Some Matrix Inequalities and Applications to Probabilistic Inequalities

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Abstract. If $A=(a_{ij})$ is an $m\times n$ real matrix, then let $S(A)$ denote the sum of the entries of A, that is $S(A)=\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$. The purpose of this article is to present inequalities involving the function $S(\cdot)$ and finite sequences of matrices with positive real entries. Then, as an application, we show how to transform these inequalities into probabilistic ones. The results so obtained are generalizations of previous results of T. F. MORI and G. J. SZEKELY, and T. TOLLIS.

1. Introduction and notation

Let $A=(a_{ij})$ be an $m\times n$ real matrix. Let $S(A)$ denote the sum of the entries of A, that is

$$S(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}.$$

In this article inequalities involving the function $S(\cdot)$ defined on matrices with positive real entries are studied. One of our objectives is to point out applications of the function $S(\cdot)$ in probability theory.

Let M denote the class of $m\times n$ matrices with positive real entries and let R denote the class of $m\times n$ random matrices with entries positive random variables. If $A=(a_{ij})$, $B=(b_{ij})$ are members of M (or R), then

$$A \leqq B \quad \text{means that} \quad a_{ij} \leqq b_{ij}$$

for all $i=1, \ldots, m$ and $j=1, \ldots, n$. (The relation $\leqq$ on random variables is used in the usual sense.) Throughout this paper φ will denote a linear map from R into M such that:

a) $\varphi(A)*\varphi(A)\leqq\varphi(A*A)$ and

b) if $A\leqq B$ then $\varphi(A)\leqq\varphi(B)$

for all $A, B\in R$, where * denotes the Hadamard (elementwise) product.

In §2 several preliminary results dealing primarily with the relation "$\leqq$", the Hadamard products and the function $S(\cdot)$ are obtained.

In §3 we are dealing with finite sequences of matrices $A_1, \ldots, A_k$, where $A_i\in R$ $(i=1, \ldots, k)$, and we find upper bounds for $\prod_{i=1}^{k} S(\varphi(A_i))$ and $\sum_{i=1}^{k} S(\varphi(A_i))$. This is accomplished by applying the results from §2. In most of the inequalities we find necessary and sufficient conditions for the case of equality.

In §4 we prove that the expectation function $E(\cdot)$ satisfies conditions a) and b) of the definition of φ. Hence, the inequalities of §3 are transformed into inequalities involving expectations of positive random variables. In particular, the inequalities of §4 are generalizations of old ones obtained by T. F. MORI and G. J. SZEKELY [4] and T. TOLLIS [6].

2. Preliminary results

Clearly $S(\cdot)$ is a linear map on M. Also, it is easy to see that the classes M and R are closed under Hadamard products.

Lemma 1. *If $A, B \in M$ (or R) then $A*B \in M$ (or R).*

Also, the following two lemmas play a significant role in proving inequalities in §3.

Lemma 2. *If $A_i, B_i \in M$ (or R) $(i=1, \dots, k)$ and $A_i \leqq B_i$, then*

$$A_1 * \dots * A_k \leqq B_1 * \dots * B_k.$$

PROOF. Let $k=2$. Let $A_1=(\overset{(1)}{a_{ij}})$, $A_2=(\overset{(2)}{a_{ij}})$, $B_1=(\overset{(1)}{b_{ij}})$ and $B_2=(\overset{(2)}{b_{ij}})$. By assumption $\overset{(1)}{a_{ij}} \leqq \overset{(1)}{b_{ij}}$ and $\overset{(2)}{a_{ij}} \leqq \overset{(2)}{b_{ij}}$ for all $i=1, \dots, n$ and $j=1, \dots, m$. Since $A_i, B_i \in M$, then if we multiply these inequalities, we obtain

$$\overset{(1)}{a_{ij}}\overset{(2)}{a_{ij}} \leqq \overset{(1)}{b_{ij}}\overset{(2)}{b_{ij}} \quad \text{for all} \quad i = 1, \dots, n \quad \text{and} \quad j = 1, \dots, m.$$

Hence $A_1 * A_2 \leqq B_1 * B_2$. The result now follows by induction on k.

