Contravariant integration, line integrals

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Abstract

In this paper an integral operator for tensor fields is introduced over the curves such that: 1° The result of integration is also a tensor field having one upper index more, 2° It is a generalization of the usual integral, and 3° The new operator acts over the differential 1-forms as a usual integration. The problem reduces to a system of Volterra's integral equations. Also some other properties of this integration are proved, and its geometrial interpretation is obtained.

1. Introduction

In this paper we shall solve the following problem. Let $\tau\colon [0,1]\to M_n$ be an arbitrary smooth curve on a differentiable manifold endowed with an affine connection, and let a differentiable tensor field A of type (r, s) be given along the curve τ. The problem is to find an integral operator $\hat{\int}$ such that in local coordinate systems along the curve τ

(i) $\hat{\int}_{\tau(0)}^{\tau(t)} A^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k$ should be components of a tensor field of type $(r+1, s)$.

(ii) In the special case when all components of the connection are identically zero, the problem should reduce to the usual integration, i.e.

$$\Gamma^i_{jk}\equiv 0 \Rightarrow \hat{\int}_{\tau(0)}^{\tau(t)} A^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k = \int_{\tau(0)}^{\tau(t)} A^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k \tag{1.1}$$

(iii) In the case of integration of an 1-form $A_i dx^i$ it should be

$$\hat{\int}_{\tau(0)}^{\tau(t)} A_i\,dx^i = \int_{\tau(0)}^{\tau(t)} A_i\,dx^i \tag{1.2}$$

In this paper a solution of this problem is found. The components

$$\hat{\int}_{\tau(0)}^{\tau(t)} A^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k$$

are solutions of a system of integral equations, and they are unique. Apart from the above three properties we give some other properties and a geometric interpretation of this integral also is given. This integration is called contravariant, because the result of its action yields one upper index more.

For the sake of simplicity we do not write the boundaries of integration.

2. Solution in E^n, and its generalization

Let M_n be an n-dimensional euclidean space. Suppose that $(y^1, \ldots, y^n) \equiv (y)$ is a rectangular Cartesian coordinate system and $(x^1, \ldots, x^n) \equiv (x)$ is a curvilinear coordinate system. According to the system (y) we shall write components as A'_i, B'^j and for (x): A_i, B^j, and so on.

From the transformation formula of the connection coefficients we obtain

$$\Gamma^i_{jk}(x) = \frac{\partial x^i}{\partial y^l} \frac{\partial^2 y^l}{\partial x^j \partial x^k}, \tag{2.1}$$

for $\Gamma'^i_{jk}(y) \equiv 0$. If we differentiate the identity

$$\delta^s_i = \frac{\partial y^s}{\partial x^u} \frac{\partial x^u}{\partial y^i}$$

by x^j, and use (2.1), we obtain the relation

$$\Gamma^l_{kj} = -\frac{\partial y^i}{\partial x^k} \frac{\partial y^r}{\partial x^j} \frac{\partial^2 x^l}{\partial y^i \partial y^r}. \tag{2.2}$$

We shall restrict ourselves to the case when A is a tensor field of type (1, 1), in order to avoid long expressions. Partial integration of the equation

$$A^j_i = \frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} A'^\lambda_\mu$$

yields

$$\int_{\tau(0)}^{\tau(t)} A^j_i \, dx^k = \int_{\tau(0)}^{\tau(t)} \frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} A'^\lambda_\mu \frac{\partial x^k}{\partial y^\nu} dy^\nu = \tag{2.3}$$

$$= \frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} \frac{\partial x^k}{\partial y^\nu} \int_{\tau(0)}^{\tau(t)} A'^\lambda_\mu \, dy^\nu - \int_{\tau(0)}^{\tau(t)} \frac{\partial \left(\frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} \frac{\partial x^k}{\partial y^\nu} \right)}{\partial y^r} \Big(\int_{\tau(0)}^{\tau(\theta)} A'^\lambda_\mu \, dy^\nu \Big) \, dy^r.$$

