

Semigroups of set-valued functions

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Abstract. It is proved that a measurable semigroup of linear continuous set-valued functions satisfying some additional assumptions is majorized by a one-parameter family of an exponential type generated by it.

Troughout the paper vector spaces are always real. The symbols \mathbb{R} and \mathbb{N} denote the set of all real numbers and the set of positive integers, respectively. Now recall some definitions connected with set-valued functions (abbreviated to “s.v. functions” in a sequel).

If X is a normed space we denote $n(X)$ the family of all non-empty subsets of X , the family $cc(X)$ consists of convex compact members of $n(X)$. Let X, Y, Z be vector spaces and let C be a convex cone in X . An s.v. function $A : C \rightarrow n(Y)$ is said to be additive iff it satisfies the condition

$$A(x + y) = A(x) + A(y) \quad \text{for all } x, y \in C.$$

An s.v. function A is said to be linear iff it is additive and

$$A(tx) = tA(x) \quad \text{for all } x \in C \quad \text{and } t \in (0, +\infty).$$

Applying Theorem 4 in [16] we define the norm of a linear s.v. function $A : C \rightarrow n(Y)$, denoted by $\|A\|$, to be the smallest element of the set

$$\{M > 0 : \|A(x)\| \leq M\|x\|, x \in C\}.$$

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For a given s.v. function $F : X \rightarrow n(Y)$ and sets $B \subset X, D \subset Y$ we define the sets

$$\begin{aligned} F(B) &= \bigcup \{F(x) : x \in B\}, \\ F^-(D) &= \{x \in X : F(x) \cap D \neq \emptyset\}, \\ F^+(D) &= \{x \in X : F(x) \subset D\}. \end{aligned}$$

They are called, respectively, the image of B , the lower inverse image of D and the upper inverse image of D under the s.v. function F .

The superposition $G \circ F$ of s.v. functions $F : X \rightarrow n(Y)$ and $G : Y \rightarrow n(Z)$ is the s.v. function defined as follows

$$(G \circ F)(x) := G(F(x)) \quad \text{for all } x \in X.$$

Assume that X and Y are two topological vector spaces. We say that an s.v. function $F : X \rightarrow n(Y)$ is lower semicontinuous (l.s.c.) iff the set $F^-(U)$ is open in X for every open set U in Y . We say that an s.v. function F is upper semicontinuous (u.s.c.) iff the set $F^+(U)$ is open in X for every open set U in Y . F is said to be continuous iff it is both l.s.c. and u.s.c.

In what follows h denotes the Hausdorff metric on $cc(X)$. If X is complete, then $cc(X)$ is complete as well.

For the properties of the Hausdorff metric and the convergence in the space $(cc(X), h)$ see [4], [8] or [9]. Some of them needed here are also collected in [11].

Let X be a nonempty set. A family $\{F^t : t \geq 0\}$ of s.v. functions $F^t : X \rightarrow n(X)$ is called an iteration semigroup iff

$$F^t \circ F^s = F^{t+s} \quad \text{for all } s, t \geq 0.$$

Let (T, \mathcal{M}, m) be a measure space, X be a metric space. An s.v. function $F : T \rightarrow n(X)$ is measurable iff for every open set U the set $F^-(U)$ belongs to \mathcal{M} (see [1]) and a function $f : T \rightarrow X$ is measurable iff the inverse image $f^{-1}(U) \in \mathcal{M}$ for every open set U . We say that an iteration semigroup $\{F^t, t \geq 0\}$ is measurable (continuous) iff for every $x \in X$ the s.v. function $t \mapsto F^t(x)$ is measurable (continuous) (see [15]).

Now let X be a separable Banach space, (T, \mathcal{M}, m) be a complete σ -finite measure space. Let $E \in \mathcal{M}$ and $F : E \rightarrow n(X)$ be an s.v. function.

We will denote by \mathcal{S}_F the set of all Bochner integrable functions $f : E \rightarrow X$ such that $f(x) \in F(x)$ almost everywhere in E . A measurable function $F : E \rightarrow c(X)$ is called Aumann integrable iff $\mathcal{S}_F \neq \emptyset$ and then we say that the set

$$\int_E F dm = \left\{ \int_E f dm : f \in \mathcal{S}_F \right\}$$

is the Aumann integral of F (see [10]). A function $F : E \rightarrow c(X)$ is integrably bounded iff there exists a Lebesgue integrable function $g : E \rightarrow \mathbb{R}$ such that $\|F(x)\| \leq g(x)$ for almost every $x \in X$. It follows easily that every measurable and integrably bounded s.v. function is Aumann integrable (see [10] and [17]). The following theorem will be useful in our considerations.