A similar argument yields the case where $A_i, B_i \in R$.

Lemma 3. *Let $A, B \in M$ (or R). If $A \leqq B$ then*

$$S(A) \leqq S(B),$$

with equality if and only if $A=B$.

PROOF. Let $A=(a_{ij})$, $B=(b_{ij})$ be members of M or R. Since $A \leqq B$ then

$$a_{ij} \leqq b_{ij} \quad \text{for all } i, j.$$

Hence $S(A) \leqq S(B)$. For the case of equality, suppose that $A \neq B$. Then $a_{ij} < b_{ij}$ for some i, j. Therefore $S(A) < S(B)$. Hence the result follows.

Kantorovich's inequality will be used in §3. We shall use a formulation of this important inequality, due to CLAUSING [1].

Let $0 < M_1 \leqq m_i \leqq M_2$ $(i=1, \dots, k)$. Suppose $0 < a_i$ $(i=1, \dots, k)$, $\sum_{i=1}^{k} a_i = 1$. Set $\gamma_I = \sum_{i \in I} a_i$ for $I \subset \{1, \dots, k\}$. Let $I_o \subset \{1, \dots, k\}$ be such that

$$\left|\gamma_{I_0} - \frac{1}{2}\right| \leqq \left|\gamma_I - \frac{1}{2}\right| \quad \text{for all} \quad I \subset \{1, \dots, k\}.$$

Define

$$C_3(m, M) \equiv 1+\gamma_{I_0}(1-\gamma_{I_0})\frac{(M_2-M_1)^2}{M_1 M_2}.$$

Lemma 4. *(Kantorovich's inequality). The following inequality holds*

$$\left[\sum_{i=1}^{k} a_i m_i\right]\left[\sum_{i=1}^{k}\frac{a_i}{m_i}\right] \leqq C_3(M_1, M_2),$$

with equality if and only if there is a subset I of $\{1, \dots, k\}$ such that $\sum_{i\in I} a_i = 1/2$.

PROOF [1].

If $n=2$, then

$$C_3(M_1, M_2) = 1+a_1 a_2\frac{(M_2-M_1)^2}{M_1 M_2}.$$

Also, in the special case $a_i=\frac{1}{k}$ $(i=1, \dots, k)$

$$C_3(M_1, M_2) = 1+\frac{(M_2-M_1)^2}{4M_1 M_2}, \quad \text{if k is even}$$

$$C_3(M_1, M_2) = 1+\left(1-\frac{1}{k^2}\right)\frac{(M_2-M_1)^2}{4M_1 M_2}, \quad \text{if } k \text{ is odd.}$$

Lemma 5. *If $A\in M$ then*

$$S(A*A) \geqq \frac{S(A)^2}{mn},$$

with equality if and only if $A=\lambda I$, $\lambda>0$.

PROOF. The result follows from Cauchy's inequality.

Definition: The numbers $(a_i), (b_i)$ $(i=1, \dots, k)$ are similarly ordered if

$$(a_i-a_j)(b_i-b_j) \geqq 0,$$

for all i, j and oppositely ordered if the inequality is always reversed.

Lemma 6. *Let $(a_i), (x_i), (y_i)$ $(i=1, \dots, k)$ be positive numbers such that*

$$\sum_{i=1}^{k} a_i = 1, \quad \sum_{i=1}^{k} a_i\frac{x_i}{y_i} \leqq 1.$$

Suppose that (x_i), $\left(\frac{1}{y_i}\right)$ $(i=1, \dots, k)$ are similarly ordered. Then the inequality

$$\sum_{i=1}^{k} a_i x_i \leqq \prod_{i=1}^{k} y_i^{a_i}$$

holds.

