Evaluation of the dedeterminant and the permanent of certain matrices

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Summary. In this paper we deal with finite matrices, which have same elements in the diagonal, over the diagonal, and under the diagonal, respectively. Such a matrix is said to be a pseudo diagonal matrix. We formulate each general theorem for the determinant, and for the permanent of these. Thus we obtain procedures to evaluate determinants, and permanents, respectively. Among others we derive a procedure to trace the evaluation of the permanents back to the evaluation of determinants. We make a detailed study of the special cases when the addition of pseudo matrices is symmetric, skew-symmetric, and skew, respectively. Finally we determine the permanental roots of the skew-symmetric Jacobi matrices in special cases.

1. Introduction and the general theorems with proofs

Denote by E the current unit matrix. Let A be a finite square matrix with real or complex entries. The roots of the equations

$$\mathrm{Det}\,(A+\lambda E) = 0,$$

and

$$\mathrm{Per}\,(A+\lambda E) = 0$$

we say to be the eigenvalues, and the permanental roots of the matrix A, respectively.

The following polynomials play a role in the formulation of the theorems.

$$D^{(n)}(z_1, z_2, \ldots, z_n) =$$

$$= \begin{vmatrix} \binom{1}{0} z_1 & \binom{1}{1} & 0 & \ldots & 0 & 0 \\ \binom{2}{0} z_2^2 & \binom{2}{1} z_2 & \binom{2}{2} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \binom{n-1}{0} z_{n-1}^{n-1} & \binom{n-1}{1} z_{n-1}^{n-2} & \binom{n-1}{2} z_{n-1}^{n-3} & \ldots & \binom{n-1}{n-2} z_{n-1} & \binom{n-1}{n-1} \\ \binom{n}{0} z_n^n & \binom{n}{1} z_n^{n-1} & \binom{n}{2} z_n^{n-2} & \ldots & \binom{n}{n-2} z_n^2 & \binom{n}{n-1} z_n \end{vmatrix}$$

$$(n = 2, 3, \ldots,).$$

$$D_1^{(2\nu+1)}(z_3, z_5, \ldots, z_{2\nu+1}) =$$

$$= \begin{vmatrix} \binom{3}{0} z_3^2 & \cdot & \binom{3}{2} & 0 & \ldots & 0 & 0 \\ \binom{5}{0} z_5^4 & \cdot & \binom{5}{2} z_5^2 & \binom{5}{4} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \binom{2\nu-1}{0} z_{2\nu-1}^{2(\nu-1)} & \cdot & \binom{2\nu-1}{2} z_{2\nu-1}^{2(\nu-2)} & \binom{2\nu-1}{4} z_{2\nu-1}^{2(\nu-3)} & \ldots & \binom{2\nu-1}{2\nu-4} z_{2\nu-1}^{2} & \binom{2\nu-1}{2\nu-2} \\ \binom{2\nu+1}{0} z_{2\nu+1}^{2\nu} & \cdot & \binom{2\nu+1}{2} z_{2\nu+1}^{2(\nu-1)} & \binom{2\nu+1}{4} z_{2\nu+1}^{2(\nu-2)} & \ldots & \binom{2\nu+1}{2\nu-4} z_{2\nu+1}^{4} & \binom{2\nu+1}{2\nu-2} z_{2\nu+1}^{2} \end{vmatrix}$$

$$(\nu = 1, 2, \ldots),$$

$$D_2^{(2\nu)}(z_2, z_4, z_{2\nu}) =$$

$$= \begin{vmatrix} \binom{2}{0} z_2^2 & \cdot & \binom{2}{2} & 0 & \ldots & 0 & 0 \\ \binom{4}{0} z_4^4 & \cdot & \binom{4}{2} z_4^2 & \binom{4}{4} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \binom{2\nu-2}{0} z_{2\nu-2}^{2\nu-2} & \cdot & \binom{2\nu-2}{2} z_{2\nu-2}^{2\nu-4} & \binom{2\nu-2}{4} z_{2\nu-2}^{2\nu-6} & \ldots & \binom{2\nu-2}{2\nu-4} z_{2\nu-2}^{2} & \binom{2\nu-2}{2\nu-2} \\ \binom{2\nu}{0} z_{2\nu}^{2\nu} & \cdot & \binom{2\nu}{2} z_{2\nu}^{2\nu-2} & \binom{2\nu}{4} z_{2\nu}^{2\nu-4} & \ldots & \binom{2\nu}{2\nu-4} z_{2\nu}^{4} & \binom{2\nu}{2\nu-2} z_{2\nu}^{2} \end{vmatrix}$$

