On orthogonally additive mappings, II

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Dedicated to Professor Hans E. Debrunner on his sixtieth birthday

0. Introduction

Many different approaches to orthogonally additive mappings are known, according to different orthogonality notions, and most of them were made under certain regularity conditions. In [7], a kind of real orthogonality vector space $(X, \perp)$ is considered. There it is possible to determine the general odd and the type of the general even orthogonally additive mapping $f\colon X \to Y$, respectively, where $(Y, +)$ is an arbitrary abelian group. Here the results in [7] are extended into different directions (cf. [8]): Scalar fields for $(X, \perp)$ other than $\mathbf{R}$ (sections 1 and 3); improvements for the determination of the even orthogonally additive mappings (section 2); a dependence of the set of orthogonally additive mappings $f\colon X \to Y$ on the group $(Y, +)$ (section 4).

Throughout the paper, $\mathbf{R}, \mathbf{Q}, \mathbf{Z}, \mathbf{N}^o, \mathbf{N}$ denote the sets of real, rational numbers, integers, nonnegative integers, positive integers, respectively. For an ordered field $\mathbf{K}$, $|\alpha| := \max\{\alpha, -\alpha\}$ $(\forall \alpha \in \mathbf{K})$, $\mathbf{K}_+ := \{\alpha \in \mathbf{K};\ \alpha \geqq 0\}$, $\mathbf{K}_+^* := \{\alpha \in \mathbf{K};\ \alpha > 0\}$. If A is a subset of a vector space, lin A stands for the linear hull (span) of A. We use o for the zero vector and 0 for both the number zero and the identity element of an abelian group. The constant mapping with value c is denoted by $\underline{c}$. Finally, for any abelian groups $(X, +)$ and $(Y, +)$, $\operatorname{Hom}[(X, +), (Y, +)]$ or sometimes more briefly $\operatorname{Hom}(X, Y)$ is the set of all solutions of the Cauchy functional equation

$$\text{(C)} \qquad f\colon X \to Y;\quad f(x_1+x_2) = f(x_1)+f(x_2) \quad (\forall x_1, x_2 \in X),$$

and g is called a *quadratic mapping* iff it satisfies the functional equation (parallelogram law; Jordan — von Neumann identity)

$$\text{(Q)} \qquad g\colon X \to Y;\quad g(x_1+x_2)+g(x_1-x_2) = 2g(x_1)+2g(x_2) \quad (\forall x_1, x_2 \in X).$$

1. K-orthogonality spaces and orthogonally additive mappings

We show that the field $\mathbf{R}$ (of real numbers) may be replaced by an ordered (i.e., totally ordered) field $\mathbf{K}$, arbitrary for some of the results in [7] (see this section), euclidean for others (cf. section 3).

Definition 1.1. Let $\mathbf{K}$ be an ordered field, X a $\mathbf{K}$-vector space with $\dim_{\mathbf{K}} X \geqq 2$, and $\perp$ a binary relation on X with the properties

(01) $x \perp o$, $o \perp x$ for every $x \in X$;
(02) if $x, y \in X \setminus \{o\}$, $x \perp y$, then x, y are linearly independent;
(03) if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbf{K}$;
(04′) if P is a 2-dimensional linear subspace of X, $x \in P$, $\lambda \in \mathbf{K}_+$, then there exists $y \in P$ such that $x \perp y$ and $x+y \perp \lambda x - y$.

Then $(X, \perp)$ is called a **K**-*orthogonality space* (cf. [7], p. 35/36, Def. 1).

Definition 1.2. If $(X, \perp)$ is a $\mathbf{K}$-orthogonality space and $(Y, +)$ an abelian group, then a mapping $f\colon X \to Y$ is called *orthogonally additive* iff

$$(*) \qquad f(x_1+x_2) = f(x_1)+f(x_2) \quad \text{for all} \quad x_1, x_2 \in X \quad \text{with} \quad x_1 \perp x_2$$

(cf. [7], p. 37, Def. 2).

Thus the orthogonally additive mappings $f\colon X \to Y$ are precisely the solutions of the conditional Cauchy functional equation (*). The set of all solutions of (*) is denoted by $\mathrm{Hom}_\perp(X, Y)$. Clearly

$$(1) \qquad \mathrm{Hom}(X, Y) \subset \mathrm{Hom}_\perp(X, Y).$$

Corresponding to Definitions 1 and 2, we also transfer Definitions 3 and 4 of [7] (pages 38, 41/42). An inspection of the proofs in [7] then shows that

(Δ) *Lemmas 1, 2, 3, 4, 8 and Theorems 5, 6 and Corollary 7 and Remark 5 of* [7] *remain valid for* **K**-*orthogonality spaces.*

In this connection, we confine ourselves to formulate explicitly the following extension of [7], Theorems 5 and 6 and Remark 3:

Theorem 1.3. *For any ordered field* $\mathbf{K}$, *any* **K**-*orthogonality space* $(X, \perp)$, *and any abelian group* $(Y, +)$, *we have:*
a) $h \in \mathrm{Hom}_\perp(X, Y)$, *h odd* $\Leftrightarrow$ $h \in \mathrm{Hom}(X, Y)$.
b) $g \in \mathrm{Hom}_\perp(X, Y)$, *g even* $\Rightarrow$ *g quadratic.*
c) $\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}(X, Y)$ $\Leftrightarrow$ *every even g in* $\mathrm{Hom}_\perp(X, Y)$ *is* $\underline{0}$.

