Is the direct square of every 2-generator simple group 2-generator?

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The problem discussed here is a special case of a more general one that has been occupying JOHN WILSON and me for some time. It stems from the following result in [5]:

Theorem 1. *Let G be an infinite simple group and Z an infinite cyclic group. Then the n-th direct power G^n of G is an epimorphic image of $G*Z$, for each n.*

In particular, for finitely generated infinite simple G one has the surprising and somewhat tantalising result:

$$\text{for all } n, \quad d(G) \leqq d(G^n) \leqq 1+d(G),$$

where d stands for minimum number of generators. We do not know of any examples where $d(G^n)=1+d(G)$ for some n; in other words, all finitely generated infinite simple groups G for which we have information are such that $d(G^n)=d(G)$ for all n.

Let us examine the most special of cases, as we did in [5], of a two-generator non-abelian simple group G, and consider G^2. Could it ever be 3-generator? Two elements $x=(a, c)$ and $y=(b, d)$ of G^2 have a chance of generating G^2 only if $G=\langle a, b\rangle=\langle c, d\rangle$. Suppose that this is so from now on.

For every two-variable word w, we have

$$w(x, y) = \big(w(a, b), w(c, d)\big).$$

This, if there exists a word w with $w(a, b)\neq 1$ and $w(c, d)=1$, we have an element $(t, 1)$ of G^2 with $t\neq 1$; and very standard arguments show that $\langle x, y\rangle$ contains $t^G\times 1=G\times 1$, the "first factor" of G, and similarly it contains $1\times G$, so that it is G^2. The existence of a word w with $w(a, b)=1$, $w(c, d)\neq 1$ yields the same conclusion.

Consequently (a, c) and (b, d) generate G^2 if and only if $G=\langle a, b\rangle=\langle c, d\rangle$ and there is no automorphism α of G with $a^\alpha=c$, $b^\alpha=d$. It follows that $d(G^2)=3$ if and only if every two ordered pairs a, b and c, d of generators for G are connected by an automorphism α: $a^\alpha=c$, $b^\alpha=d$. In [1], (not necessarily simple) 2-generator groups *without* this property are called J-groups; we shall call groups *with* the property $\bar{J}$-groups. There are plenty of non-simple $\bar{J}$-groups, for example the direct power S^h of a finite 2-generator non-abelian simple groups S such that $d(S^h)=2$, $d(S^{h+1})=3$ (see P. HALL [2]).

Our first problem was posed for finite groups by J. NEUBÜSER.

Problem 1. Does there exist a two-generator non-abelian simple $\mathcal{J}$-group?

As far as I know (modulo the classification theorem), every finite non-abelian simple group G can be generated by a generating pair a, b with a of order 2 and b (of course) of some other order. Thus there is no automorphism of G interchanging a, b, that is, taking the ordered pair of generators a, b to b, a. It is a pity that this simple result cannot be proved directly!

In fact, in a $\mathcal{J}$-group G, there is a constant k depending only on G such that for every generating pair $\{a, b\}$, $|a|=|b|=|a^m b|=|ab^n|=k$ for all integers m, n. WILSON and I thought immediately of Tarski groups of prime exponent p, that is, 2-generator infinite groups of exponent p with all non-trivial proper subgroups of order p. The existence of such monsters for very large p has finally been established by Ol'shanskii, in a truly remarkable paper [4].

We had thought that some Tarski group might be a $\mathcal{J}$-group, on the grounds that there seemed nothing obvious against such a hypothesis. The main point of this article is to prove that no Tarski group is a $\mathcal{J}$-group, and for a reson that *is* obvious.

Take any Tarski monster $G=\langle a, b\rangle$ of exponent p. Then $\langle a, a^b\rangle$ must be G too, else $a^b=a^\lambda$ for some integer λ, which would make G finite. Thus G is generated by two conjugates of a single element. Such groups are studied in [1]. [3] contains a proof that any finitely generated simple group with all proper subgroups nilpotent has the same property.

Suppose then that $G=\langle a, a^b\rangle\neq 1$. As in [1], we prove that there is no automorphism α of G such that $a^\alpha=a, b^\alpha=ab$. If there were, one would have $(a^b)^\alpha=(a\alpha)^{b\alpha}= =a^{ab}=a^b$, so that α fixes a and a^b. But $G=\langle a, a^b\rangle$, so that α is the identity map; clearly, it is not, since $b^\alpha=ab$ and $a\neq 1$.

Problem 2. Can every two-generator simple group be generated by two conjugates of a single element?

See [1] for some information on the finite case. In general, one has the following result.

Theorem 2. *Let G be a non-abelian simple group generated by two conjugates a and a^b of the element a. Then G^n is generated by the two elements $u=(a, a, \ldots, a)$ and $v=(b, ab, a^2b, \ldots, a^{n-1}b)$, where n is the order of a (or is arbitrary if a has infinite order). In particular, $d(G^p)=2$ for a Tarski group G of exponent p.*

PROOF. There is no automorphism $a\to a$, $a^r b\to a^s b$ unless $a^r=a^s$ (same reasoning as before), so that the group H generated by u and v contains elements of the form $(\ldots, x, \ldots, y, \ldots)$, where x and y are arbitrary elements of G, in the r-th and s-th positions in the strings. In other words, we can prescribe elements of G in any two given positions. It will be enough of an indication of the proof if we show how to prescribe elements in any three positions. Consider the first three positions for definiteness, and ignore the remainder. We can obtain elements

$$c = (*, c_2, c_3)$$

$$d = (d_1, *, d_3)$$

where the c_i, d_j are arbitrary elements of G. Choosing $c_2=d_1=1$ but c_3, d_3 arbitrary, we find that

$$[c, d] = (1, 1, [c_3, d_3]).$$

Thus we can of course provide the whole of the third factor, and the rest is routine.

I have tried to improve this result for Tarski groups of prime exponent, but failed. It could be that an example exists with $d(G^{p+1})=3$, but I have no idea how to prove it. Indeed, I hope that the answer to the final problem is yes.

Problem 3. If G is a Tarski group, is $d(G^n)=2$ for all n?

References

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