On π-t groups*

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In this paper we deal with t-groups, and we introduce the concept of π-t group. We generalize the results concerning solvable t-groups for the case of π-solvable π-t groups.

We recall the following definitions and theorems.

Definition. A t-groups is a group G whose subnormal subgroups are all normal in G.

Definition. We say that a subgroup H of a group G is pronormal in G if and only if for all x in G, H and H^x are conjugate in $\langle H, H^x \rangle$.

The following result is well known:

Lemma A. *A subgroup H of G is normal in G if and only if it is both subnormal and pronormal in G.*

Lemma B. (PENG [1]). *Let M be a normal p'-subgroup of G and let P be any p-subgroup of G. Then P is pronormal in G if and only if PM/M is pronormal in G/M*

In 1957 GASCHÜTZ [2] proved the following

Theorem A. *The subgroups of solvable t-groups are again t-groups.*

In 1969 PENG [1] showed.

Theorem B. *G is a solvable t-group if and only if all subgroups of prime power order of G are pronormal in G.*

In 1977 ASAAD, M. [3] proved the following

Theorem C. *If each subgroup of prime power order of G' is pronormal in G and G' is a π-Hall subgroup of G, then G is a solvable t-group.*

The converse of this theorem is not true. ASAAD gave the following example: $G=A\times B$, where A is a quaternion group and B is an abelian group of odd order.

This theorem may be stated in the following more general form:

Theorem D. (PENG, T.: Personal communication)
Let N be a normal π-Hall subgroup of G. If G/N is a solvable t-group, and each subgroup of prime power order of N is pronormal in G, then G is a solvable t-group.

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We prove that if the order of the group G is odd, then the converse implication of Asaad's Theorem also holds.

Theorem 1. *Let G be a finite group of odd order. G is a solvable t-group if and only if each subgroup of prime power order of G' is pronormal in G, and G' is a π-Hall subgroup in G.*

For the proof we need the following Lemma.

Lemma 1. *Let G be a finite group, $G'<G$, $P\in \mathrm{Syl}\, p(G)$, $p\in\pi(G')$, $p\neq 2$. Suppose, that each subnormal p-subgroup of $N_G(P)$ is normal in $N_G(P)$. Then either $P\leqq G'$ or $C_G(P)=N_G(P)$ or both.*

Proof. Obviously P is a t-group. As $p\neq 2$, it is well known, that P is abelian. Assume, that $P\nleqq G'$. Thus $PG'\lhd G$, whence $N_G(P)G'=G$ follows by a Frattini argument.

The Schur-Zassenhaus' theorem implies that there exists a subgroup K such that $N_G(P)=P\cdot K$, $P\cap K=1$. Denote $P\cap G'=S$. Clearly $S\neq 1$. Obviously each element of K induces an automorphism of P by conjugation, and clearly they induce the identity on P/S. Applying a theorem of Glauberman [4] $C_P(K)S=P$ follows. Denote $L=C_P(K)$. We distinguish two cases

a) $\Omega_1(P)\leqq L$

In this case $K\leqq C_G(\Omega_1(P))$. As P is abelian, $K\leqq C_G(P)$ follows. Thus $N_G(P)=C_G(P)$.

b) $\Omega_1(P)\nleqq L$

There exists a subgroup T such that

$$\Omega_1(P)=[L\cap\Omega_1(P)]\times T$$

If $r\in T$, $r\neq e$, then $\langle r\rangle\lhd\lhd N_G(P)$. By assumptions $\langle r\rangle\lhd N_G(P)$ follows. Obviously $K\nleqq C_G(\langle r\rangle)$. Consider $u\in L\cap\Omega_1(P)$, $u\neq e$. As $|ur|=p$, $\langle ur\rangle\lhd\lhd N_G(P)$, $\langle ur\rangle\lhd N_G(P)$ by assumptions. Let $k\in K$ be, such that, $k\in C_G(u)$ but $k\notin C_G(r)$. Since $K\leqq N_G(\langle ur\rangle)$, there exists a natural number m such that

$$(ur)^k=(ur)^m,\quad \text{where}\quad 2\leqq m\leqq p-1\quad (k\notin C_G(ur))$$

Thus $ur^k=u^mr^m$, whence $u^{m-1}=r^k(r^m)^{-1}$ follows. As $K<N_G(\langle r\rangle)$, $r^k=r^n$, where $2\leqq n\leqq p-1$. So $u^{m-1}=r^{n-m}$, but $\langle u\rangle\cap\langle r\rangle=1$, thus $u^{m-1}=e$, a contradiction.

Now we turn to the proof of our theorem.

Proof of Theorem 1. i, Assume, that each subgroup of prime power order of G_r is pronormal in G, and G' is a π-Hall subgroup of G. Then G is a solvable t-group by Asaad's Theorem.

ii, Conversely, suppose that G is a solvable t-group. It is well known, that G is supersolvable, and all its Sylow-subgroups are abelian. So G' is nilpotent. Thus each subgroup of prime power order of G' is subnormal in G, whence it is normal in G by assumption.