(For "$\Leftarrow$" in c), let be $f \in \mathrm{Hom}_\perp(X, Y)$ and define $g(x) = f(x) + f(-x)$ $(\forall x \in X)$. Then $g \in \mathrm{Hom}_\perp(X, Y)$ even, and by hypothesis $g = \underline{0}$, i.e., $f(-x) = -f(x)$ $(\forall x \in X)$, and by part a) $f \in \mathrm{Hom}(X, Y)$. Therefore $\mathrm{Hom}_\perp(X, Y) \subset \mathrm{Hom}(X, Y)$, and the converse inclusion is ensured by (1); cf. [11], Thm. 1.8).

The foregoing theorem is an invitation for further specification of $\mathrm{Hom}_\perp(X, Y)$ and $\mathrm{Hom}(X, Y)$ in special situations. This will be done in the following sections.

Unless otherwise stated, $\mathbf{K}$ will denote an ordered field, $(X, \perp)$ or X a $\mathbf{K}$-orthogonality space, and $(Y, +)$ or Y an abelian group.

2. Improvements in the determination of the even solutions of (*)

It is important to note that we have "$\Leftrightarrow$" in part a), but only "$\Rightarrow$" in part b) of Theorem 1.3. This is not astonishing since (*) is in a way much closer to (C) than to (Q). The more it is necessary to undertake special efforts for the determination of the even solutions of (*). Lemma 2.1 below expresses that these solutions are sensitive in the respect that their behavior in a (possibly) small part of the domain space X may determine them completely. It is therefore hopeless to think of constructing an even g in $\mathrm{Hom}_\perp(X, Y)$ by making independent choices on direct summands of X and "pasting" these together.

Lemma 2.1. *If $g \in \mathrm{Hom}_\perp(X, Y)$ even and M is a 1-dimensional linear subspace of X such that $g|M$ is additive, then $g=\underline{0}$.*

Proof. Let be $x \in X \setminus \{o\}$, $M = \mathrm{lin}\,\{x\}$, $z \in X$, and P a 2-dimensional linear subspace of X with $x, z \in P$. By (04′) there exists $y \in P$ with $x \perp y$ and $x+y \perp x-y$. (02) and $x \neq o$ make the case $y=o$ impossible, thus, by (02), x and y are linearly independent, i.e., $P = \mathrm{lin}\,\{x, y\}$. There are $\lambda, \mu \in \mathbf{K}$ such that $z = \lambda x + \mu y$. (03) and $x+y \perp x-y$ imply $\mu x + \mu y \perp \mu x - \mu y$ and then $g(\mu x) = g(\mu y)$ (cf. [7], p. 39, step (iii)). By $x \perp y$ and (03) we also have $\lambda x \perp \mu y$, so $g(z) = g(\lambda x + \mu y) = g(\lambda x) + g(\mu y) = g(\lambda x) + g(\mu x)$. If $\gamma \in \mathbf{K}$, we conclude from additivity of $g|M$ and evenness of g that $g(\gamma x) = g\left(\frac{\gamma}{2}x + \frac{\gamma}{2}x\right) = g\left(\frac{\gamma}{2}x\right) + g\left(\frac{\gamma}{2}x\right) = g\left(\frac{\gamma}{2}x\right) + g\left(-\frac{\gamma}{2}x\right) = g\left(\frac{\gamma}{2}x - \frac{\gamma}{2}x\right) = g(o) = 0$, hence $g(z) = g(\lambda x) + g(\mu x) = 0 + 0 = 0$. Since $z \in X$ was arbitrary, $g = \underline{0}$.

Definition 2.2 The orthogonality relation $\perp$ of $(X, \perp)$ is called
a) *left-unique* iff for all $x \in X \setminus \{o\}$ and $z \in X$ there is at most one $\alpha \in \mathbf{K}$ for which $\alpha x + z \perp x$,
b) *right-unique* iff for all $x \in X \setminus \{o\}$ and $z \in X$ there is at most one $\beta \in \mathbf{K}$ for which $x \perp \beta x + z$.
(For these notions in connection with Birkhoff-James orthogonality in real normed spaces cf. [3] (p. 273, Def. 4.1, 4.2; p. 268, Cor. 2.2; p.. 269, Thm. 2.3)).