Lemma 1 (Theorem 3.5 in [7]). *Let (T, \mathcal{M}) be a measurable space with complete and σ -finite measure, let (X, d) be a complete separable metric space and let $F : T \rightarrow cl(X)$. Then the following conditions are equivalent:*

- (a) $F^-(U) \in \mathcal{M}$ for every open U
- (b) $F^-(B) \in \mathcal{M}$ for every Borel B
- (c) *there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of measurable selections of F such that $F(t) = cl\{f_n(t) : n \in \mathbb{N}\}$.*

In what follows $T = [0, \infty)$ and \mathcal{M} is the σ -algebra of Lebesgue measurable subsets of $[0, \infty)$.

Applying above lemma and properties of the Bochner integral we can prove analogous properties of the Aumann one given in below lemmas. (For other informations on the Aumann integral we refer the reader to [1], [2], [4], [8] and [10].)

Lemma 2. *Let X be a separable Banach space. If $F : [\xi, t + \xi] \rightarrow cc(X)$ is an Aumann integrable function, then*

$$(1) \quad \int_{\xi}^{t+\xi} F(x) dx = \int_0^t F(x + \xi) dx.$$

Lemma 3. *Let X be a separable Banach space, $A \in cc(X)$ and let $t_1 < t_2$. If $F : [t_1, t_2] \rightarrow cc(X)$ is an s.v. function such that*

$$F(t) = A \quad \text{for every } t \in [t_1, t_2]$$

then

$$\int_{t_1}^{t_2} F(t) dt = (t_2 - t_1)A.$$

A consequence of the Fubini theorem (Theorem 3. 7. 13 in [7]) is the next

Lemma 4. *Let X be a separable Banach space, let $n \in \mathbb{N}$, $t > 0$. If $F : [0, t] \rightarrow cc(X)$ is measurable and integrably bounded then*

$$\iint_E (t-x)^n F(y) dx dy = \frac{1}{n+1} \int_0^t (t-y)^{n+1} F(y) dy,$$

where $E = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq t, 0 \leq y \leq x\}$.

Proofs of above three lemmas are left to the reader.

Lemma 5. *Let X and Y be separable Banach spaces, C be an open convex cone in X . If $A : C \rightarrow cc(Y)$ is a linear continuous s.v. function and $f : [t_1, t_2] \rightarrow C$, ($0 < t_1 < t_2$) is Bochner integrable, then the composition $A \circ f$ is Aumann integrable and*

$$A \left[\int_{t_1}^{t_2} f(x) dx \right] = \int_{t_1}^{t_2} (A \circ f)(x) dx.$$

PROOF. An easy computation shows that the composition $A \circ f$ is a measurable s.v. function. First we will show that $\mathcal{S}_{A \circ f} \neq \emptyset$. Since A is linear and continuous, by the corollary after Theorem 2 in [13]

$$A(x) = \{a(x) : a \in \mathcal{F}_A\} \quad \text{for } x \in C$$

where \mathcal{F}_A denotes the family of all linear continuous selections of A .

Take an $a \in \mathcal{F}_A$, then the function $a \circ f$ is measurable. Moreover, since f is Bochner integrable,

$$\int_{t_1}^{t_2} \|(a \circ f)(x)\| dx \leq \int_{t_1}^{t_2} \|a\| \|f(x)\| dx = \|a\| \int_{t_1}^{t_2} \|f(x)\| dx < \infty$$

hence $a \circ f \in \mathcal{S}_{A \circ f}$.

It suffices to show the equality of integrals.