$$(\nu = 1, 2, \ldots).$$

Definition 1.1. Let $n \geqq 2$ be an integer. The $n \times n$ matrix $B_n = (a_{jk})_1^n$ is said to be a pseudo scalar matrix generated by the real or complex numbers a, b, c, if conditions

$$(1.1) \qquad a_{ii} = a; \quad a_{ij} = b, \quad i > j; \quad a_{ij} = c, \quad i < j \quad (i, j = 1, \ldots, n)$$

are satisfied.

The two general Theorems, which have an importance in Section 2., are the following.

Theorem 1.1. *If B_n is a $n \times n$ pseudo scalar matrix generated by the numbers a, b, c, then*

$$(1.2) \qquad \operatorname{Det} B_n = (-1)^n D^{(n)}(\omega_1, \omega_2, \ldots, \omega_n),$$

where ω_k is an arbitrary eigenvalue of the pseudo diagonal matrix B_k also generated by the numbers a, b, c, with $k = 1, \ldots, n$.

Theorem 1.2. *Under the assumption of Theorem 1.1. we get*

$$(1.3) \qquad \operatorname{Per} B_n = (-1)^n D^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where λ_k is an arbitrary permanental root of B_k.

PROOF. Since ω_k and λ_k is an eigenvalue, and a permanental root of B_k, respectively, we have

$$\operatorname{Det}(B_k+\omega_k E) = \sum_{j=0}^{k} \binom{k}{j} \omega_k^{k-j} \operatorname{Det} B_j = 0,$$

$$\operatorname{Per}(B_k+\lambda_k E) = \sum_{j=0}^{k} \binom{k}{j} \lambda_k^{k-j} \operatorname{Per} B_j = 0$$

$$(k = 1, \ldots, n),$$

which are linear equation systems for the unknowns Det B_j and Per B_j $(j=1, \ldots, n)$, respectively. Solving these for Det B_n, and for Per B_n, respectively, we get solutions (1.2) and (1.3) after elementary determinantal transformations.

The use of Theorem 1.1. is mainly that the determinant of the left hand side of (1.2) can sometimes be evaluated easily, thus the value of the determinant on the right hand side is known too.

However the use of Theorem 1.2. is in fact that the evaluation of a permanent can be traced back to the evaluation of a determinant. This circumstance has an importance, since the evaluation of a determinant of a given matrix is an easier problem than the calculation of the permanent of the same matrix. On the other hand it seems that this process is difficult, since it is necessary to count the permanent roots, but one only among the roots of B_k $(k=1, \ldots, n)$. This can be obtained in many cases easily by an ad hoc consideration.

2. Applications of Theorems 1.1. and 1.2.

In this section we apply Theorems 1.1., and 1.2. in the cases when the pseudoscalar matrices are symmetric, skew, and skew symmetric, respectively.

2.1. The case of the symmetric pseudo scalar matrices

Let $b=c$ in (1.1). Then

$$B_k = (a-b)E + bM_k \quad (k = 1, 2, \ldots),$$

where M_k is the $k\times k$ matrix with all entrics 1. Since the eigenvalues of B_k are $-[a+(k-1)b]$, and $b-a$, we get the following result by Theorem 1.1. choosing $\omega_1=a$, $\omega_k=b-a$ $(k=2, \ldots, n)$:

Theorem 2.1.1. *If the symmetric pseudo-scalar matrices B_k $(k=1, 2, \ldots)$ are generated by numbers a and b, then*

$$D^{(n)}(-a, b-a, \ldots, b-a) = -[a+(m-1)b](a-b)^{n-1}$$

for all positive integers n.

