAKANO [24] concerning topologicalness.

Theorem 3.8 has again been suggested by EFRĔMOVIČ—ŠVARC [8] and HUŠEK [11, p. 261]. The origin of Theorem 3.15 goes back to NIEMYTZKI [26, p. 512]. This latter theorem may also be compared to Theorem 7.18 of FLETCHER—LINDGREN [9]. Theorem 4.3 greatly improves the first assertions of Theorem 4 of DAVIS [5] and Theorem 1.4 of WILLIAMS [36]. Theorem 4.5 and 4.6 are essential generalizations of Theorems 1.43 and 1.45 of MURDESHWAR—NAIMPALLY [21].

Finally, we remark that our subsequent paper "Inverse, symmetric and neighbourhood relators" will also contain several theorems about topological and transitive relators.

References

[1] E. M. ALFSEN—O. NJÅSTAD, Proximity and generalized uniformity, Fund. Math. 52 (1963), 235—252.

[2] E. ČECH, Topological Spaces, Prague, 1966. (Revised edition by Z. FROLIK and M. KATETOV.)

[3] Á. CSÁSZÁR, Foundation of General Topology, *London*, 1963.

[4] Á. CSÁSZÁR, General Topology, *Bristol*, 1978.

[5] A. S. DAVIS, Indexed systems of neighbourhoods for general topological spaces, *Amer. Math. Monthly* **68** (1961), 886—893.

[6] A. S. DAVIS, Total boundedness, Gal's theorem, and completion for regular spaces, *Amer. Math. Monthly* **70** (1963), 1075—1079.

[7] D. DOIČINOV, A unified theory of topological spaces, proximity spaces and uniform spaces, *Dokl. Akad. Nauk. SSSR* **156** (1964), 21—24 (Russian).
[8] V. A. EFRĚMOVIČ—A. S. ŠVARC, A new definition of uniform spaces. Metrization of proximity spaces, *Dokl. Akad. Nauk SSSR* **89** (1953), 393—396 (Russian).
[9] P. FLETCHER—W. F. LINDGREN, Quasi-Uniform Spaces, New York, 1982.
[10] S. GACSÁLYI, Remarks on a paper of C. J. Mozzochi, *Publ. Math. (Debrecen)* **21** (1974), 295—303.
[11] M. HUŠEK, Generalized proximity and uniform spaces I, II, *Comment. Math. Univ. Caroline* **5** (1964), 247—266; 6 (1965), 119—139.
[12] F. ISHIKAWA, Counter examples in the theory of generalized uniform spaces, *Proc. Japan Acad.* **42** (1966), 335—338.
[13] J. L. KELLEY, General Topology, New York, 1955.
[14] I. KONISHI, On uniform topologies in general spaces, *J. Math. Soc. Japan* **4** (1952), 166—188.
[15] V. S. KRISHNAN, A note on semi-uniform spaces, *J. Madras Univ.* **29** (1955), 123—124.
[16] V. S. KRISHNAN, Additivity and summetry for generalized uniform structures, and characterizations of semi-uniform structures, *J. Madras Univ.* **25** (1955), 201—212.
[17] N. LEVINE, On Pervin's quasi uniformity, *Math. J. Okayama Univ.* **14** (1970), 97—102.
[18] Z. P. MAMUZIČ, Introduction to general topology, *Groningen*, 1963.
[19] A. G. MORDKOVIČ, Test for correctness of uniform spaces *Dokl. Akad. Nauk SSSR* **169** (1966), 276—279.
[20] C. J. MOZZOCHI—M. S. GAGRAT—S. A. NAIMPALLY, Symmetric Generalized Topological Structures, *New York*, 1976.
[21] M. G. MURDESHWAR—S. A. NAIMPALLY, Quasi-Uniform Topological Spaces, *Groningen*, 1966.
[22] L. NACHBIN, Topology and Order, *Princeton*, 1965.
[23] S. A. NAIMPALLY—B. D. WARRACK, Proximity Spaces, *Cmbridge*, 1970.
[24] H. NAKANO—K. NAKANO, Connector theory, *Pacific J. Math.* **56** (1975), 195—213.
[25] K. NAKANO, A notion of completeness of topological structures, *Questiones Math.* **2** (1977), 235—243.
[26] V. W. NIEMYTSKI, On the third axiom of a metric space, *Trans. Amer. Math. Soc.* **29** (1927), 507—513.
[27] W. PAGE, Topological Uniform Structures, *New York*, 1978.
[28] W. J. PERVIN, Uniformization of neighbourhood axioms, Math. Ann. **147** (1962), 313—315.
[29] W. J. PERVIN, Quasi-uniformization of topological spaces, *Math. Ann.* **147** (1962), 316—317.
[30] W. J. PERVIN, Foundation of General Topology, *New York*, 1964.
[31] Á. SZÁZ, Coherences instead of convergences, *Proceedings of the Conference on Convergence and Generalized Functions, Katowice* 1983, 141—148.
[32] Á. SZÁZ, Continuities in relator spaces, Report of the International Conference on Generalized Functions, Debrecen 1984, *Publ. Math. (Debrecen)* **33** (1986), 188—189.
[33] Á. SZÁZ, Basic tools and mild continuities in relator spaces, *Acta Math. Hungar* **50** (1987), 177—201.
[34] Á. SZÁZ, Projective and inductive generations of relator spaces, *Acta Math. Hungar.* **52** (1989),
[35] D. V. THAMPURAN, Regularity for bitopological spaces, *Publ. Math. (Debrecen)* **20** (1973), 41—44.
[36] J. WILLIAMS, Locally uniform spaces, *Trans. Amer. Math. Soc.* **168** (1972), 435—469.

(Received July 30, 1985)