Assume that $Y = \mathbb{R}$. Then a linear continuous function $A : C \rightarrow cc(\mathbb{R})$ is of the form

$$A(x) = [a(x), b(x)] \quad \text{for } x \in C,$$

where $a, b : C \rightarrow \mathbb{R}$ are linear and continuous. Then

$$(2) \quad \begin{aligned} A\left[\int_{t_1}^{t_2} f(x)dx\right] &= \left[a\left(\int_{t_1}^{t_2} f(x)dx\right), b\left(\int_{t_1}^{t_2} f(x)dx\right)\right] \\ &= \left[\int_{t_1}^{t_2} (a \circ f)(x)dx, \int_{t_1}^{t_2} (b \circ f)(x)dx\right]. \end{aligned}$$

Observe that $a \circ f, b \circ f \in \mathcal{S}_{A \circ f}$ and for every $y \in \int_{t_1}^{t_2} (A \circ f)(x)dx$ there exists $\lambda \in [0, 1]$ such that

$$y = \lambda \int_{t_1}^{t_2} (a \circ f)(x)dx + (1 - \lambda) \int_{t_1}^{t_2} (b \circ f)(x)dx.$$

Since the Aumann integral is a convex set (see [10]),

$$\int_{t_1}^{t_2} (A \circ f)(x)dx = \left[\int_{t_1}^{t_2} (a \circ f)(x)dx, \int_{t_1}^{t_2} (b \circ f)(x)dx\right]$$

which with (2) gives the equality

$$A\left[\int_{t_1}^{t_2} f(x)dx\right] = \int_{t_1}^{t_2} (A \circ f)(x)dx.$$

If Y is any separable Banach space take an $y^* \in Y^*$. Then $y^* \circ A$ is a linear continuous function with convex compact images in \mathbb{R} and in this case we have

$$(3) \quad (y^* \circ A)\left[\int_{t_1}^{t_2} f(x)dx\right] = \int_{t_1}^{t_2} [(y^* \circ A) \circ f](x)dx.$$

On the other hand

$$(4) \quad (y^* \circ A)\left[\int_{t_1}^{t_2} f(x)dx\right] = y^*\left[A\left(\int_{t_1}^{t_2} f(x)dx\right)\right]$$

and

$$(5) \quad \int_{t_1}^{t_2} (y^* \circ A)(f(x))dx = y^* \left[\int_{t_1}^{t_2} (A \circ f)(x)dx \right]$$

Combining (3),(4) with (5) we obtain

$$y^* \left[A \left(\int_{t_1}^{t_2} f(x)dx \right) \right] = y^* \left[\int_{t_1}^{t_2} (A \circ f)(x)dx \right]$$

for every $y^* \in Y^*$. The Aumann integral being weakly closed (see Corollary after Proposition 3.1 in [10]) it is closed. Moreover it is also convex. Finally the Second separation theorem (see [3] p. 251) ends the proof. \square

Lemma 6. *Let X, Y be separable Banach spaces and let C be an open convex cone in X . If $A : C \rightarrow cc(Y)$ is a linear continuous s.v. function, $F : [t_1, t_2] \rightarrow cc(C)$ ($0 < t_1 < t_2$) is measurable and integrably bounded, then the composition $A \circ F$ is Aumann integrable and*

$$A \left[\int_{t_1}^{t_2} F(x)dx \right] = \int_{t_1}^{t_2} (A \circ F)(x)dx.$$

PROOF. Observe that the function $A \circ F : [t_1, t_2] \rightarrow cc(Y)$ is measurable and analogously to the proof of the Lemma 4 $\mathcal{S}_{A \circ F} \neq \emptyset$.

We only need to show the equality of integrals.

Take an $y \in A \left[\int_{t_1}^{t_2} F(x)dx \right]$, there exists a $w \in \int_{t_1}^{t_2} F(x)dx$ such that $y \in A(w)$. Thus there exist $f \in \mathcal{S}_F$ and $a \in \mathcal{F}_A$ which satisfy conditions

$$w = \int_{t_1}^{t_2} f(x)dx \quad \text{and} \quad y = a(w),$$

hence

$$y = a \left(\int_{t_1}^{t_2} f(x)dx \right) = \int_{t_1}^{t_2} (a \circ f)(x)dx.$$

Since $a \circ f \in \mathcal{S}_{A \circ F}$ (cf. the proof of the Lemma 5)

$$y \in \int_{t_1}^{t_2} (A \circ F)(x)dx$$

and the inclusion

$$A\left[\int_{t_1}^{t_2} F(x)dx\right] \subset \int_{t_1}^{t_2} (A \circ F)(x)dx$$

is showed.