Theorem 2.3. *For any ordered field $\mathbf{K}$, any $\mathbf{K}$-orthogonality space $(X, \perp)$, and any abelian group $(Y, +)$, the following holds: If $\perp$ is not left-unique or not right-unique or not symmetric, then $\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}\,(X, Y)$.*

Proof. 1) Assume that $\perp$ is not left-unique. Then there exist $x \in X \setminus \{o\}$, $z \in X$, and $\alpha, \alpha' \in \mathbf{K}$ with $\alpha \neq \alpha'$ such that $\alpha x + z \perp x$ and $\alpha' x + z \perp x$. For $u := (\alpha' - \alpha)x$, $w := \alpha x + z$ we have $u \in X \setminus \{o\}$, $\alpha' x + z = u + w$, $w \perp x$ and $u + w \perp x$, and therefore by (03) $w \perp u$ and $u + w \perp u$; u is a nonzero σ-element in the sense of [7], p. 42, Def. 4b).

2) If $\perp$ is not right-unique, an analogous argument leads to a nonzero σ-element u of X.

3) Let $\perp$ be non-symmetric. Thus there are $u, v' \in X$ with $v' \perp u$ but not $u \perp v'$. By (01), $u \neq o$. Let P be a 2-dimensional linear subspace of X for which $u, v' \in P$. By (Δ) and [7], p. 37, Lemma 1, there exists $y \in P$ such that $u \perp y$ and $\mathrm{lin}\,\{u, y\} = P$. Then $v' = \lambda u + \mu y$ for suitable $\lambda, \mu \in \mathbf{K}$. By (03) $u \perp \mu y$, i.e., $u \perp -\lambda u + v'$. The fact that not $u \perp v'$ makes $\lambda = 0$ impossible, and by (03) we get $u \perp u - \frac{1}{\lambda} v'$. But from $v' \perp u$ we also obtain $-\frac{1}{\lambda} v' \perp u$, and $v := -\frac{1}{\lambda} v'$ now satisfies $v \perp u$ and $u \perp u + v$ which shows that u is a nonzero ϱ-element of X ([7], p. 42, Def. 4a)).

4) In either case, u is a nonzero ϱ- or σ-element of X. Let be $g \in \mathrm{Hom}_\perp(X, Y)$ even. By (Δ) and [7], p. 42, Lemma 8, $g|\mathrm{lin}\,\{u\}$ is additive. Lemma 2.1 and $u \neq o$ now imply $g = \underline{0}$, and $\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}\,(X, Y)$ follows from Theorem 1.3c).

Corollary 2.4. *Let be $(X, \|\,.\,\|)$ a real normed vector space with $\dim_R X \geqq 2$ and $\perp_{BJ}$ the Birkhoff-James orthogonality defined by*

(BJ) $$x, y \in X; \quad x \perp_{BJ} y :\Leftrightarrow \|x + \beta y\| \geqq \|x\| \quad (\forall\, \beta \in \mathbf{R}),$$

and $(Y, +)$ an abelian group. Then each of the following conditions is sufficient for $\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}\,(X, Y)$:

(i) $\|\,.\,\|$ *is not strictly convex.*
(ii) $\|\,.\,\|$ *is not Gateaux differentiable at some nonzero element of X.*
(iii) $\dim_R X \geqq 3$, *and $(X, \|\,.\,\|)$ is not an inner product space.*
(iv) $\dim_R X \geqq 2$, *and $\perp_{BJ}$ is not symmetric.*

Proof. $(X, \perp_{BJ})$ is an $\mathbf{R}$-orthogonality space ([7], p. 36, Example C). (i): By [3], p. 275, Thm. 4.3, $\perp_{BJ}$ is not left-unique. (ii): By [3], p. 274, Thm. 4.2, $\perp_{BJ}$ is not right-unique. In these two cases, the assertion follows from Theorem 2.3, and for (iii), (iv) we refer to [7], p. 47, Thm. 16; the proof for (iii) rests on Kakutani's projection theorem which has no analogue for $\dim_R X = 2$.

Remark 2.5. Corollary 2.4 improves [7], Thm. 16, but the case when $\dim_R X = 2$, $\perp_{BJ}$ is symmetric and $\|\,.\,\|$ is strictly convex and Gateaux differentiable at every $x \in X \setminus \{o\}$ remains uncovered, and this case does occur ([4], lower half of p. 561). However, by a direct attack of $\perp_{BJ}$, G. Szabó succeeded in showing that $\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}\,(X, Y)$ whenever $(X, \|\,.\,\|)$ is not an inner product space ([11], Thm. 1.8).

3. K-inner product spaces

In this section we generalize the classical inner product orthogonality ([7], p. 36, Example B) from the real case to that of an ordered field $\mathbf{K}$. In order to stay within the framework of Definition 1.1, we look for those bilinear functionals $\varphi\colon X \times X \to \mathbf{K}$ for which the relation $\perp_\varphi$ defined by

(2) $$x, y \in X; \quad x \perp_\varphi y :\Leftrightarrow \varphi(x, y) = 0$$

satisfies the axioms (01) to (04′). For orientation, we first make some remarks.