From conditions (i) and (ii) follows that

$$\int_{\tau(0)}^{\tau(t)} A'^\lambda_\mu \, dy^\nu = \frac{\partial x^p}{\partial y^\mu} \frac{\partial y^\lambda}{\partial x^q} \frac{\partial y^\nu}{\partial x^r} \hat{\int}_{\tau(0)}^{\tau(t)} A^q_p \, dx^r$$

and

$$\frac{\partial x^j}{\partial y^\lambda} \frac{\partial y^\mu}{\partial x^i} \frac{\partial x^k}{\partial y^\nu} \int_{\tau(0)}^{\tau(\theta)} A'^\lambda_\mu \, dy^\nu = \hat{\int}_{\tau(0)}^{\tau(\theta)} A^j_i \, dx^k.$$

Putting this in (2.3), we obtain

$$\hat{\int}_{\tau(0)}^{\tau(t)} A_i^j\, dx^k = \int_{\tau(0)}^{\tau(t)} A_i^j\, dx^k +$$

$$+ \int_{\tau(0)}^{\tau(t)} \frac{\partial\left(\dfrac{\partial x^j}{\partial y^\lambda}\dfrac{\partial y^\mu}{\partial x^i}\dfrac{\partial x^k}{\partial y^\nu}\right)}{\partial y^t} \frac{\partial x^p}{\partial y^\mu}\frac{\partial y^\lambda}{\partial x^q}\frac{\partial y^\nu}{\partial x^r}\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q\, dx^r\Big)\frac{\partial y^t}{\partial x^s}\, dx^s =$$

$$= \int_{\tau(0)}^{\tau(t)} A_i^j\, dx^k + \int_{\tau(0)}^{\tau(t)} \frac{\partial^2 x^k}{\partial y^\nu \partial y^t}\delta_q^j \frac{\partial y^\nu}{\partial x^r}\frac{\partial y^t}{\partial x^s}\delta_i^p \Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q\, dx^r\Big)\, dx^s +$$

$$+ \int_{\tau(0)}^{\tau(t)} \frac{\partial^2 x^j}{\partial y^\lambda \partial y^t}\delta_i^p \delta_r^k \frac{\partial y^t}{\partial x^s}\frac{\partial y^\lambda}{\partial x^q}\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q\, dx^r\Big)\, dx^s +$$

$$+ \int_{\tau(0)}^{\tau(t)} \frac{\partial^2 y^\mu}{\partial x^i \partial x^u}\delta_q^j \delta_s^u \delta_r^k \frac{\partial x^p}{\partial y^\mu}\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^q\, dx^r\Big)\, dx^s.$$

From (2.1) and (2.2) we obtain finally

$$\hat{\int}_{\tau(0)}^{\tau(t)} A_i^j\, dx^k = \int_{\tau(0)}^{\tau(t)} A_i^j\, dx^k - \int_{\tau(0)}^{\tau(t)} \Gamma_{rs}^k\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_i^j\, dx^r\Big)\, dx^s - \tag{2.4}$$

$$- \int_{\tau(0)}^{\tau(t)} \Gamma_{qs}^j\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_i^q\, dx^k\Big)\, dx^s + \int_{\tau(0)}^{\tau(t)} \Gamma_{ir}^p\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_p^j\, dx^k\Big)\, dx^r.$$

In the general case by the same way we obtain the following system of equations

$$\hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1\dots j_s}^{i_1\dots i_r}\, dx^k = \int_{\tau(0)}^{\tau(t)} A_{j_1\dots j_s}^{i_1\dots i_r}\, dx^k - \int_{\tau(0)}^{\tau(t)} \Gamma_{l\lambda}^k\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{j_1\dots j_s}^{i_1\dots i_r}\, dx^l\Big)\, dx^\lambda - \tag{2.5}$$

$$- \int_{\tau(0)}^{\tau(t)} \Gamma_{l\lambda}^{i_1}\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{j_1\dots j_s}^{l i_2\dots i_r}\, dx^k\Big)\, dx^\lambda - \dots - \int_{\tau(0)}^{\tau(t)} \Gamma_{l\lambda}^{i_r}\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{j_1\dots j_s}^{i_1\dots i_{r-1} l}\, dx^k\Big)\, dx^\lambda +$$

$$+ \int_{\tau(0)}^{\tau(t)} \Gamma_{j_1\lambda}^l\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{l j_2\dots j_s}^{i_1\dots i_r}\, dx^k\Big)\, dx^\lambda + \dots + \int_{\tau(0)}^{\tau(t)} \Gamma_{j_s\lambda}^l\Big(\hat{\int}_{\tau(0)}^{\tau(\theta)} A_{j_1\dots j_{s-1} l}^{i_1\dots i_r}\, dx^k\Big)\, dx^\lambda$$

where $i_1, \dots, i_r, j_1, \dots, j_s, k \in \{1, 2, \dots, n\}$.