The proof of the other inclusion will be done in two steps. Take an element $y \in \int_{t_1}^{t_2} (A \circ F)(x)dx$. There exists $g \in \mathcal{S}_{A \circ F}$ such that $y = \int_{t_1}^{t_2} g(x)dx$. Define the set

$$T := \{x \in [t_1, t_2] : g(x) \in (A \circ F)(x)\}$$

and the s.v. functions for $x \in T$

$$\begin{aligned} G(x) &:= \{t \in F(x) : g(x) \in A(t)\}, \\ H(x) &:= \{t \in \overline{C} : g(x) \in \overline{A}(t)\} \end{aligned}$$

where \overline{A} is a continuous additive extension of the function A on the cone \overline{C} . Hence the functions G and H have nonempty closed values.

Now assume that $Y = \mathbb{R}$ and take an open ball $K(x_0, r) \subset C$. The following conditions hold

$$\begin{aligned} H^-(K(x_0, r)) &= \{x \in [t_1, t_2] : H(x) \cap K(x_0, r) \neq \emptyset\} \\ &= \{x \in [t_1, t_2] : \exists t \in K(x_0, r) \ t \in H(x)\} \\ &= \{x \in [t_1, t_2] : \exists t \in K(x_0, r) \ g(x) \in \overline{A}(t) = A(t)\} \\ &= \{x \in [t_1, t_2] : g(x) \in A(K(x_0, r))\} = g^-[A(K(x_0, r))]. \end{aligned}$$

Since the image of a connected set by a connected-valued continuous function is connected as well (see [3]), the set $A(K(x_0, r))$ is an interval. Therefore it is a borel set. Applying the condition (b) in Lemma 1 we have that the set

$$H^-(K(x_0, r)) = g^-[A(K(x_0, r))]$$

is measurable because g is a measurable function.

If U is any open subset of C in the separable space X then there exists a countable family $\{K(x_n, r_n) : n \in \mathbb{N}\}$ of open balls in X such that $U = \bigcup_{n \in \mathbb{N}} K(x_n, r_n)$. Therefore the lower inverse image

$$H^-(U) = H^-\left(\bigcup_{n \in \mathbb{N}} K(x_n, r_n)\right) = \bigcup_{n \in \mathbb{N}} H^-(K(x_n, r_n))$$

belongs to the σ -algebra \mathcal{M} , hence H is measurable.

Observe that $G = F \cap H$ and by Th. 8.2.4 and 8.1.3 in [1] the s.v. function G is measurable and there exists a measurable selection f of G , it means

$$f(x) \in F(x) \quad \text{and} \quad g(x) \in A[f(x)]$$

for all $x \in T$. The function f is weakly measurable, because for every $x^* \in X^*$ the composition $x^* \circ f$ is a strongly measurable function with values in a separable space \mathbb{R} . Therefore by Pettis measurability theorem (see Th. II.1.2 in [5]) it is strongly measurable. Moreover, since F is integrably bounded

$$\int_{t_1}^{t_2} \|f(x)\| dx \leq \int_{t_1}^{t_2} \|F(x)\| dx < \infty,$$

hence f is Bochner integrable selection of F .

Applying Lemma 5, we conclude that in this case

$$y = \int_{t_1}^{t_2} g(x) dx \in \int_{t_1}^{t_2} A[f(x)] dx = A \left[\int_{t_1}^{t_2} f(x) dx \right] \subset A \left[\int_{t_1}^{t_2} F(x) dx \right].$$

If Y is any separable Banach space we will use the same method as in the proof of the above lemma.

For every $y^* \in Y^*$ we have the equality

$$(y^* \circ A) \left[\int_{t_1}^{t_2} F(x) dx \right] = \int_{t_1}^{t_2} [(y^* \circ A) \circ F](x) dx,$$

hence

$$y^* \left[A \left(\int_{t_1}^{t_2} F(x) dx \right) \right] = y^* \left[\int_{t_1}^{t_2} (A \circ F)(x) dx \right]$$

and consequently

$$A \left[\int_{t_1}^{t_2} F(x) dx \right] = \int_{t_1}^{t_2} (A \circ F)(x) dx. \quad \square$$