As special cases we get

$$D^{(n)}(-1, 1, \ldots, 1) = 1-2n,$$

$$D^{(n)}(1, 1, \ldots, 1) = 1.$$

Theorem 2.1.2. *If λ_k is an arbitrary root of the polynomial*

$$P_k(\lambda) = \sum_{j=0}^{k} \frac{\lambda^j}{j!},$$

then

$$D^{(n)}(\lambda_1, \ldots, \lambda_n) = (-1)^n n!.$$

PROOF. If $B_k = M_k$, then $\operatorname{Per} B_k = k!$, thus

$$\operatorname{Per}(B_k + \lambda E) = k!\, P_k(\lambda).$$

2.2. Skew symmetric, and skew pseudo-scalar matrices

In this section we deal with pseudo-scalar matrices, which are simultaneously skew symmetric, and skew, respectively.

It is known, that the $n \times n$ matrix A is skew symmetric if $A^* = -A$, or in other words if $A = iB$, where $B^* = B$. A^* denotes the transpose of A, and i is the imaginary-unit.

From the Definition we get immediately that $\operatorname{Det} A = \operatorname{Per} A = 0$ if n is an odd positive integer. Let S_1, and S_2 be the sum of the terms belonging to the even, and to the odd permutations, respectively, in the expansion of $\operatorname{Det} A$. From the foregoing we get $S_1 = S_2 = 0$ if n is odd.

Similarly if A is a $n \times n$ skew symmetric matrix with real elements, and if $-iA$ is a $n \times n$ positive definite Hermite symmetric matrix, then $\operatorname{Per} A < 0$ for $n \equiv 2 \bmod 4$, and $\operatorname{Per} A > 0$ for $n \equiv 0 \bmod 4$.

It is obvious that the diagonal elements of a skew symmetric matrix are equal to zero.

In the following we consider skew symmetric pseudo scalar matrices in the special case when $a = 0$, $b = -c = 1$ in (1.1). We denote by A_n such a $n \times n$ matrix. We have already mentioned that the determinant and the permanent of these matrices are equal to zero for odd n, and it is easiy to verify that their determinant equals to 1 for even n.

Lemma 2.2.1. *The eigenvalues of A_n are given by*

$$\omega_k = i \operatorname{tg} \frac{(2k+1)\pi}{2n} \quad \left(k = 0, 1, \ldots, n-1;\ k \neq \frac{n-1}{2}\right) \tag{2.1}$$

and $\omega = 0$ is also an eigenvalue if n is odd. Moreover the components of the eigenvector belonging to ω_k are given by

$$\chi_j^{(k)} = \left(-\exp \frac{(2k+1)\pi}{n}\right)^j \chi_n^{(k)} \quad (j = 1, \ldots, n-1), \quad \chi_n^{(k)}, \tag{2.2}$$

and if n is odd, the those of the eigenvector belonging to $\omega = 0$ are

$$\chi_j^{(0)} = (-1)^j \chi_n^{(0)} \quad (j = 1, \ldots, n-1), \quad -\chi_n^{(0)}. \tag{2.3}$$

PROOF. Since

$$\mathrm{Det}\,(A_n+\lambda E) = \sum_{k=0} \binom{n}{2k} \lambda^{n-k} = \lambda^{\delta}\frac{1}{2}[(1+\lambda)^n+(1-\lambda)^n],$$

where $\delta=0$ if n is even, and $\delta=1$ if n is odd, the eigenvalues of A_n are the solutions of equation

$$1+\lambda = (1-\lambda)\,\alpha \tag{2.4}$$

(except the eigenvalue $\lambda=0$ in the case of odd n), where α runs over the numbers

$$\alpha_k = \exp\,(i\varphi_k), \quad \varphi_k = \frac{(2k+1)\pi}{n} \tag{2.5}$$

$$(k = 0, 1, \ldots, n-1).$$

We have by (2.4)

$$(1+\alpha)\,\lambda = \alpha-1,$$

and we got from here

$$\lambda \cos\frac{\varphi}{2} = i\sin\frac{\varphi}{2} \tag{2.6}$$

using known trigonometric relations, where φ is an angle from (2.5). If $\cos\frac{\varphi}{2}=0$ then $\sin\frac{\varphi}{2}=0$ too by (2.6). But this is a contradiction, which can be occurred in the case only if $k=\frac{n-1}{k}$, i.e. if n is an odd number. Thus by (2.6) the eigenvalues of A_n are equal to (2.1), and $\omega=0$ is also an eigenvalue if n is odd.