PROOF. The left-hand inequality is Tchebychef's inequality [2, p. 43].

$$\left[\sum_{i=1}^{k} a_i \frac{1}{y_i}\right]\left[\sum_{i=1}^{k} a_i x_i\right] \leqq \sum_{i=1}^{k} a_i \frac{x_i}{y_i} \leqq 1.$$

Using the arithmetic-geometric mean inequality in the left hand inequality above, we obtain

$$\prod_{i=1}^{k}\left(\frac{1}{y_i}\right)^{a_i}\left[\sum_{i=1}^{k} a_i x_i\right] \leqq 1,$$

hence the result.

3 Main results

We apply the results of §2 to find new inequalities involving the function $S(\cdot)$ defined on M. The following is the first result of this section.

Proposition 1. *For any sequence of matrices $A_1, \ldots, A_k$ such that $A_i \in R$ $(i=1, \ldots, k)$ and for any sequence $a_1, \ldots, a_k$ of positive numbers such that $\sum_{i=1}^{k} a_i = 1$, the inequality*

$$\prod_{i=1}^{k} S[\varphi(A_i)]^{2a_i} \leqq kmn\left[\sum_{i=1}^{k} a_i^2\right] \prod_{i=1}^{k} S[\varphi(A_i * \bar{A})] \tag{1}$$

holds, where $\bar{A} = \frac{1}{k}(A_1 + \ldots + A_k)$.

PROOF. Using the linearity of $S(\cdot)$ and $\varphi(\cdot)$ we make the following steps

$$\sum_{i=1}^{k} a_i^2 = S\left[\sum_{i=1}^{k} a_i^2 \frac{\varphi(A_i * \bar{A})}{S(\varphi(A_i * \bar{A}))}\right] = S\left[\varphi\left(\sum_{i=1}^{k} a_i^2 \frac{A_i * \bar{A}}{S(\varphi(A_i * \bar{A}))}\right)\right] =$$

$$= \frac{1}{k} S\left[\varphi\left(\sum_{i=1}^{k}\sum_{j=1}^{k} a_i^2 \frac{A_i * A_j}{S(\varphi(A_i * \bar{A}))}\right)\right] = \frac{1}{k} S\left\{\varphi\left[\left(\sum_{i=1}^{k} a_i \frac{A_i}{S(\varphi(A_i * \bar{A}))^{1/2}}\right)^{*2} + \right.\right.$$

$$\left.\left. + \sum_{1 \leqq i < j \leqq k} A_i * A_j \left(\frac{a_i}{S(\varphi(A_i * \bar{A}))^{1/2}} - \frac{a_j}{S(\varphi(A_j * \bar{A}))^{1/2}}\right)^2\right]\right\} =$$

$$= \frac{1}{k} S\left\{\varphi\left[\left(\sum_{i=1}^{k} a_i \frac{A_i}{S(\varphi(A_i * \bar{A}))^{1/2}}\right)^{*2}\right] + \right.$$

$$\left. + \sum_{1 \leqq i < j \leqq k} \varphi(A_i * A_j)\left[\frac{a_i}{S(\varphi(A_i * \bar{A}))^{1/2}} - \frac{a_j}{S(\varphi(A_j * \bar{A}))^{1/2}}\right]^2\right\}$$

$$\geqq \frac{1}{k} S\left\{\varphi\left[\left(\sum_{i=1}^{k} a_i \frac{A_i}{S(\varphi(A_i * \bar{A}))^{1/2}}\right)^{*2}\right]\right\} \tag{2}$$

$$\geqq \frac{1}{k} S\left[\varphi\left(\sum_{i=1}^{k} a_i \frac{A_i}{S(\varphi(A_i * \bar{A}))^{1/2}}\right) * \varphi\left(\sum_{i=1}^{k} a_i \frac{A_i}{S(\varphi(A_i * \bar{A}))^{1/2}}\right)\right] \tag{3}$$

(by the definition of φ and Lemma 3)

$$(4) \qquad \geqq \frac{1}{kmn} S\left[\varphi\left(\sum_{i=1}^{k} a_i \frac{A_i}{S(\varphi(A_i*\bar{A}))^{1/2}}\right)\right]^2 \quad \text{(by Lemma 5)}$$

$$= \frac{1}{kmn}\left[\sum_{i=1}^{k} a_i \frac{S(\varphi(A_i))}{S(\varphi(A_i*\bar{A}))^{1/2}}\right]^2$$

$$(5) \qquad \geqq \frac{1}{kmn} \prod_{i=1}^{k}\left[\frac{S(\varphi(A_i))}{S(\varphi(A_i*\bar{A}))^{1/2}}\right]^{2a_i}$$

(by the arithmetic-geometric mean inequality).
This proves the validitity of our statement (1).