So the problem reduces to solve the system of integral equations (2.5) with unknown functions

$$\hat{\int}_{\tau(0)}^{\tau(t)} A_{j_1\dots j_s}^{i_1\dots i_r}\, dx^k.$$

It has a unique solution, because it is a Volterra system of integral equations and it can be solved by successive approximation ([1], [2], [3]). In fact (2.5) is a special case of Volterra system where the scores are functions only from the variable θ. So we see that if in the euclidean space the problem raised in § 1 has solution, then it is a solution of the system (2.5), and this solution is unique.

Let us consider an M_n. Let U be a coordinate neighbourhood of it with local coordinates x^i. Consider (2.5) in U, where the tensor A and the curve τ are given, and the integral

$$\hat{\int}_{\tau(0)}^{\tau(t)} A^{i_1\dots i_r}_{j_1\dots j_s}\, dx^k$$

is unknown. (2.5) has a unique solution for this integral.

Proposition. *This integral is a solution of our problem in U.*

PROOF. Condition (i) will be considered in § 3. Condition (ii) is obviously satisfied. From (2.5) we obtain

$$\hat{\int} A_j\, dx^k = \int A_j\, dx^k - \int \Gamma^k_{l\lambda}\Big(\hat{\int} A_j\, dx^l\Big)\, dx^\lambda + \int \Gamma^l_{j\lambda}\Big(\hat{\int} A_l\, dx^k\Big)\, dx^\lambda.$$

If we put here $j=k$ and sum up over k, we receive

$$\hat{\int} A_k\, dx^k = \int A_k\, dx^k,$$

and the condition (iii) is satisfied. ∎

We see that if V is a scalar, then

$$\hat{\int} V dx^k = \int V dx^k - \int \Gamma^k_{l\lambda}\Big(\hat{\int} V dx^l\Big)\, dx^\lambda.$$

We see that in this case we do not integrate a scalar but a contravariant vector $V dx^k$. Integration of a scalar appears when we integrate the contraction $A_i dx^i$ and so in this case the contravariant integral is equal to the usual integral.

3. Tensor character of the contravariant integral

We prove that the solutions of the system (2.5) are tensor components, i.e. condition (i) holds. We shall restrict ourselves to the case when A is a covariant vector in order to avoid large expressions. In the general case when A is a tensor field of type (r, s), the same result can be obtained in a similar way.

Let $x^1, \dots, x^n$ and $\bar{x}^1, \dots, \bar{x}^n$ be two coordinate systems. The components $\hat{\int} A_i dx^k$ and $\hat{\int} \bar{A}_i d\bar{x}^k$ are obtained as solutions of the following two systems

$$\hat{\int} A_i\, dx^k = \int A_i dx^k - \int \Gamma^k_{rl}\Big(\hat{\int} A_i\, dx^r\Big)\, dx^l + \int \Gamma^r_{il}\Big(\hat{\int} A_r\, dx^k\Big)\, dx^l \tag{3.1}$$

and

$$\hat{\int} \bar{A}_i\, d\bar{x}^k = \int \bar{A}_i d\bar{x}^k - \int \bar{\Gamma}^k_{rl}\Big(\hat{\int} \bar{A}_i\, d\bar{x}^r\Big)\, d\bar{x}^l + \int \bar{\Gamma}^r_{il}\Big(\hat{\int} \bar{A}_r\, d\bar{x}^k\Big)\, d\bar{x}^l \tag{3.2}$$

Our aim is to prove that

$$\hat{\int} A_i\, dx^k = \frac{\partial \bar{x}^j}{\partial x^i}\frac{\partial x^k}{\partial \bar{x}^s}\,\hat{\int} \bar{A}_j\, d\bar{x}^s \tag{3.3}$$

for $i, k \in \{1, \ldots, n\}$. It is sufficient to assume that (3.2) and (3.3) hold and to prove that the system (3.1) is satisfied, because the system (3.1) has a unique solution.