Remark 3.1. The property (02) of $\perp_\varphi$ dictates that φ be *nonisotropic*, i.e., that we have

(NI) $$x \in X, \quad \varphi(x, x) = 0 \Rightarrow x = o.$$

Remark 3.2. If X is a $\mathbf{K}$-vector space with $\dim_\mathbf{K} X \geqq 2$ and Hamel base $\{b_i;\ i \in I\}$, and if the bilinear functional $\varphi\colon X \times X \to \mathbf{K}$ is given by $\varphi(b_i, b_j) = \delta_{ij}$ $(\forall i, j \in I)$ (Kronecker symbol), then the property (04′) for $\perp_\varphi$ implies that K is *euclidean*, that is that every nonnegative element of $\mathbf{K}$ is a square of an element of $\mathbf{K}$ ([12], p. 20). In fact, let be $\lambda \in \mathbf{K}_+$ arbitrary, assume that $1, 2 \in I$, and put $P := \mathrm{lin}\,\{b_1, b_2\}$ and $x := b_1$. By (04′) there exists $y \in P$ such that $b_1 \perp_\varphi y$ and $b_1 + y \perp_\varphi \lambda b_1 - y$. From $b_1 \perp_\varphi y$ we conclude $y \in \mathrm{lin}\,\{b_2\}$, say $y = \alpha b_2$ for a suitable $\alpha \in \mathbf{K}$, i.e.,

$b_1+\alpha b_2 \perp_\varphi \lambda b_1-\alpha b_2$, $\varphi(b_1+\alpha b_2, \lambda b_1-\alpha b_2)=0$, $\lambda-\alpha^2=0$, $\lambda=\alpha^2$, which establishes euclidicity of $\mathbf{K}$.

Remark 3.3. Let be $\mathbf{K}=\mathbf{Q}$, $X=\mathbf{Q}^2$, $\varphi((\xi_1, \xi_2), (\eta_1, \eta_2)):=2\xi_1\eta_1-\xi_2\eta_2$ $(\forall \xi_{1,2}, \eta_{1,2}\in\mathbf{Q})$. Then φ is symmetric and non-isotropic. Furthermore, let be $P:=\mathbf{Q}^2$, $e_1:=(1,0)$, $e_2:=(0,1)$, and $x:=e_1$, $\lambda:=1$. If $y\in\mathbf{Q}^2$, $x\perp_\varphi y$, then $y\in\operatorname{lin}\{e_2\}$, and $\varphi(e_1+\alpha e_2, e_1-\alpha e_2)=2+\alpha^2$ for all $\alpha\in\mathbf{Q}$. Hence there is no $y\in\mathbf{Q}^2$ for which $x\perp_\varphi y$ and $x+y\perp_\varphi x-y$, i.e., (04′) is violated.

Remark 3.4. It may be shown by standard techniques in field theory that every ordered field is contained in a euclidean ordered field. Every real closed field is euclidean by the Euler—Lagrange Theorem ([13], p. 493). Of course, $\mathbf{Q}$ is not euclidean, and here is the reason for the failure in Remark 3.3.

Lemma 3.5. *If* $\mathbf{K}$ *is a euclidean ordered field,* X *a* $\mathbf{K}$*-vector space,* $\dim_{\mathbf{K}} X\geqq 2$, $\varphi\colon X\times X\to\mathbf{K}$ *a non-isotropic bilinear functional, then* φ *is definite, and* $\perp_\varphi$ *satisfies the axioms* (01), (02), (03), (04′). (Cf. [10]). $(X, \perp_\varphi)$ then is called a $\mathbf{K}$-inner product space.

Proof. (01) and (03) directly follow from bilinearity of φ. — (02): Let be $x, y\in X\setminus\{o\}$, $\varphi(x,y)=0$, and assume linear dependence of x, y, say $y=\alpha x$. Then $\alpha\neq 0$, $\alpha\varphi(x,x)=\varphi(x,y)=0$, so $\varphi(x,x)=0$, and (NI) implies $x=o$, a contradiction. Thus, (02) holds. — (04′): Let be P a 2-dimensional linear subspace of X, $x\in P$, $\lambda\in\mathbf{K}_+$ arbitrary. If $\lambda x=o$, then $y:=o$ has the properties required in (04′). Now let be $\lambda x\neq o$, i.e., $\lambda\in\mathbf{K}_+^*$, $x\neq o$, so by (NI) $\varphi(x,x)\neq 0$. By $\psi:=\varphi(x,\cdot)|P$ we obtain a linear functional on P with $\psi\neq\underline{0}$ since $\psi(x)=\varphi(x,x)\neq 0$. Since $\ker\psi$ is a 1-dimensional linear subspace of P, there exists $y'\in P\setminus\{o\}$ with $\varphi(x,y')=\psi(y')=0$, i.e., with $x\perp_\varphi y'$. We then get

$$\varphi(x+\alpha y', \lambda x-\alpha y') = \lambda\varphi(x,x)+\alpha\lambda\varphi(y',x)-\alpha^2\varphi(y',y') \quad (\forall\alpha\in\mathbf{K}). \tag{3}$$