We will now proceed to determine the eigenvectors belonging to the eigenvalues of A_n. If $\chi=(\chi_j)$ is an eigenvector belonging to the eigenvalue ω of A_n, then the equation system

$$(A_n+\omega E)\,\chi = 0$$

is satisfied. Adding the last equation to the previous ones, we obtain an equivalent equation system

$$-2(\chi_1+\ldots+\chi_{k-1})+(\omega-1)\,\chi_k = -(1+\omega)\,\chi_n \tag{2.7}$$

$$(k = 1, \ldots, n-1).$$

Since 1 is not an eigenvalue of A_n, the determinant $(\omega-1)^{n-1}$ of the equation system (2.7) does not wanish. Using Cramer's rule we get after conversions that

$$\chi_j = -\left(\frac{\omega+1}{\omega-1}\right)^j \chi_m \quad (j=1, \ldots, n-1), \quad \chi_n \tag{2.8}$$

are the components of the eigenvector belonging to the eigenvalue ω of A_n. If we substitute $\omega=\omega_k$ defined by (2.1) and $\omega=0$ in (2.8), then (2.2) and (2.3) give us the components of the vectors belonging to ω_k, and to $\omega=0$, respectively.

By Theorem 1.1. and by Lemma 2.2.1. we get the following

Theorem 2.2.1. *If* ω_{2k} *is taken from*

$$i\,\mathrm{tg}\,\frac{(2j+1)\pi}{4k} \quad (j = 0, 1, \ldots, 2k-1),$$

then

$$D_2^{(2\nu)}(\omega_2, \omega_4, \ldots, \omega_{2\nu}) = (-1)^\nu$$

holds for all positive integers ν.

Theorem 2.2.2. *If* ω_{2k+1} *is taken from*

$$0,\ i\,\mathrm{tg}\,\frac{(2j+1)\pi}{2(2k+1)} \quad (j = 0, 1, \ldots, 2k;\ j \neq k),$$

then

$$D_1^{(2\nu+1)}(\omega_3, \omega_5, \ldots, \omega_{2\nu+1}) = 0$$

holds for all positive integers ν.

By Theorem 1.2. we get similar Theorems, but we did'nt know explicity the permanental roots of A_n in these cases. It is a conjecture that 1 is a root if n is even. Thus we have the following two statements:

Theorem 2.2.3. *If* λ_{2k+1} *is an arbitrary permantal root of* A_{2k+1}, *then*

$$D_1^{(2\nu+1)}(\lambda_3, \lambda_5, \ldots, \lambda_{2\nu+1}) = 0$$

holds for all positive integers ν.

Conjecture 2.2.1. The identity

$$\mathrm{Per}\, A_{2\nu} = (-1)^\nu D_2^{(2\nu)}(1, 1, \ldots, 1)$$

holds for all positive integers ν.

In the following we consider the skew matrix $A_n - E$. If ω_k is an eigenvalue of this matrix, then

$$\omega_k = 1 + i\,\mathrm{tg}\,\psi_k = \frac{e^{i\psi_k}}{\cos\psi_k}, \quad \psi_k = \frac{(2k+1)\pi}{2n}$$

$$\left(k = 0, 1, \ldots, n-1;\ k \neq \frac{n-1}{2}\right),$$

and 1 too is an eigenvalue if n is odd. It can be verified easily that

$$\mathrm{Det}\,(A_n - E) = (-1)^n 2^{n-1}.$$

By Theorem 1.1. we get the following statement:

Theorem 2.2.4. *Let* ω_k *be an value from*

$$\left(\cos\frac{(2j+1)\pi}{2k}\right)^{-1} \exp\left\{i\frac{(2j+1)\pi}{2k}\right\}$$

$$\left(j = 0, 1, \ldots, k-1;\ j \neq \frac{k-1}{2}\right)$$

(adding to these the number 1 *if k is odd). Then*

$$D^{(n)}(\omega_1, \omega_2, \dots, \omega_n) = 2^{n-1}$$

holds for all positive integers $n \geqq 2$.