Putting

$$\bar{\Gamma}^l_{ij} = \frac{\partial \bar{x}^l}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^t}{\partial \bar{x}^j} \Gamma^r_{st} + \frac{\partial \bar{x}^l}{\partial x^r} \frac{\partial^2 x^r}{\partial \bar{x}^i \partial \bar{x}^j} \tag{3.4}$$

in (3.2) we get

$$\hat{\int} \bar{A}_i d\bar{x}^k = \int \bar{A}_i d\bar{x}^k - \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^r} \frac{\partial x^\nu}{\partial \bar{x}^l} \Gamma^\lambda_{\mu\nu} \left(\hat{\int} \bar{A}_i d\bar{x}^r \right) d\bar{x}^l -$$

$$- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \left(\hat{\int} \bar{A}_i d\bar{x}^r \right) d\bar{x}^l + \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^l} \Gamma^\lambda_{\mu\nu} \left(\hat{\int} \bar{A}_r d\bar{x}^k \right) d\bar{x}^l +$$

$$+ \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^i \partial \bar{x}^l} \left(\hat{\int} \bar{A}_r d\bar{x}^k \right) d\bar{x}^l.$$

Expressing the integrals in the coordinate system (x)

$$\frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^v} \hat{\int} A_u dx^v = \int \frac{\partial x^s}{\partial \bar{x}^i} A_s \frac{\partial \bar{x}^k}{\partial x^r} dx^r -$$

$$- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^r} \frac{\partial x^\nu}{\partial \bar{x}^l} \Gamma^\lambda_{\mu\nu} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^r}{\partial x^\omega} \left(\hat{\int} A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha -$$

$$- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^r}{\partial x^\omega} \left(\hat{\int} A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha +$$

$$+ \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^r} \frac{\partial \bar{x}^k}{\partial x^\omega} \Gamma^\lambda_{\mu\nu} \left(\hat{\int} A_s dx^\omega \right) dx^\nu +$$

$$+ \int \frac{\partial \bar{x}^r}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial \bar{x}^i \partial \bar{x}^l} \frac{\partial x^s}{\partial \bar{x}^r} \frac{\partial \bar{x}^k}{\partial x^\omega} \left(\hat{\int} A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha =$$

$$= \int \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^k}{\partial x^\lambda} A_s dx^\lambda - \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \Gamma^\lambda_{\omega\nu} \left(\hat{\int} A_s dx^\omega \right) dx^\nu -$$

$$- \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^r}{\partial x^\omega} \frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \left(\hat{\int} A_s dx^\omega \right) \frac{\partial \bar{x}^l}{\partial x^\alpha} dx^\alpha +$$

$$+ \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^i} \Gamma^s_{\mu\nu} \left(\hat{\int} A_s dx^\lambda \right) dx^\nu + \int \frac{\partial \bar{x}^k}{\partial x^\lambda} \frac{\partial \bar{x}^l}{\partial x^\alpha} \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^l} \left(\hat{\int} A_s dx^\lambda \right) dx^\alpha.$$

Putting here

$$\frac{\partial^2 x^\lambda}{\partial \bar{x}^r \partial \bar{x}^l} \frac{\partial \bar{x}^r}{\partial x^\omega} \frac{\partial \bar{x}^l}{\partial x^\alpha} = - \frac{\partial x^\lambda}{\partial \bar{x}^\mu} \frac{\partial^2 \bar{x}^\mu}{\partial x^\omega \partial x^\alpha}$$

we get

$$\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^\lambda}\,\hat{}\int A_s\,dx^\lambda=\int\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^\lambda}A_s\,dx^\lambda-\int\frac{\partial \bar{x}^k}{\partial x^\lambda}\frac{\partial x^s}{\partial \bar{x}^i}\Gamma^\lambda_{\omega\nu}\Big(\hat{}\int A_s\,dx^\omega\Big)\,dx^\nu+$$
$$+\int\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial^2 \bar{x}^k}{\partial x^\lambda\partial x^\nu}\Big(\hat{}\int A_s\,dx^\lambda\Big)\,dx^\nu+\int\frac{\partial \bar{x}^k}{\partial x^\lambda}\frac{\partial x^s}{\partial \bar{x}^i}\Gamma^\mu_{s\nu}\Big(\hat{}\int A_\mu\,dx^\lambda\Big)\,dx^\nu+$$
$$+\int\frac{\partial \bar{x}^k}{\partial x^\lambda}\frac{\partial}{\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\Big(\hat{}\int A_s\,dx^\lambda\Big)\,dx^\nu.$$