Corollary 1. *For any sequence of matrices $A_1, \ldots, A_k$ such that $A_i \in M$ $(i=1, \ldots, k)$ and for any sequence $a_1, \ldots, a_k$ of positive numbers such that $\sum_{i=1}^{k} a_i = 1$, the inequality*

$$(6) \qquad \prod_{i=1}^{k} S(A_i)^{2a_i} \leqq kmn\left(\sum_{i=1}^{k} a_i^2\right) \prod_{i=1}^{k} S(A_i*\bar{A})^{a_i}$$

holds, where

$$\bar{A} = \frac{1}{k}(A_1 + \ldots + A_k).$$

If $A_{\sigma(1)} \leqq \ldots \leqq A_{\sigma(k)}$ for some permutation σ of $\{1, \ldots, k\}$, then equality holds if and only if $A_i = \lambda I$ and $a_i = \frac{1}{k}$ $(i=1, \ldots, k)$, where $\lambda > 0$.

Proof. If we restrict φ on M and if we take $\varphi(A) = A$ for all $A \in M$ in (1) we obtain (6).
Next equality holds in (2) if and only if

$$(7) \qquad \frac{a_i}{S(A_i*\bar{A})^{1/2}} = \frac{a_j}{S(A_j*\bar{A})^{1/2}} \quad \text{for all} \quad 1 \leqq i, j \leqq k,$$

since $S(A_i*A_j) > 0$ for all $1 \leqq i, j \leqq k$ (Lemma 1).
Also, $\varphi(A*A) = \varphi(A)*\varphi(A)$ since φ is the identity on M. Hence equality holds in (3). Now, by the previous remarks and Lemma 5 equality holds in (4) if and only if

$$(8) \qquad \sum_{i=1}^{k} A_i = \mu I, \quad \mu > 0.$$

Then equality holds in (5) if and only if

$$S(A_1) = \ldots = S(A_k),$$

and therefore by Lemma 3

$$A_1 = \ldots = A_k.$$

By (8) we obtain that

$$A_i = \lambda I \quad (i = 1, \ldots, k),$$

where $\lambda=\frac{\mu}{k}$. Hence, $S(A_i*\bar{A})=S(A_j*\bar{A})$ for all $1\leqq i,j\leqq k$ and so by (7)

$$a_i = a_j \quad \text{for all} \quad 1 \leqq i,\ j \leqq k.$$

This completes the proof.

The following is an immediate consequence of Corollary 1.

Corollary 2. *For any sequence of matrices $A_1, \dots, A_k$ such that $A_i\in M$ $(i=1, \dots, k)$, the inequality holds*

$$\prod_{i=1}^{k} S(A_i)^2 \leqq (mn)^k \prod_{i=1}^{k} S(A_i*\bar{A}),$$

where $\bar{A}=\frac{1}{k}(A_1+\dots+A_k)$.

If we combine Lemma 6 and (4) we obtain the following result

Proposition 2. *If $A_1, \dots, A_k$ is a sequence of matrices such that $A_i\in R$ $(i=1, \dots, k)$ and*

$$[S(\varphi(A_i))], \quad [S(\varphi(A_i*\bar{A}))] \quad (i = 1, \dots, k)$$

are oppositely ordered, the inequality

$$(\sum_{i=1}^{k} a_i S[\varphi(A_i)])^2 \leqq kmn \prod_{i=1}^{k} S[\varphi(A_i*\bar{A})]^{a_i} \tag{9}$$

holds, where $\bar{A}=\frac{1}{k}\ (A_1+\dots+A_k)$.