Lemma A implies, that each subgroup of prime power order of G' is pronormal in G. Now we prove, that G' is a π-Hall subgroup in G. Assume, that there exists a

$Q \in \mathrm{Syl}_q(G)$ such that $Q \not\leqq G'$, $Q \cap G' = Q^* \neq 1$. Using Theorem A, by Lemma 1. $N_G(Q) = C_G(Q)$. It is easy to see, that $N_G(Q)G' = G$, thus $C_G(Q)G' = G$. As Q is abelian and G' is nilpotent, $G' \leqq C_G(Q^*)$ follows. Thus $C_G(Q^*) = G$. It is easy to show, that there exists a subgroup T in G such that

$$G = TQ, \quad T \lhd G, \quad T \geqq G' \quad \text{and} \quad T \cap Q = Q^*.$$

Since $C_G(Q^*) = G$ it follows by the Schur-Zassenhaus theorem that there exists a subgroup L in T such that $T = LQ^*$, and $L \lhd T$. It is easy to see, that $L \lhd G$ and $G = LQ$. As Q is abelian therefore $L \geqq G'$. Thus $Q \cap G' = 1$ contrary to the assumption $Q^* \neq 1$.

We introduce now the concept of π-t group.

Definition. Let G be a finite group, $\pi \subset \pi(G)$. G is called a π-t group if and only if its subnormal π-subgroups are all normal in G.

We construct a π-solvable π-t group, which possesses a subgroup that is not a π-t group. Let G be a finite group with following properties:

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd G_3 = G, \quad G_2/G_1 \cong P, \quad P \in \mathrm{Syl}\, p(G)$$

$$G_3/G_2 \cong Q, \quad Q \in \mathrm{Syl}\, q(G), \quad G_1 \in \mathrm{Syl}\, r(G)$$

$P = P_0 \times P_1$ where $|P_0| = p$ and P_1 is abelian, $G_1 \cdot P_0$ is a one step nonabelian group.

$$G_1 P_1 = G_1 \times P_1, \quad P_0 Q = P_0 \times Q, \quad G_1 Q = G_1 \times Q$$

furthermore $P_1 \cdot Q$ is a one step nonabelian group.

Let $\pi = \{p, q\}$. It is easy to show, that G is a π-solvable π-t group. Let X be a subgroup of P, such that $X \not\leqq P_0$, $X \not\leqq P_1$, $|X| = p$. We have $X \lhd P \lhd PQ$. Suppose $X \lhd PQ$, $X = \langle ab \rangle$, where $a \in P_1$, $b \in P_0$ $a \neq e$, $b \neq e$. Consider $c \in Q$, $c \neq e$. Then $(ab)^c = a^* b$ where $a^* \in P_1$. On the other hand $(ab)^c = (ab)^k$ where $2 \leqq k \leqq p-1$ $(c \notin C_G(ab))$. Thus $a^* b = a^k b^k$, hence $(a^k)^{-1} a^* = b^{k-1}$ follows.

But $\langle a \rangle \cap \langle b \rangle = 1$, so $b^{k-1} = e$, a contradiction. Thus PQ is not a π-t group.

In the following we shall derive some properties of π-solvable π-t groups. First of all we generalize the result of Peng.

Theorem 2. *Let G be a finite group, $\pi \subset \pi(G)$. Let N be a π-solvable normal subgroup of G such that $(|G : N|, p) = 1$ for all $p \in \pi \cap \pi(N)$. If G/N is a π-t group, and each p-subgroup of N, where $p \in \pi$ is pronormal in G, then G is a π-t group.*

For the proof we need the following

Lemma 2. *Let U be a finite group, $\pi \subset \pi(u)$. Let L be a solvable π-subgroup of U and $L \lhd\lhd U$. Suppose that each p-subgroup of L, is pronormal in U. Then $L \lhd U$.*

Proof. We argue by induction on $|u|$. The solvability of L implies, that there exists a p-subgroup S $|p \in \pi|$ of L such that $S \lhd L$. We have $S \lhd\lhd U$ and S is pronormal in U, hence $S \lhd U$ follows by Lemma A. Set $\overline{U} = U/S$, $\overline{L} = L/S$. $\overline{L} \lhd\lhd \overline{U}$ holds. With help of Lemma B it is easy to show that each p-subgroup of $\overline{L}$ is pronormal in $\overline{U}$. As $|\overline{U}| < |U|$, $\overline{L} \lhd \overline{U}$ follows by induction. Thus $L \lhd U$.

Now we turn to the proof of our theorem.