Using partial integration at several terms we obtain

$$\text{(3.5)}\qquad \frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^\lambda}\,\hat{}\int A_s\,dx^\lambda=\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^\lambda}\int A_s\,dx^\lambda-\int\frac{\partial^2 \bar{x}^k}{\partial x^\lambda\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\Big(\int A_s\,dx^\lambda\Big)\,dx^\nu-$$
$$-\int\Big(\frac{\partial}{\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\Big)\frac{\partial \bar{x}^k}{\partial x^\lambda}\Big(\int A_s\,dx^\lambda\Big)\,dx^\nu-\frac{\partial \bar{x}^k}{\partial x^\lambda}\frac{\partial x^s}{\partial \bar{x}^i}\int\Gamma^\lambda_{\omega\nu}\Big(\hat{}\int A_s\,dx^\omega\Big)\,dx^\nu+$$
$$+\int\frac{\partial^2 \bar{x}^k}{\partial x^\lambda\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\Big(\int\Gamma^\lambda_{\omega\alpha}\Big(\hat{}\int A_s\,dx^\omega\Big)\,dx^\alpha\Big)\,dx^\nu+$$
$$+\int\Big(\frac{\partial}{\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\Big)\frac{\partial \bar{x}^k}{\partial x^\lambda}\Big(\int\Gamma^\lambda_{\omega\alpha}\Big(\hat{}\int A_s\,dx^\omega\Big)\,dx^\alpha\Big)\,dx^\nu+$$
$$+\int\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial^2 \bar{x}^k}{\partial x^\lambda\partial x^\nu}\Big(\hat{}\int A_s\,dx^\lambda\Big)\,dx^\nu+\frac{\partial \bar{x}^k}{\partial x^\lambda}\frac{\partial x^s}{\partial \bar{x}^i}\int\Gamma^\mu_{s\nu}\Big(\hat{}\int A_\mu\,dx^\lambda\Big)\,dx^\nu-$$
$$-\int\frac{\partial^2 \bar{x}^k}{\partial x^\nu\partial x^\lambda}\frac{\partial x^s}{\partial \bar{x}^i}\Big(\int\Gamma^\mu_{s\alpha}\Big(\hat{}\int A_\mu\,dx^\lambda\Big)\,dx^\alpha\Big)\,dx^\nu-$$
$$-\int\frac{\partial}{\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^\lambda}\Big(\int\Gamma^\mu_{s\alpha}\Big(\hat{}\int A_\mu\,dx^\lambda\Big)\,dx^\alpha\Big)\,dx^\nu+$$
$$+\int\Big(\frac{\partial}{\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\Big)\frac{\partial \bar{x}^k}{\partial x^\lambda}\Big(\hat{}\int A_s\,dx^\lambda\Big)\,dx^\nu.$$

Let us denote

$$M^k_i=\hat{}\int A_i\,dx^k-\int A_i\,dx^k+\int\Gamma^k_{rl}\Big(\hat{}\int A_i\,dx^r\Big)\,dx^l-\int\Gamma^r_{il}\Big(\hat{}\int A_r\,dx^k\Big)\,dx^l.$$

Then (3.5) can be brought to the form

$$\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^\lambda}M^\lambda_s=\int\frac{\partial^2 \bar{x}^k}{\partial x^\lambda\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}M^\lambda_s\,dx^\nu+\int\Big(\frac{\partial}{\partial x^\nu}\frac{\partial x^s}{\partial \bar{x}^i}\Big)\frac{\partial \bar{x}^k}{\partial x^\lambda}M^\lambda_s\,dx^\nu,$$
$$\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^\lambda}M^\lambda_s=\int M^\lambda_s\,d\Big(\frac{\partial \bar{x}^k}{\partial x^\lambda}\frac{\partial x^s}{\partial \bar{x}^i}\Big).$$

Differentiating this equality by the parameter t we obtain

$$\frac{\partial \bar{x}^k}{\partial x^\lambda}\frac{\partial x^s}{\partial \bar{x}^i}\frac{dM_s^\lambda}{dt}=0.$$

So all the components M_s^λ must be constants along the curve $\tau(t)$. Since $M_s^\lambda=0$ at the point $\tau(0)$, it follows that $M_s^\lambda=0$ along the whole curve. So we have proved that the system (3.1) is satisfied.