Since $\mathbf{K}$ is euclidean, φ is definite ([5], p. 12, Exercise; [6], p. 251, Lemma 1c)), so that $x\neq o$, $y'\neq o$ imply $\varphi(x,x)\cdot\varphi(y',y')>0$. Therefore the discriminant δ of the right-hand side of (3) is positive, and since $\sqrt{\delta}\in\mathbf{K}_+^*$, the mapping $\alpha\mapsto\varphi(x+\alpha y', \lambda x-\alpha y')$ has two zeros $\alpha_{1,2}$ in $\mathbf{K}$. For $y:=\alpha_1 y'$ we then have $x\perp_\varphi y$ and $x+y\perp_\varphi \lambda x-y$, i.e., (04′) holds.

The alternative $\perp$ symmetric/non-symmetric will be important in the sequel.

Remark 3.6. a) If φ is symmetric, so is $\perp_\varphi$. b) The mapping $\varphi\colon \mathbf{R}^2\times\mathbf{R}^2\to\mathbf{R}$ given by $\varphi((\zeta_1,\zeta_2),(\eta_1,\eta_2)):=\zeta_1\eta_2-\zeta_2\eta_1$ shows that the foregoing statement has no converse; this φ violates (NI). c) If φ is non-isotropic and $\perp_\varphi$ is symmetric, then φ must be symmetric. In fact, let be $x, y\in X$ arbitrary. If $\varphi(y,y)=0$, (NI) implies $y=o$, i.e., $\varphi(x,y)=0=\varphi(y,x)$. If $\varphi(y,y)\neq 0$, define $z:=x-\dfrac{\varphi(x,y)}{\varphi(y,y)}\cdot y$ to obtain $\varphi(z,y)=0$, i.e., $z\perp_\varphi y$, hence $y\perp_\varphi z$, i.e., $\varphi(y,z)=0$, and this immediately leads to $\varphi(x,y)=\varphi(y,x)$.

Remark 3.7. If $\mathbf{K}$ is euclidean, X a $\mathbf{K}$-vector space, $\varphi\colon X\times X\to\mathbf{K}$ positive definite bilinear (but not necessarily symmetric), and $p\colon X\to\mathbf{K}$ is given by

$$p(x):=[\varphi(x,x)]^{1/2} \quad (\forall x\in X), \tag{4}$$

then the following facts are easily established:

(N1) $$p(o) = 0; \quad p(x) > 0 \quad \text{for all} \quad x \in X \setminus \{o\};$$

(N2) $$p(\lambda x) = |\lambda| \cdot p(x) \quad \text{for all} \quad \lambda \in \mathbf{K}, x \in X;$$

(N3) $$p(x+y) \leqq p(x) + p(y) \quad \text{for all} \quad x, y \in X;$$

(N4) $$[p(x+y)]^2 + [p(x-y)]^2 = 2[p(x)]^2 + 2[p(y)]^2 \quad \text{for all} \quad x, y \in X.$$

For the proof of (N3) we may use the symmetric positive denite bilinear functional $\psi\colon X \times X \to \mathbf{K}$ defined by $\psi(x, y) := \frac{1}{2}[\varphi(x, y) + \varphi(y, x)]$ $(\forall x, y \in X)$ and the fact that the Cauchy—Schwarz inequality is valid for ψ (but in general not for φ; consider $X = \mathbf{R}^2$, $\varphi_1((\xi_1, \xi_2), (\eta_1, \eta_2)) := \xi_1\eta_1 + \xi_1\eta_2 + \xi_2\eta_2$). For $\mathbf{K} = \mathbf{R}$, p becomes an ordinary norm on X, and its Birkhoff-James orthogonality (cf. Corollary 2.4 above) coincides with $\perp_\psi$ and therefore is symmetric while $\perp_\varphi$ need not be so. Not only are $\perp_\psi$ and $\perp_\varphi$ unequal in general, it even occurs that neither is a subset of the other one (e.g., for the example φ_1 above). But nevertheless such a $(X, \perp_\varphi)$ always satisfies (01), (02), (03), (04′) by Lemma 3.5, no matter whether the euclidean ordered field $\mathbf{K}$ is $\mathbf{R}$ or not, and many different φ's may lead to the same $\perp_\psi$ (cf. [9]).