3. Matrices of the Jacobi type

In this section we deal with the permanents and with the permanental roots of Jacobi matrices.

It is known that the eigenvalues of a skew symmetric matrix with real entries are imaginary numbers. At the some time we do not know much about the permanent, in particular about the permanental roots of matrices, especially of skew symmetric ones. But more can be said if matrices of Jacobi type are considered.

Let n be a positive integer, and let

$$a_j, b_k, c_k \quad (j = 1, \dots, n;\ k = 1, \dots, n-1)$$

be real numbers. As it is known, the matrix

$$J_n = (a_{jk})_1^n$$

with

$$a_{jj} = a_j, \quad a_{jj+1} = b_j, \quad a_{j-1j} = c_j$$
$$a_{jk} = 0 \quad \text{for} \quad |j-k| \geqq 2$$

is said to be a matrix of the Jacobi type. Introducing the notation

$$P_k(\lambda) = \operatorname{Per}(J_k + \lambda E) \quad (k = 1, \dots, n), \quad P_0(\lambda) \equiv 1,$$

it can be shown ([1], p. 77) that the relation

$$P_k(\lambda) = (a_k + \lambda) P_{k-1}(\lambda) + b_{k-1} c_{k-1} P_{k-2}(\lambda) \tag{3.1}$$
$$(k = 2, \dots, n)$$

holds.

In the following we consider only the case when

$$b_k c_k < 0 \quad (k = 1, \dots, n-1). \tag{3.2}$$

Thus by (3.1) the same can be said about the roots of $P_n(\lambda) = 0$ as about the roots of

$$\operatorname{Det}(J_n + \lambda E) = 0$$

if in the last equation J_n is a normal matrix, i.e. $b_k > 0$, $c_n > 0$ for $k = 1, \dots, n-1$. Therefore the following statement holds ([1], p. 80. Satz 1.):

Theorem 3.1. *Assume that condition* (3.2) *is satisfied. Then the permanental roots of* J_n *are real numbers with multiplicity one.*

Theorem 3.2. *If the diagonal elements of* J_n *are zero, then*

$$\operatorname{Per} J_n = \begin{cases} 0 \\ b_1 \dots b_{n-1} c_1 \dots c_{n-1} \end{cases} \text{if } n \text{ is} \begin{cases} \text{odd} \\ \text{even.} \end{cases}$$

PROOF. Under the given condition

$$P_k(0) = b_{k-1}c_{k-1}P_{k-2}(0) \tag{3.3}$$

holds for $k=2, \dots, n$. By (3.3)

$$P_{2\nu+1}(0) = \prod_{k=1}^{\nu} b_{2k}c_{2k}P_1(0),$$

and

$$P_{2\nu}(0) = \prod_{k=1}^{\nu} b_{2k-1}c_{2k-1}P_0(0).$$

Since $P_1(0)=0$ and $P_0(0)=1$, we obtain the statement of the Theorem.

We say that the matrix J_n of Jacobi type is a skew matrix with parameters a and b, if conditions

$$a_k = a \quad (k = 1, \dots, n).$$

$$b_k = -c_k = b > 0 \quad (k = 1, \dots, n-1) \tag{3.4}$$

are satisfied.