4. Some properties of the contravariant integration

Since the system (2.5) has a unique solution, it follows that *the following two properties are satisfied*

(4.1) $\qquad 1^\circ\ \hat{\int}(A^{i_1\dots i_r}_{j_1\dots j_s}+B^{i_1\dots i_r}_{j_1\dots j_s})\,dx^k=\hat{\int}A^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k+\hat{\int}B^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k$

and

(4.2) $\qquad 2^\circ\ \hat{\int}CA^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k=C\,\hat{\int}A^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k$

where $C=const.$

(4.3) $\qquad 3^\circ\ \nabla_{\dot{\tau}(t)}\left(\hat{\int}A^{i_1\dots i_r}_{j_1\dots j_s}\,dx^k\right)=A^{i_1\dots i_r}_{j_1\dots j_s}\dfrac{dx^k}{dt}$

PROOF. We shall prove it for $r=1$ and $s=2$. The general case can then be proved in the same way. (4.3) means

$$\frac{d}{dt}\left(\hat{\int}A^i_{jl}\,dx^k\right)+\left(\Gamma^k_{\lambda\mu}\,\hat{\int}A^i_{jl}\,dx^\lambda\right)\frac{dx^\mu}{dt}+\left(\Gamma^i_{\lambda\mu}\,\hat{\int}A^\lambda_{jl}\,dx^k\right)\frac{dx^\mu}{dt}-$$

$$-\left(\Gamma^\lambda_{j\mu}\,\hat{\int}A^i_{\lambda l}\,dx^k\right)\frac{dx^\mu}{dt}-\left(\Gamma^\lambda_{l\mu}\,\hat{\int}A^i_{j\lambda}\,dx^k\right)\frac{dx^\mu}{dt}=A^i_{jl}\frac{dx^k}{dt}.$$

In order to prove this we consider the equation

$$\hat{\int}_{\tau(0)}^{\tau(t)}A^i_{jl}\,dx^k=\hat{\int}_{\tau(0)}^{\tau(t)}A^i_{jl}\,dx^k-\hat{\int}_{\tau(0)}^{\tau(t)}\Gamma^k_{rs}\left(\hat{\int}_{\tau(0)}^{\tau(\theta)}A^i_{jl}\,dx^r\right)dx^s-$$

$$-\int_{\tau(0)}^{\tau(t)}\Gamma^i_{rs}\left(\hat{\int}_{\tau(0)}^{\tau(\theta)}A^r_{jl}\,dx^k\right)dx^s+\int_{\tau(0)}^{\tau(t)}\Gamma^r_{js}\left(\hat{\int}_{\tau(0)}^{\tau(\theta)}A^i_{rl}\,dx^k\right)dx^s+\int_{\tau(0)}^{\tau(t)}\Gamma^r_{ls}\left(\hat{\int}_{\tau(0)}^{\tau(\theta)}A^i_{jr}\,dx^k\right)dx^s$$

which is a special case of (2.5). Differentiating by t we obtain

$$\frac{d}{dt}\,\hat{\int}A^i_{jl}\,dx^k=A^i_{jl}\frac{dx^k}{dt}-\Gamma^k_{rs}\left(\hat{\int}A^i_{jl}\,dx^r\right)\frac{dx^s}{dt}-\Gamma^i_{rs}\left(\hat{\int}A^r_{jl}\,dx^k\right)\frac{dx^s}{dt}+$$

$$+\Gamma^r_{js}\left(\hat{\int}A^i_{rl}\,dx^k\right)\frac{dx^s}{dt}+\Gamma^r_{ls}\left(\hat{\int}A^i_{jr}\,dx^k\right)\frac{dx^s}{dt}$$

which was to be proved. ∎

4° The equality

$$\text{(4.4)} \qquad \hat{\int\limits_{\tau(0)}^{\tau(t)}} (\nabla_{\dot{\tau}(t)} A^{i_1 \ldots i_r}_{j_1 \ldots j_s})\, dx^k = A^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{dx^k}{dt} + T^{i_1 \ldots i_r k}_{j_1 \ldots j_s}$$

holds, where T is a tensor field parallel along the curve $\tau(t)$, if and only if $\tau(t)$ is a geodesic line.