We now come to the main result of the section which complements and generalizes [7], p. 43, Theorem 9:

Theorem 3.8. Hypotheses: 1) $\mathbf{K}$ *is a euclidean ordered field.* 2) X *is a* $\mathbf{K}$*-vector space,* $\dim_{\mathbf{K}} X \geqq 2$. 3) $\varphi\colon X \times X \to \mathbf{K}$ *is a non-isotropic bilinear functional.* 4) $(Y, +)$ *is an abelian group.* Assertions: a) $(X, \perp_\varphi)$ *is a* $\mathbf{K}$*-orthogonality space.* b) *If* φ *is not symmetric, then* $\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}(X, Y)$. c) *If* φ *is symmetric, then* g *is an even solution of* (*) *if and only if there exists* $l\colon \mathbf{K} \to Y$ *additive with* $g(x) = l[\varphi(x, x)]$ $(\forall x \in X)$ (cf. [10]).

Proof. a) is ensured by Lemma 3.5. — b) By Remark 3.6c), $\perp_\varphi$ is not symmetric, and the assertion follows from a) and Theorem 2.3. — c) By Remark 3.6a), $\perp_\varphi$ is symmetric. If l exists, then $l \circ \varphi(\cdot, \cdot)$ is even. $x_1 \perp_\varphi x_2$ implies $x_2 \perp_\varphi x_1$ and $\varphi(x_1 + x_2, x_1 + x_2) = \varphi(x_1, x_1) + \varphi(x_2, x_2)$, i.e., $l \circ \varphi(\cdot, \cdot) \in \mathrm{Hom}_\perp(X, Y)$; so far, (NI) was not used. Conversely, assume that g be an even solution of (*). If $u, v \in X$ are such that $\varphi(u, u) = \varphi(v, v)$, symmetry of φ yields $u + v \perp_\varphi u - v$, and as in [7], p. 39/40, step (iii), we obtain $g(u) = g(v)$. By Lemma 3.5, φ is definite, without loss of generality positive definite. $\bar{l}\colon \mathbf{K}_+ \to Y$ is well-defined by

(5) $$\bar{l}[\varphi(u, u)] = g(u) \quad (\forall u \in X).$$

Let be $x' \in X \setminus \{o\}$ arbitrary. As shown in the proof of Lemma 3.5, verification part for (04′), there exists $y' \in X \setminus \{o\}$ such that $\varphi(x', y') = 0$. Let be $\lambda, \mu \in \mathbf{K}_+$ arbitrary. Since $\varphi(x', x') > 0$, $\varphi(y', y') > 0$, the vectors $x := (\lambda/\varphi(x', x'))^{1/2} x'$, $y := (\mu/\varphi(y', y'))^{1/2} y'$ are available, and $\varphi(x, x) = \lambda$, $\varphi(y, y) = \mu$, $\varphi(x, y) = 0$, thus also $\varphi(y, x) = 0$. Hence $\varphi(x+y, x+y) = \lambda + \mu$, i.e., $\bar{l}(\lambda + \mu) = \bar{l}[\varphi(x+y, x+y)] = (5) = g(x+y) = g(x) + g(y) = (5) = \bar{l}[\varphi(x, x)] + \bar{l}[\varphi(y, y)] = \bar{l}(\lambda) + \bar{l}(\mu)$, i.e., $\bar{l}$ is additive. Since $\mathbf{K}_+ - \mathbf{K}_+ = \mathbf{K}$, there exists a unique additive mapping $l\colon \mathbf{K} \to Y$ such that $l|\mathbf{K}_+ = \bar{l}$ ([1], p. 265, Thm. 2), and $g(x) = l[\varphi(x, x)]$ $(\forall x \in X)$ holds by (5).

4. A dependence of $\mathrm{Hom}_\perp(X, Y)$ on $(Y, +)$

So far we considered how $\mathrm{Hom}_\perp(X, Y)$ depends on the domain space $(X, \perp)$ (cf., e.g., Theorems 2.3 and 3.8) when $(Y, +)$ is an arbitrary abelian group. In [7], p. 39, Remark 2, we briefly looked at aspects of the dependence of $\mathrm{Hom}(X, Y)$ on $(Y, +)$. This question will now be treated more systematically (Theorem 4.6 below). We begin by some auxiliary statements which are formulated somewhat wider than actually needed for Theorem 4.6. The first one is a preparation for dealing with the even solutions of (*) (cf. Theorem 1.3b)).

Lemma 4.1. *For any abelian groups $(X, +)$ and $(Y, +)$ and any quadratic mapping $g\colon X\to Y$ we have:*

a) $2g(o)=0$.

b) $g(-x)=g(x)$ $(\forall x\in X)$.

c) *If* $g(o)=0$, *then* $g(nx)=n^2g(x)$ $(\forall x\in X, \forall n\in \mathbf{Z})$.

d) *If Y contains an element c of order 2, then $g(nx)=n^2g(x)$ $(\forall x\in X, \forall n\in\mathbf{Z})$ need not hold.*