Lemma 7. *Let X be a separable Banach space, Y be a normed space and C be an open convex cone in X . If $A : C \rightarrow cc(Y)$ is a linear continuous s.v. function then there exists a real constant $M \geq 0$ such that for every $x, y \in C$*

$$h(A(x), A(y)) \leq M\|x - y\|.$$

PROOF. First we show that there exists an $x_0 \in C$ satisfying condition

$$(6) \quad \forall v \in S \quad \exists u \in C \quad x_0 + v = u,$$

where S is a closed unit ball in X . Indeed, take an $z \in C$. Then there exists an $\epsilon > 0$ such that

$$z + \epsilon S \subset C.$$

Therefore, defining $x_0 := \frac{1}{\epsilon}z$ we have

$$x_0 + S \subset \frac{1}{\epsilon}C = C,$$

and (6) holds true.

Now take $x, y \in C$ and $w \in A(x)$. Then there exists $a \in \mathcal{F}_A$ such that $w = a(x)$ and $\|a(v)\| \leq \|A\|\|v\|$ for every $v \in C$. Since $C - C = X$, a function $\hat{a} : C - C \rightarrow Y$ defined by the formula $\hat{a}(v) := a(v_1) - a(v_2)$ for $v = v_1 - v_2$ is a linear continuous extension of the function a . By (6) for every $v \in S$ there exists an $u \in C$ such that $v = u - x_0$ and $\|u\| \leq 1 + \|x_0\|$ therefore

$$\begin{aligned} \|\hat{a}(v)\| &= \|a(u) - a(x_0)\| \leq \|a(u)\| + \|a(x_0)\| \\ &\leq \|a\|(\|u\| + \|x_0\|) \leq \|A\|(1 + 2\|x_0\|) \end{aligned}$$

for all $v \in S$ thus $\|\hat{a}\| \leq \|A\|(1 + 2\|x_0\|) =: M$. It follows that for every $a \in \mathcal{F}_A$

$$\begin{aligned} d(a(x), A(y)) &\leq \|a(x) - a(y)\| = \|\hat{a}(x) - \hat{a}(y)\| \\ &\leq \|\hat{a}\|\|x - y\| \leq M\|x - y\|, \end{aligned}$$

thus

$$e(A(x), A(y)) \leq M\|x - y\|$$

and analogously

$$e(A(y), A(x)) \leq M\|x - y\|$$

which completes the proof. \square

A consequence of the above theorem and the Theorem 1.4 in [15] is the next

Remark 1. Let X be a separable Banach space, C be an open convex cone in X . If $\{A^t : t > 0\}$ is a measurable semigroup of linear continuous s.v. functions $A^t : C \rightarrow cc(C)$ then it is continuous.

Lemma 8. *Let X be a separable Banach space, C be an open convex cone in X . If $F : [0, \infty) \rightarrow cc(C)$ is a continuous s.v. function then for every $t \geq 0$*

$$\lim_{\xi \rightarrow 0} \frac{1}{\xi} \int_t^{t+\xi} F(s) ds = F(t).$$

PROOF. Fix $t \geq 0$. By continuity of F the function $s \mapsto h(F(t), F(s))$ is continuous as well and therefore

$$(7) \quad \lim_{\xi \rightarrow 0} \frac{1}{\xi} \int_t^{t+\xi} h(F(t), F(s)) ds = 0.$$

By the Lemma 3

$$h\left(\frac{1}{\xi} \int_t^{t+\xi} F(s) ds, F(t)\right) = \frac{1}{\xi} h\left(\int_t^{t+\xi} F(s) ds, \int_t^{t+\xi} F(t) ds\right).$$

Since it is easy to check that Lemma 9 in [14] is true for integrably bounded functions it follows

$$h\left(\frac{1}{\xi} \int_t^{t+\xi} F(s) ds, F(t)\right) \leq \frac{1}{\xi} \int_t^{t+\xi} h(F(s), F(t)) ds$$

for all $t \geq 0$, which with (7) gives our assertion. \square

The main goal is to prove

Theorem. *Let X be a separable Banach space, C be an open convex cone in X . Let $\{A^t : t \geq 0\}$ be a measurable semigroup of linear continuous functions $A^t : \overline{C} \rightarrow cc(C)$ satisfying conditions*