PROOF. As in 3° we shall assume that $r=1$ and $s=2$. Using the property 3°, from (4.4) we obtain

$$\nabla_{\dot{\tau}(t)} T^i_{jl} = \nabla_{\dot{\tau}(t)} \left[\hat{\int\limits_{\tau(0)}^{\tau(t)}} (\nabla_{\dot{\tau}(t)} A^i_{jl})\, dx^k - A^i_{jl} \frac{dx^k}{dt} \right] =$$

$$= (\nabla_{\dot{\tau}(t)} A^i_{jl}) \frac{dx^k}{dt} - (\nabla_{\dot{\tau}(t)} A^i_{jl}) \frac{dx^k}{dt} - A^i_{jl} \nabla_{\dot{\tau}(t)} \frac{dx^k}{dt} = - A^i_{jl} \nabla_{\dot{\tau}(t)} \dot{\tau}(t)$$

We see that the tensor field T is parallel if and only if $\nabla_{\dot{\tau}(t)} \dot{\tau}(t)=0$, i.e. $\tau(t)$ is a geodesic line. ∎

5° If the affine connection Γ admits parallel metric field g_{ij} (torsion tensor may not be equal to zero), then the metric tensor behaves as a constant for contravariant integration satisfying property 2°.

This property is an immediate consequence of 3°.

6° Contravariant integration commutes with the contraction for any of the indices $i_1, \ldots, i_r, k$ with $j_1, \ldots, j_s$.

This can be seen directly from the (2.5).

7° Let the tensor field $A^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ be symmetric (antisymmetric) with respect to two lower or upper indices. Then $\hat{\int} A^{i_1 \ldots i_r}_{j_1 \ldots j_s}\, dx^k$ is also symmetric (antisymmetric) with respect to the same indices.

PROOF. For example, let A_{ij} be an antisymmetric tensor field. Adding the equations

$$\hat{\int} A_{ij}\, dx^k = \int A_{ij}\, dx^k - \int \Gamma^k_{\lambda s} \left(\hat{\int} A_{ij}\, dx^\lambda \right) dx^s +$$
$$+ \int \Gamma^\lambda_{is} \left(\hat{\int} A_{\lambda j}\, dx^k \right) dx^s + \int \Gamma^\lambda_{js} \left(\hat{\int} A_{i\lambda}\, dx^k \right) dx^s$$

and

$$\hat{\int} A_{ji}\, dx^k = \int A_{ji}\, dx^k - \int \Gamma^k_{\lambda s} \left(\hat{\int} A_{ji}\, dx^\lambda \right) dx^s +$$
$$+ \int \Gamma^\lambda_{js} \left(\hat{\int} A_{\lambda i}\, dx^k \right) dx^s + \int \Gamma^\lambda_{is} \left(\hat{\int} A_{j\lambda}\, dx^k \right) dx^s$$

we obtain the relation

$$\text{(4.5)} \qquad y^k_{ij} = - \int \Gamma^k_{\lambda s} y^\lambda_{ij}\, dx^s + \int \Gamma^\lambda_{is} y^k_{\lambda j}\, dx^s + \int \Gamma^\lambda_{js} y^k_{i\lambda}\, dx^s,$$

where

$$y^k_{ij} = \hat{\int} A_{ij}\, dx^k + \hat{\int} A_{ji}\, dx^k$$

The system (4.5) can be considered as a Volterra system of integral equations for y^k_{ij}. Thus the solution of (4.5) is unique. However $y^k_{ij}=0$ is also a solution. Therefore

$$\hat{\int} A_{ij}\, dx^k = - \hat{\int} A_{ji}\, dx^k. \quad ∎$$

5. Geometrical interpretation

Independently from the previous results we shall now introduce an integral operator independent from the choice of the coordinate system and we shall prove that it is the same operator which we have introduced earlier.