Proof. a) Put $x_1=x_2=o$ in (Q). — b) Put $x_1=o$, $x_2=x$ in (Q) and use part a). — c) Clearly $g(nx)=n^2g(x)$ $(\forall x\in X, \forall n\in\{0, 1\})$. If $n\in\mathbf{N}$ is such that $g(nx)=n^2g(x)$, $g((n-1)x)=(n-1)^2g(x)$ $(\forall x\in X)$, then $g((n+1)x)=(n+1)^2g(x)$ $(\forall x\in X)$, thus by induction, $g(nx)=n^2g(x)$ $(\forall x\in X, \forall n\in\mathbf{N}^0)$. Finally, if $x\in X$, $n\in\mathbf{Z}$, $n<0$, then by the foregoing $g(nx)=g((-n)(-x))=(-n)^2g(-x)=n^2g(x)$. — d) $g:=\underline{c}$ satisfies (Q) but violates $g(o)=0$ as $c\neq 0$, and $g(nx)=c\neq 0=n^2c=n^2g(x)$ $(\forall x\in X, \forall n\in\mathbf{Z}$ even). (This shows how essential $g(o)=0$ is for the $\mathbf{Z}$-homogeneity of degree 2 of g).

The next result concerns the question of existence of nontrivial solutions of (C).

Lemma 4.2. *For any abelian group $(Y, +)$, the following statements are equivalent:*

(i) $\mathrm{Hom}\,[(\mathbf{Q}, +), (Y, +)]\neq\{\underline{0}\}$.
(ii) $\mathrm{Hom}\,[(X, +), (Y, +)]\neq\{\underline{0}\}$ *for every $\mathbf{Q}$-vector space X with* $\dim_{\mathbf{Q}} X\geqq 1$.
(iii) $\mathrm{Hom}\,[(X, +), (Y, +)]\neq\{\underline{0}\}$ *for some $\mathbf{Q}$-vector space X with* $\dim_{\mathbf{Q}} X\geqq 1$.
(iv) $(Y, +)$ *has a divisible subgroup Y_0 different from* $\{0\}$.

Proof. (i)$\Rightarrow$(ii): Let be $X\neq\{o\}$ and $\{b_i;\ i\in I\}$ a Hamel base of X over $\mathbf{Q}$. Then $I\neq\emptyset$, and we choose $i_0\in I$ arbitrary. The mapping $\tilde h\colon X\to\mathbf{Q}$ defined by $\tilde h(x):=\xi_{i_0}$ for $x=\sum_{i\in I}\xi_i b_i$ is an epimorphism from $(X, +)$ onto $(\mathbf{Q}, +)$. If $h\in\mathrm{Hom}(\mathbf{Q}, Y)\setminus\{\underline{0}\}$, then $h\circ\tilde h\in\mathrm{Hom}(X, Y)\setminus\{\underline{0}\}$. — (ii)$\Rightarrow$(iii): trivial. — (iii)$\Rightarrow$(iv): For any $h\in\mathrm{Hom}(X, Y)\setminus\{\underline{0}\}$, $h(X)$ is divisible since X is. — (iv)$\Rightarrow$(i): Choose $y_0\in Y_0\setminus\{0\}$ and define $f\in\mathrm{Hom}(\mathbf{Z}, Y_0)$ by $f(q):=qy_0$ $(\forall q\in\mathbf{Z})$. Since, by virtue of a theorem of R. Baer ([2], p. 99, Thm. 21.1; p. 105, Exercise 5), divisible groups are injective, there exists $\tilde f\in\mathrm{Hom}(\mathbf{Q}, Y_0)$ such that $\tilde f|\mathbf{Z}=f$, and $f\neq\underline{0}$ ensures $\tilde f\neq\underline{0}$. If finally $j\colon Y_0\hookrightarrow Y$, then $j\circ\tilde f\in\mathrm{Hom}(\mathbf{Q}, Y)\setminus\{\underline{0}\}$.

Definition 4.3. An abelian group $(Y, +)$ is called *reduced* if it has no divisible subgroup other than $\{0\}$ (cf. [2], p. 100).

Lemma 4.2 concerns the functional equation (C); (iv) expresses that $(Y, +)$ is not reduced. We now develop (iv) into a direction relevant also for other functional equations than (C).

Lemma 4.4. *For any abelian group $(Y, +)$, the following statements are equivalent:*

(iv) *$(Y, +)$ is not reduced.*

(v) *For every $k\in\mathbf{N}$, there exists a sequence (y_n) of elements of Y with the properties*

$$y_1 \neq 0; \quad (n+1)^k y_{n+1} = y_n \text{ for all } n\in\mathbf{N}. \tag{6}$$

(vi) *There exists a $k\in\mathbf{K}$ and a sequence (y_n) of elements of Y satisfying (6).*