Proposition 3. *Let $A_1, \dots, A_k$ be a sequence of matrices such that $A_i\in R$ $(i=1, \dots, k)$ and $A_{\sigma(1)}\leqq\dots\leqq A_{\sigma(k)}$ for some permutation σ of $\{1, \dots, k\}$. Let $a_1, \dots, a_k$ be positive numbers such that $\sum_{i=1}^{k}a_i=1$. Then the following inequality holds*

$$\sum_{i=1}^{k} a_i S[\varphi(A_i)] \leqq \frac{C_3(M_1, M_2)}{(kmn \sum_{i=1}^{k} a_i^2)^{1/2}} \sum_{i=1}^{k} a_i S[\varphi(A_i*\bar{A})]^{1/2},$$

where $\bar{A}=\frac{1}{k}(A_1+\dots+A_k)$, $M_1=S(\varphi(A_{\sigma(1)}))$ and $M_2=S(\varphi(A_{\sigma(k)}))$. Equality holds if and only if k is even, $A_1=\dots=A_k=\lambda I$ and $a_1=\dots=a_k=\frac{1}{k}$, where $\lambda>0$.

Proof. Using the arithmetic-harmonic mean inequality in (4) we obtain

$$\sum_{i=1}^{k} a_i \frac{S[\varphi(A_i*\bar{A})]^{1/2}}{S[\varphi(A_i)]} \leqq (kmn \sum_{i=1}^{k} a_i^2)^{1/2}. \tag{10}$$

If $A_{\sigma(i)}\leqq A_{\sigma(j)}$ then $A_{\sigma(i)}*\bar{A}\leqq A_{\sigma(j)}*\bar{A}$ by Lemma 2. Hence, by Lemma 3 and the properties of φ $S[\varphi(A_{\sigma(i)})]\leqq S[\varphi(A_{\sigma(j)})]$ and $S[\varphi(A_{\sigma(i)}*\bar{A})]\leqq S[\varphi(A_{\sigma(j)}*\bar{A})]$.

This means that the sequences

$$[S(\varphi(A_{\sigma(i)}))], \quad [S(\varphi(A_{\sigma(i)}*\bar{A}))]^{1/2} \quad (i=1, \ldots, k)$$

are oppositely ordered and so Tchebychef's inequality [2, p. 43] applies

$$(11) \qquad \Big(\sum_{i=1}^{k} a_i S[\varphi(A_i*\bar{A})]^{1/2}\Big)\Big(\sum_{i=1}^{k} \frac{a_i}{S[\varphi(A_i)]}\Big) \geqq \sum_{i=1}^{k} a_i \frac{S[\varphi(A_i*\bar{A})]^{1/2}}{S[\varphi(A_i)]}.$$

By Lemma 4 we have

$$(12) \qquad \sum_{i=1}^{k} \frac{a_i}{S[\varphi(A_i)]} \leqq \frac{C_3(M_1, M_2)}{\sum_{i=1}^{k} a_i S[\varphi(A_i)]}.$$

If we combine (10), (11) and (12) we obtain the result.

As in the proof of Proposition 1 equality holds in (10) if and only if $A_1=\ldots$ $\ldots=A_k=\lambda I$ and $a_1=\ldots=a_k=\frac{1}{k}$. Also, by Lemma 4 equality holds in (12) if and only if there is a subset I_0 of $\{1, \ldots, k\}$ such that $\sum_{i\in I_0} a_i=1/2$. The last condition on a_i's forces k to be even.

If we consider the special case of Proposition 2 where $a_i=\frac{1}{k}\,(i=1, \ldots, k)$, then we obtain an upper bound for $S(\varphi(\bar{A}))$. In particular, we have the following result.

Corollary 3. *Under the assumptions of Proposition 2, the following inequality holds*

$$S[\varphi(\bar{A})] \leqq \frac{C_3(M_1, M_2)}{k(mn)^{1/2}} \sum_{i=1}^{k} S[\varphi(A_i*\bar{A})]^{1/2},$$

where $\bar{A}=\frac{1}{k}(A_i+\ldots+A_k)$.