Proof of Theorem 2. Let H be a π-subgroup of G, such that $H \lhd \lhd G$. Obviously $H \cap N \lhd \lhd N$. Apply Lemma 2. to N and $H \cap N$, we obtain $H \cap N \lhd N$. As $H \cap N \lhd N \lhd G$, using Lemma 2. to G and $H \cap N$, we get $H \cap N \lhd G$. $(|H \cap N|, |H: H \cap N|)=1$ by the conditions of the Theorem. The Schur—Zassenhaus theorem implies, that there exists a subgroup K in H such that $H=(H \cap N)K$ and $(H \cap N) \cap K=1$. Clearly $H \cap N \lhd NK$, $H \lhd \lhd NK$. Obviously $(|K|, |N|)=1$. Using Sylow's theorems it is very easy to see that $H \lhd NK$. We have

$$HN/N \cong H/H \cap N \cong K$$

As $H \lhd \lhd G$, therefore $NH/H \lhd \lhd G/N$. G/N is a π-t group by assumption, hence $HN/N \lhd G/N$, so $HN \lhd G$ follows. Thus $H \lhd NK \lhd G$. As $H \cap N \lhd G$, K is a Hall subgroup in NK, using Sylow's theorems it is easy to see that $H \lhd G$.

Theorem 3. *Let G be a π-solvable finite group of odd order, where $\pi \subset \pi(G)$. G is a π-t group such that each subgroup of G is again a π-t group if and only if each p-subgroup of G', where $p \in \pi$, is pronormal in G and $(|G: G'|, p)=1$ for all $p \in \pi \cap \pi(G')$.*

Proof of Theorem 3. First, suppose that G' satisfies the above conditions. Theorem 2. is applicable with $N=G'$, thus we obtain that G is a π-t group. Let L be an arbitrary subgroup of G. We have

$$L/L \cap G' \cong LG'/G'.$$

As $LG'/G' \leqq G/G'$, $L/L \cap G'$ is abelian. Clearly $(|L: L \cap G'|, p)=1$ for all $p \in \pi$. Obviously each p-subgroup of $L \cap G'$ — where $p \in \pi$ — is pronormal in L. Applying Theorem 2. to L and $L \cap G'$, we obtain that L is a π-t group.

Conversely, assume that G is a π-solvable π-t group such that, each subgroup is a π-t group again.

a) Set $S \leqq G'$, $|S|=p^k$, $p \in \pi$.

We show that S is pronormal in G. Let $x \in G$ be arbitrary. Consider $T=\langle S, S^x \rangle$. If $S \in \mathrm{Syl}_p(T)$, then $S^x=S^u$, where $u \in T$. Thus we can assume that $S<P^*$ and $P^* \in \mathrm{Syl}_p(T)$. Obviously there exists a $t \in T$ such that $S^{xt}<P^*$, $P^* \leqq P \in \mathrm{Syl}_p(G)$. We apply now the Theorem of Alperin [5] for G with S and S^{xt}. Thus there exist elements x_i and p-Sylow subgroups Q_i of G, $1 \leqq i \leqq n$, furthermore an element y of $N_G(P)$ which satisfy the following conditions:

(i) $xt=x_1 x_2 \dots x_n y$

(ii) $P \cap Q_i$ is a tame intersection, $1 \leqq i \leqq n$.

(iii) x_i is a p-element of $N_G(P \cap Q_i)$, $1 \leqq i \leqq n$

(iv) $S \leqq P \cap Q_1$, while $S^{x_1 \dots x_i} \leqq P \cap Q_{i+1}$, where $1 \leqq i \leqq n-1$.

Clearly $S \lhd \lhd P \cap Q_1 \lhd N_G(P \cap Q_1)$. By assumptions $N_G(P \cap Q_1)$ is a π-t group, hence $S \lhd N_G(P \cap Q_1)$. As $x_1 \in N_G(P \cap Q_1)$, $S=S^{x_1}$. Similarly we obtain $S^{x_1 \dots x_n}=S$. Thus $S^{xt}=S^y$, where $y \in N_G(P)$. Obviously $S \lhd \lhd P \lhd N_G(P)$. Since $N_G(P)$ is a π-t group, it follows that $S \lhd N_G(P)$. So $S^{xt}=S$. Thus $S^x=S^{t^{-1}}$ and S is pronormal in G.

b) Let $r \in \pi(G')$, $R \in \mathrm{Syl}_r(G)$. Assume that $R \not\leqq G'$, then $N_G(R)=C_G(R)$ by Lemma 1.

R is a t-group of odd order, so R is abelian. As $N_G(R)=C_G(R)$, G has a normal r-complement K by the Theorem of Burnside. Thus $RK=G$, $R\cap K=1$, $K\lhd G$. Clearly $G/K\cong R$. As R is abelian, $G'\leqq K$, whence $G'\cap R=1$ follows, a contradiction. The proof is complete.

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