Theorem 3.3. *If J_n is a skew matrix of Jacobi type with parameters a and b, then the permanental roots of J_n are given by*

$$\lambda_k = 2b\cos\frac{k\pi}{n+1} - a \quad (k = 1, \dots, n).$$

PCOOF. First of all we consider the polynomial $P_n(\lambda)$. By (3.4) this polynomial satisfies the homogenenous linear difference equation

$$P_n(\lambda) = (a+\lambda)P_{n-1}(\lambda) - b^2P_{n-2}(\lambda) \tag{3.5}$$

with contant coefficients, and with initial conditions

$$P_0(\lambda) \equiv 1, \quad P_1(\lambda) = a+\lambda. \tag{3.6}$$

Substituting $a+\lambda=2y$ the characteristic equation of (3.5) is

$$z^2 - 2yz + b^2 = 0$$

with roots

$$z_1 = y+\sqrt{y^2-b^2}, \quad z_2 = y-\sqrt{y^2-b^2}. \tag{3.7}$$

Thus

$$P_n(\lambda) = C_1z_1^n + C_2z_2^n, \tag{3.8}$$

where the quantities C_1 and C_2 are independent of n, but can depend on λ. Taking the intial condition (3.6) into consideration we obtain by (3.8) that

$$2\sqrt{y^2-b^2}\,P_n(\lambda) = z_1^{n+1} - z_2^{n+1},$$

$$\lambda = 2y - a. \tag{3.9}$$

The roots of (3.9) are those of the equation

$$y+\sqrt{y^2-b^2} = \left(y-\sqrt{y^2-b^2}\right)\alpha_k, \tag{3.10}$$

where

$$\alpha_k = \exp\left\{\frac{2k\pi}{n+1} i\right\} \quad (k = 0, 1, \ldots, n).$$

In the case of $k=0$ we get from (3.10) that $y=\pm b$, which are the roots of the first factor of the left hand side of (3.9). Thus the roots of $P_n(\lambda)$ can be obtained by (3.10) if k runs over $1, \ldots, n$. Since α_k for $k\neq 0$ is a complex number, the equality (3.10) is satisfied if and only if $y^2<b^2$. Using the substitution $y=\cos t$, we have by (3.10) that

$$t = \frac{k\pi}{n+1} i \quad (k = 1, \ldots, n).$$

i.e. the permanental roots of J_n are the numbers defined by Theorem 3.3. It can be verified easily that all these numbers satisfy the equality $P_n(\lambda)=0$.

By Theorem 3.3. we have the following results:

Corollary 3.1. *If J_n is a $n\times n$ skew matrix of Jacobi type with parameters a and b, then*

$$\operatorname{Per} J_n = \prod_{k=1}^{n}\left(a-2b\cos\frac{k\pi}{n+1}\right)$$

$$(n = 1, 2, \ldots).$$

Corollary 3.2. *If J_n is a $n\times n$ skew matrix of Jacobi type with parameters* 1 *and* 1/2, *then*

$$\operatorname{Per} I_n = \frac{n+1}{2^n} \quad (n = 0, 1, \ldots).$$

PROOF. A known identity says that

$$2^{n-1}\prod_{k=0}^{n-1}\sin\left(x+\frac{k\pi}{n}\right) = \sin(nx).$$

From here

$$2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \lim_{x\to 0}\frac{\sin(nx)}{\sin x} = n.$$

Thus

$$2^{2n-1}\prod_{k=1}^{2n-1}\sin\frac{k\pi}{2n} = 2^{2n-1}\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{2n} = 2n.$$

Using Corollary 3.1. we get for $a=1$, $b=1/2$ that

$$\operatorname{Per} J_n = 2^n\prod_{k=1}^{n}\sin^2\frac{k\pi}{2(n+1)} = \frac{n+1}{2^n}$$

corresponding to the statement of Corollary 3.1.

The proof of the following Theorem is similar to that of Theorem 3.3.

Theorem 3.4. *Assume that the entries of the* $n \times n$ *Jacobi matrix* J_n *satisfy the conditions*

$$\begin{cases} a_k = a \quad (k = 1, \ldots, n). \\ b_k = c_k = b > 0 \quad (k = 1, \ldots, n-1). \end{cases}$$

Then the permanental roots of J_n *are given by*

$$\lambda_k = 2ib \cos \frac{k\pi}{n+1} - a \quad (k = 1, \ldots, n).$$

It would seem that the permanental roots of all real symmetric matrices are complex numbers, and those of all real skew matrices are real numbers. But this statement is not true. It is not difficult to construct a counter-example.

References

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