- (i) $A^0(x) = \{x\}$ for $x \in \overline{C}$
- (ii) $A^t(x) - x \subset \overline{C}$ for $x \in \overline{C}$, $t \geq 0$
- (iii) *the family of s.v. functions $\{\frac{1}{\xi}(A^\xi - A^0) : \xi > 0\}$ is uniformly convergent to an s.v. function G for $\xi \rightarrow 0$ on each compact subset of \overline{C} .*
Then for every $x \in C$ and $t \geq 0$

$$A^t(x) \subset B^t(x) := \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x).$$

PROOF. By assumption (iii), G is a linear continuous function on \overline{C} with convex compact values, therefore the formula

$$B^t(x) := \sum_{i=1}^{\infty} \frac{t^i}{i!} G^i(x) \quad \text{for } x \in \overline{C}$$

defines the family $\{B^t : t \geq 0\}$ of linear s.v. functions $B^t : \overline{C} \rightarrow cc(\overline{C})$ which are continuous on C (see Theorem and Corollary 2 in [11]).

Fix an $x \in C$ and define $F(t) := A^t(x)$ for $t \geq 0$. By the Remark 1 F is continuous on $(0, \infty)$. We will show that it is continuous at the origin. Take an $\epsilon > 0$, then by (iii) there exists $0 < t_0 < 1$ such that for every $0 < t < t_0$

$$h\left(\frac{A^t(x) - x}{t}, G(x)\right) < \frac{\epsilon}{2} \quad \text{and} \quad t\|G\|\|x\| < \frac{\epsilon}{2},$$

thus

$$A^t(x) - x \subset tG(x) + \frac{\epsilon}{2}tS \subset tG(x) + \frac{\epsilon}{2}S.$$

Therefore for $t < t_0$

$$\begin{aligned} h(F(t), F(0)) &= h(A^t(x) - x, \{0\}) = \|A^t(x) - x\| \\ &\leq \|tG(x) + \frac{\epsilon}{2}S\| \leq t\|G\|\|x\| + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Now observe that

$$(8) \quad F(t+s) = A^{t+s}(x) = A^t(A^s(x)) = A^t(F(s))$$

which implies

$$(9) \quad \int_0^t F(\xi+s)ds = \int_0^t A^\xi[F(s)]ds.$$

By Lemma 2, we have

$$(10) \quad \int_0^t F(\xi+s)ds = \int_\xi^{\xi+t} F(s)ds.$$

Lemma 6 gives

$$(11) \quad \int_0^t A^\xi[F(s)]ds = A^\xi \left[\int_0^t F(s)ds \right].$$

Combining (10) and (9) with (11) we obtain

$$\int_\xi^{\xi+t} F(s)ds = A^\xi \left[\int_0^t F(s)ds \right],$$

and adding to the both sides of the above equality an integral $\int_0^\xi F(s)ds$ we have

$$\int_\xi^{t+\xi} F(s)ds + \int_0^\xi F(s)ds = A^\xi \left[\int_0^t F(s)ds \right] + \int_0^\xi F(s)ds.$$

Thus and by Theorem 1.1.8 in [2]

$$(12) \quad \int_0^t F(s)ds + \int_t^{t+\xi} F(s)ds = A^\xi \left[\int_0^t F(s)ds \right] + \int_0^\xi F(s)ds,$$

for every $t, \xi \geq 0$. Fix an $\epsilon > 0$, $t \geq 0$ and put $K := \int_0^t F(s)ds$. If $f \in \mathcal{S}_F$ then by Corollary II.2.8 in [5] $\int_0^t f(s)ds \subset t\overline{\text{co}}f([0, t])$. Therefore $K = \int_0^t F(s)ds \subset t\overline{\text{co}}F([0, t]) =: K_0$. Continuous image $F([0, t])$ of $[0, t]$ by F is compact and by Mazur's theorem (Th. II.2.12 in [5]) K_0 is compact too. Since the Aumann integral K is a closed (cf. the proof of the Lemma 4) subset of the compact set it is compact.