PROOF. (iv)$\Rightarrow$(v): Assume that $k\in\mathbf{N}$ arbitrary, Y_0 a divisible subgroup $\neq\{0\}$ of Y, and $y_1\in Y_0\setminus\{0\}$. Divisibility of Y_0 and an inductive argument immediately provide a sequence (y_n) satisfying (6). (v)$\Rightarrow$(vi) is trivial. (vi)$\Rightarrow$(iv): Let Y_0 be the subgroup of Y generated by $\{y_n;\ n\in\mathbf{N}\}$. $y_1\neq 0$ ensures $Y_0\neq\{0\}$. Let be $z\in Y_0$ arbitrary. Then there exist $r\in\mathbf{N}$ and $n_1, \dots, n_r\in\mathbf{Z}$ for which $z=n_1y_1+\dots+n_ry_r$. (6) implies $y_{r-1}=r^ky_r$, $y_{r-2}=(r-1)^ky_{r-1}=(r-1)^kr^ky_r, \dots, y_1=2^k\cdot\dots\cdot r^ky_r=(r!)^ky_r$, i.e., $z=n_1(r!)^ky_r+\dots+n_ry_r=:qy_r$, i.e., for every $z\in Y_0$ there exist $r\in\mathbf{N}$ and $q\in\mathbf{Z}$ with $z=qy_r$. Let be $m\in\mathbf{N}$ arbitrary. Since one of the integers $r+1, \dots, r+m$ is divisible by m, we get $m|(r+1)^k\cdot\dots\cdot(r+m)^k$, say $ms=(r+1)^k\cdot\dots\cdot(r+m)^k$, therefore $z=qy_r=q(r+1)^k\cdot\dots\cdot(r+m)^ky_{r+m}=qmsy_{r+m}=m\cdot qsy_{r+m}$ with $qsy_{r+m}\in Y_0$. Since $z\in Y_0$ and $m\in\mathbf{K}$ were arbitrary, Y_0 is divisible.

Lemma 4.5. *For any abelian group $(Y, +)$, the following statements are equivalent:*

(vii) *$(Y, +)$ is reduced.*

(viii) *For any $\mathbf{Q}$-vector space X with $\dim_{\mathbf{Q}} X\geqq 1$, any $k\in\mathbf{N}$ and any $f\colon X\to Y$ with $f(nx)=n^kf(x)$ $(\forall x\in X, \forall n\in\mathbf{N})$ we have $f=\underline{0}$.*

PROOF. (vii)$\rangle\Rightarrow$(viii): Assume that there are X, k, f of the type required except that $f\neq\underline{0}$. Hence there exists $x_1\in X$ with $y_1:=f(x_1)\neq 0$. Define $x_{n+1}=(n+1)^{-1}\cdot x_n$ $(\forall n\in\mathbf{N})$. Then $(n+1)x_{n+1}=x_n$, $(n+1)^k\cdot f(x_{n+1})=f(x_n)$, $(n+1)^ky_{n+1}=y_n$ where $y_n:=f(x_n)$ $(\forall n\in\mathbf{N})$, and by Lemma 4.4, (vi)$\Rightarrow$(iv), $(Y, +)$ is not reduced, contradicting (vii). — (viii)$\Rightarrow$(vii): Let X be an arbitrary $\mathbf{Q}$-vector space with $\dim_{\mathbf{Q}} X\geqq 1$ and $h\in\operatorname{Hom}[(X, +), (Y, +)]$. Then $h(nx)=nh(x)$ $(\forall x\in X, \forall n\in\mathbf{N})$. By (viii) $h=\underline{0}$. By Lemma 4.2, (iv)$\Rightarrow$(iii), $(Y, +)$ is reduced.

Theorem 4.6. *For any ordered field $\mathbf{K}$, any $\mathbf{K}$-orthogonality space $(X, \perp)$, and any abelian group $(Y, +)$, the following statements are equivalent:*

(a) $\operatorname{Hom}_\perp(X, Y)=\{\underline{0}\}$;
(b) $\operatorname{Hom}(X, Y)=\{\underline{0}\}$;
(c) *$(Y, +)$ is reduced.*

PROOF. Since $\mathbf{Q}$ is a subfield of $\mathbf{K}$, X may be considered as a $\mathbf{Q}$-vector space, and $\dim_{\mathbf{Q}} X\geqq\dim_{\mathbf{K}} X\geqq 2$. — (a)$\Rightarrow$(b) follows from (1), and (b)$\Rightarrow$(c), (c)$\Rightarrow$(b) directly from Lemma 4.2, (iv)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv), respectively. — (b)$\Rightarrow$(a): Let be

$g \in \mathrm{Hom}_\perp(X, Y)$ even. By (Δ) and [7] (p. 37, Lemma 2a) and p. 39, Thm. 6), $g(o)=0$ and q is quadratic. Hence by Lemma 4.1c) $g(nx)=n^2g(x)$ ($\forall n \in \mathbf{N}$, $\forall x \in X$). Since we already know that (b) implies (c), Lemma 4.5, (vii)$\Rightarrow$(viii), guarantees that $g=\underline{0}$. Therefore, by Theorem 1.3c), $\mathrm{Hom}_\perp(X, Y)=\mathrm{Hom}(X, Y)=\{\underline{0}\}$.

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