4. Applications

Let $X=(X_{ij})$ be an $m\times n$ random matrix with X_{ij} arbitrary positive random variables with positive variances. Then, see e.g. [3], we have

$$E^2X_{ij} \leqq EX_{ij}^2 \quad \text{for all} \quad 1 \leqq i \leqq m \quad \text{and} \quad 1 \leqq j \leqq n.$$

Hence $E^2X \leqq E(X*X)$. Also, if $Y=(Y_{ij})$ is an $m\times n$ random matrix with Y_{ij} arbitrary positive random variables with positive variances such that $X\leqq Y$, then

$$EX_{ij} \leqq EY_{ij} \quad \text{for all} \quad 1 \leqq i \leqq m \quad \text{and} \quad 1 \leqq j \leqq n.$$

Hence $EX \leqq EY$. Thus, $E(\cdot)$ satisfies the same conditions as $\varphi(\cdot)$ does. Therefore the inequalities of §3 can be transformed into probabilistic inequalities. In particular, Propositions 4 and 5 below are obtained from Propositions 1 and 3 respectively.

Proposition 4. *Let* $X^{(l)}=(X_{ij}^{(l)})$ $(l=1, \dots, k)$ *be a sequence of* $m\times n$ *random matrices with* $X_{ij}^{(l)}$ *arbitrary positive random variables with positive variances. Let* $a_1, \dots, a_k$ *be positive numbers such that* $\sum_{i=1}^{k} a_i=1$. *Then the following inequality holds*

$$\prod_{l=1}^{k} \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^{(l)}\Big)^{2a_l} \leqq kmn\Big(\sum_{i=1}^{k} a_i^2\Big) \prod_{l=1}^{k} \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^{(l)} \overline{X}_{ij}\Big)^{a_l}, \tag{13}$$

where $\overline{X}_{ij}=\frac{1}{k} \sum_{i=1}^{k} X_{ij}^{(l)}$.

Proposition 5. *Let* $X^{(l)}=(X_{ij}^{(l)})$ $(l=1, \dots, k)$ *be a sequence of* $m\times n$ *random matrices with* $X_{ij}^{(l)}$ *arbitrary positive random variables with positive variances such that* $X^{(1)} \leqq \dots \leqq X^{(k)}$. *Then, the following inequality holds*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} E\overline{X}_{ij} \leqq \frac{C_3(M_1, M_2)}{k(mn)^{1/2}} \sum_{l=1}^{k} \sum_{i=1}^{m} \sum_{j=1}^{n} (EX_{ij}^{(l)} \overline{X}_{ij})^{1/2},$$

where $\overline{X}_{ij}=\frac{1}{k} \sum_{l=1}^{k} X_{ij}^{(l)}$, $M_1=\sum_{i=1}^{m} \sum_{j=1}^{n} EX_{ij}^{(l)}$ *and* $M_2=\sum_{i=1}^{m} \sum_{j=1}^{u} EX_{ij}^{(k)}$.

Remark. Proposition 4 is a generalization of an inequality obtained by T. F. Mori and G. J. Szekely [4]. Also, Proposition 5 is a generalization of results obtained by T. Tollis [6].

Let A_{lj} $(l=1, \dots, k$ and $j=1, \dots, n)$ be an arbitrary double sequence of events in a probability space, $P(A_{lj})>0$ for all l, j. Replacing $X_i^{(l)}=\left(\frac{I(A_{l1})}{P(A_{l1})}, \dots, \frac{I(A_{ln})}{P(A_{ln})}\right)$ for $l=1, \dots, k$ in (13) (where $I(A_{lj})$ denotes the indicator function of the event A_{lj}, i.e., $I(A_{lj})=1$ on the event A_{lj} and 0 otherwise) we obtain the inequality

$$n^k \leqq \prod_{l=1}^{k} \frac{1}{k} \sum_{j=1}^{n} \sum_{i=1}^{k} \frac{P(A_{ij} A_{lj})}{P(A_{ij}) P(A_{lj})},$$

which generalizes a problem conjectured by Laslett and solved by T. F. Mori and G. J. Szekely in [4].

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