By assumption (iii), there exists $\delta > 0$ such that for $y \in K$, $\xi \in (0, \delta)$

$$h\left(\frac{A^\xi(y) - y}{\xi}, G(y)\right) < \epsilon.$$

Taking an $y \in K$ and $\xi \in (0, \delta)$, we have

$$h(A^\xi(y) - y, \xi G(y)) < \epsilon \xi$$

which implies

$$A^\xi(y) - y \subset \xi G(y) + \epsilon \xi S,$$

therefore for every $y \in K$

$$A^\xi(y) \subset y + \xi G(y) + \epsilon \xi S.$$

It follows that for $y \in K$

$$A^\xi(y) \subset K + \xi G(K) + \epsilon \xi S$$

and therefore

$$(13) \quad A^\xi(K) \subset K + \xi G(K) + \epsilon \xi S.$$

Substituting K we can rewrite (13) as

$$(14) \quad A^\xi\left[\int_0^t F(s)ds\right] \subset \int_0^t F(s)ds + \xi G\left[\int_0^t F(s)ds\right] + \epsilon \xi S \quad \text{for } \xi \in (0, \delta),$$

which with (12) yields

$$\begin{aligned} \int_t^{t+\xi} F(s)ds + \int_0^t F(s)ds &= A^\xi\left[\int_0^t F(s)ds\right] + \int_0^\xi F(s)ds \\ &\subset \int_0^t F(s)ds + \xi G\left[\int_0^t F(s)ds\right] + \epsilon \xi S + \int_0^\xi F(s)ds. \end{aligned}$$

Thus and by Lemma 1 in [12]

$$\frac{1}{\xi} \int_t^{t+\xi} F(s)ds \subset G\left[\int_0^t F(s)ds\right] + \epsilon S + \frac{1}{\xi} \int_0^\xi F(s)ds,$$

for $\xi \in (0, \delta)$. When we let $\xi \rightarrow 0$ then we obtain

$$F(t) \subset G \left[\int_0^t F(s) ds \right] + \epsilon S + F(0) \quad \text{for } \epsilon > 0,$$

(see Lemma 8) because $G(\int_0^t F(s) ds) \in cc(X)$, hence $t \geq 0$ and $x \in C$

$$(15) \quad F(t) \subset G \left[\int_0^t F(s) ds \right] + x.$$

Observe that (15) with the properties of the Aumann integral yield

$$\begin{aligned} & \frac{1}{n!} G^{n+1} \left[\int_0^t (t-s)^n F(s) ds \right] \\ & \subset \frac{1}{n!} G^{n+1} \left\{ \int_0^t (t-s)^n \left[G \left(\int_0^s F(u) du \right) + x \right] ds \right\} \\ & = \frac{1}{n!} G^{n+2} \left[\int_0^t \int_0^s (t-s)^n F(u) du ds \right] + \frac{1}{n!} G^{n+1} \left[\int_0^t (t-s)^n x ds \right] \\ & = \frac{1}{(n+1)!} G^{n+2} \left[\int_0^t (t-u)^{n+1} F(u) du \right] + \frac{t^{n+1}}{(n+1)!} G^{n+1}(x), \end{aligned}$$

for every $n \in \mathbb{N}_0$, $t \geq 0$ and $x \in C$. Therefore

$$\begin{aligned} F(t) & \subset G \left[\int_0^t F(s) ds \right] + x \subset G^2 \int_0^t (t-s) F(s) ds + tG(x) + x \\ & \subset \frac{1}{2!} G^3 \left[\int_0^t (t-s)^2 F(s) ds \right] + \frac{t^2}{2!} G^2(x) + tG(x) + x. \end{aligned}$$

hence substituting in such way we can write

$$F(t) \subset x + tG(x) + \frac{t^2}{2} G^2(x) + \dots + \frac{1}{n!} G^{n+1} \left[\int_0^t (t-s)^n F(s) ds \right]$$

for every $n \in \mathbb{N}$, $t \geq 0$ and $x \in C$. Since

$$\begin{aligned} \left\| \frac{1}{n!} G^{n+1} \left[\int_0^t (t-s)^n F(s) ds \right] \right\| & \leq \frac{\|G\|^{n+1}}{n!} \int_0^t (t-s)^n \|F(s)\| ds \\ & \leq \frac{t^n \|G\|^{n+1}}{n!} \int_0^t \|A^s(x)\| ds \end{aligned}$$

and $\int_0^t \|A^s(x)\| ds < \infty$ the rest of the series converges to $\{0\}$, when $\xi \rightarrow 0$, therefore

$$A^t(x) \subset B^t(x) \quad \text{for } x \in C, \quad t \geq 0$$

and the proof is complete. \square

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