Let A be a tensor field of type (r, s) along the smooth curve τ. At the points $\tau(0), \tau(t/m), \tau(2t/m), \ldots, \tau((m-1)t/m)$ and $\tau(t)$ we consider the tangent vectors to the curve τ

$$X_0 = \frac{t}{m}\dot\tau(0)\in T_{\tau(0)}(M), \quad X_1 = \frac{t}{m}\dot\tau\left(\frac{t}{m}\right)\in T_{\tau(t/m)}(M),$$

$$\ldots X_{m-1} = \frac{t}{m}\dot\tau\left((m-1)\frac{t}{m}\right)\in T_{\tau((m-1)t/m)}(M) \quad \text{and} \quad X_m = \frac{t}{m}\dot\tau(t)\in T_{\tau(t)}(M)$$

Let $\varphi^{\tau(a)}_{\tau(b)}$ denote a parallel displacement from the point $\tau(a)$ to the point $\tau(b)$ along the curve τ. It can be proved that there exists the following limit

$$(5.1) \quad \lim_{m\to\infty}[\varphi^{\tau(0)}_{\tau(t)}A\otimes X_0 + \varphi^{\tau(t/m)}_{\tau(t)}A\otimes X_1 + \ldots + \varphi^{\tau((m-1)t/m)}_{\tau(t)}A\otimes X_{m-1} + A\otimes X_m]$$

In fact the term in brackets is a tensor of type $(r+1, s)$ at the point $\tau(t)$, and so the limit (5.1) defines a tensor $B^{i_1\ldots i_r k}_{j_1\ldots j_s}(\tau(t))$ of type $(r+1, s)$ at the point $\tau(t)$.

Now we shall prove the following equality

$$(5.2) \qquad \hat{\int\limits_{\tau(0)}^{\tau(t)}} A^{i_1\ldots i_r}_{j_1\ldots j_s}\,dx^k = B^{i_1\ldots i_r k}_{j_1\ldots j_s}(\tau(t)).$$

This equality is obviously satisfied for $t=0$. So it is sufficient to prove that

$$\nabla_{\dot\tau(t)}\hat{\int\limits_{\tau(0)}^{\tau(t)}} A^{i_1\ldots i_r}_{j_1\ldots j_s}\,dx^k = \nabla_{\dot\tau(t)} B^{i_1\ldots i_r k}_{j_1\ldots j_s}(\tau(t)).$$

From (5.1) and (4.3) we obtain

$$\begin{aligned}
\nabla_{\dot\tau(t)} B^{i_1\ldots i_r k}_{j_1\ldots j_s}(\tau(t)) &= \left\{\lim_{m\to\infty}\left[\varphi^{\tau(t(m+1)/m)}_{\tau(t)} B\left(\tau\left(\frac{t(m+1)}{m}\right)\right) - B(\tau(t))\right]\frac{m}{t}\right\}^{i_1\ldots i_r k}_{j_1\ldots j_s} = \\
&= \{\lim_{m\to\infty}[\varphi^{\tau(t(m+1)/m)}_{\tau(t)}(\varphi^{\tau(0)}_{\tau(t(m+1)/m)}A\otimes X_0 + \varphi^{\tau(t/m)}_{\tau(t(m+1)/m)}A\otimes X_1 + \ldots \\
&+ \varphi^{\tau(t)}_{\tau(t(m+1)/m)}A\otimes X_m + A\otimes X_{m+1}) - (\varphi^{\tau(0)}_{\tau(t)}A\otimes X_0 + \\
&+ \varphi^{\tau(t/m)}_{\tau(t)}A\otimes X_1 + \ldots + A\otimes X_m)]\frac{m}{t}\Big\}^{i_1\ldots i_r k}_{j_1\ldots j_s} = \\
&= \left\{\lim_{m\to\infty}\varphi^{\tau(t(m+1)/m)}_{\tau(t)}(A\otimes X_{m+1})\frac{m}{t}\right\}^{i_1\ldots i_r k}_{j_1\ldots j_s} = \\
&= (A\otimes\dot\tau(t))^{i_1\ldots i_r k}_{j_1\ldots j_s} = A^{i_1\ldots i_r}_{j_1\ldots j_s}\frac{dx^k}{dt} = \nabla_{\dot\tau(t)}\hat{\int\limits_{\tau(0)}^{\tau(t)}} A^{i_1\ldots i_r}_{j_1\ldots j_s}\,dx^k
\end{aligned}$$

and the proof is finished.

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