A hybrid algebra with duality

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In the Jacobson point of view certain algebras derived from the original algebra can best be studied by defining other binary operations in terms of the original product.

The objective is to study an algebra in which the product is replaced by a derived product which is defined in terms of the Jordan type and Lie type products.

The attempt is to develop the structure and the representation theory of the algebra.

1. Definition

Let F be a field of characteristic not equal to two, and let L be a non-associative linear algebra over F with identity.

The field F may be supposed to be a subfield of the algebra L with 0 and 1 as the additive and the multiplicative identities of F and, hence, also of L.

Consider a set of elements x_i and y_i $(i=1, \dots, n)$ of L satisfying the operation (written: $\square$) (called: square dot product) with multiplicative properties

$$(1) \qquad x_i x_j = \begin{cases} -x_j x_i, & i \neq j \\ x_i^2 = 0, & i = j \end{cases} \quad i, j = 1, \dots, n$$

$$(2) \qquad y_i y_j = \begin{cases} y_j y_i, & i \neq j \\ y_i^2 = 1, & i = j \end{cases} \quad i, j = 1, \dots, n$$

(3) If p and q are elements of $\{1, \dots, n\}$, if i_k $(k=1, \dots, p)$ are different elements of $\{1, \dots, n\}$ and if j_k $(k=1, \dots, q)$ are different elements of $\{1, \dots, n\}$, then

$$\Big(\sum_{k=1}^{p} x_{i_k}\Big)\Big(\sum_{k=1}^{q} y_{j_k}\Big) = \Big(\sum_{k=1}^{q} y_{j_k}\Big)\Big(\sum_{k=1}^{p} x_{i_k}\Big) =$$
$$= \begin{cases} \sum_{k=1}^{p} x_{i_k} & \text{if } \{i_k\colon k=1, \dots, p\} = \{j_k\colon k=1, \dots, q\} \\ 0 & \text{otherwise} \end{cases}$$

(4) Multiplication is associative on the closure of $\{x_i\colon i=1, \dots, n\}$ under multiplication.

(5) Multiplication is associative on the closure of $\{y_i\colon i=1, \dots, n\}$ under multiplication.

(6) If $n=1$, $x_1 \neq 0$ and $y_1 \neq 1$, and if $n \geqq 2$, $\sum_{i=1}^{n} x_i \neq 0$.

It follows from (2) and (5) that, on the closure of $\{y_i\colon i=1, \dots, n\}$ under multiplication, the operation is both commutative and associative.

Axioms (1), (2), (4) and (5) imply that the meaning of an expression like $y_2y_3y_4$ is clear without any parenthesis, and that, for example

$$x_1x_2x_3x_4x_1 = 0$$

and

$$y_1y_2y_3y_4y_1 = y_2y_3y_4.$$

Proposition 1. *The x-products with the subscripts in their natural order),* 1 *and the y-products are linearly independent.*

PROOF. This is an easy verification.

Proposition 2. *The set* $\bigcup_{i=1}^{n} \{x_i, y_i\}$ *generates a subalgebra* $C_n(x, y)$ *of the algebra L over F consisting of all linear combinations of products of x's only and of products of y's only.*

PROOF. This is an easy verification.

By an abuse of terminology, we refer to the subalgebra $C_n(x, y)$ of the algebra L over F simply as the algebra.

Definition. The algebra $C_n(x, y)$ over the field F generated by the set $\bigcup_{i=1}^{n} \{x_i, y_i\}$ and satisfying axioms (1)—(6) is called *NI-algebra.*

The NI-algebra $C_n(x, y)$ is commutative if $n=1$ and non-commutative if $n\geqq 2$.

The following is an example of an algebra L which has elements x_i and y_i $(i=1, \dots, n)$ with the properties (1)–(6).

Example 1. Let P be the power set of $\{1, \dots, n\}$. Let α_A $(A\in P-\{\varphi\})$ and $\bar{\alpha}_A$ $(A\in P-\{\varphi\})$ and β_A $(A\in P)$ be different things. (Their cardinality is $1+2(2^n-1)+$ $+2^n$.) Let

$$\Gamma' = \{\alpha_A\colon A\in P-\{\varphi\}\}\cup\{\beta_A\colon A\in P\}.$$

Define multiplication of elements of Γ as follows:

$$\alpha_A\alpha_B = \begin{cases} \zeta & \text{if } A\cap B\neq\varphi \\ \alpha_{A\cup B} & \begin{cases}\text{if } A\cap B=\varphi \text{ and} \\ \{\{p, q\}\colon p\in A, q\in B, q-p<0\} \text{ has an even} \\ \text{number of elements}\end{cases} \\ \bar{\alpha}_{A\cup B} & \begin{cases}\text{if } A\cap B=\varphi \text{ and} \\ \{\{p, q\}\colon p\in A, q\in B, q-p<0\}\} \\ \text{has an odd number of elements}\end{cases} \end{cases}$$

$$\alpha_A\beta_B = \beta_B\alpha_A = \begin{cases} \alpha_A & \text{if } B=\varphi \\ \alpha_A & \text{if } A=B \quad (\text{hence } B\neq\varphi) \\ \zeta & \text{if } B\neq 0 \text{ and } A\neq B \end{cases}$$

$$\beta_A\beta_B = \beta_{A\Delta B} \quad \text{where } \Delta \text{ is the symmetric difference.}$$

Let

$$\overline{(\bar{\alpha}_A)} = \alpha_A.$$

Let

$$x_i = \alpha_{\{i\}}$$

$$y_i = \beta_{\{i\}}$$

for $i=1, \ldots, n$. Then all the conditions are satisfied with the modification that 0 is replaced by ζ, 1 by β_φ and $-x_j x_i$ by $\overline{x_j x_i}$. The operation defined on Γ so far can be extended to an operation on

$$\{\bar{\alpha}_A\colon A \in P - \{\varphi\}\} \cup \{\zeta\} \cup \Gamma$$

in an obvious way.

Let L be the set of all mappings of Γ into F (or of all Γ-vectors with components in F). Define addition and multiplication of elements of L in a suitable way. Then L will become a non-associative ring having a subring isomorphic to F. If $i \in \{1, \ldots, n\}$, let x_i' and y_i' be the Γ-vectors defined as follows:

$$x_i'(\gamma) = \begin{cases} 1 & \text{for } \gamma = \alpha_{\{i\}} \\ 0 & \text{for } \gamma \in \Gamma - \{\alpha_{\{i\}}\} \end{cases}$$

$$y_j'(\gamma) = \begin{cases} 1 & \text{for } \gamma = \beta_{\{i\}} \\ 0 & \text{for } \gamma \in \Gamma - \{\beta_{\{i\}}\} \end{cases}$$

Let $0'$ and $1'$ be the Γ-vectors defined as follows

$$0'(\gamma) = 0 \quad \text{for every element } \gamma \text{ of } F.$$

$$l'(\gamma) = \begin{cases} 1 & \text{for } \gamma = \beta_\varphi \\ 0 & \text{for } \gamma \in \Gamma - \{\beta_\varphi\} \end{cases}$$

(0 and 1 are, respectively, the additive and the multiplicative identities of F.) Then the conditions are satisfied with x_i, y_i, 0 and 1 replaced by x_i', x_i', $0'$ and $1'$, respectively.

Example 2. Let $C_1(x, y)$ be generated by

$$1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then by direct verification

$$x^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$y^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$xy = yx = x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

2. Basis and dimension

The NI-algebra is a finite-dimensional non-associative algebra.

Proposition 3. *The basis of the algebra $C_n(x, y)$ is*

$$\{x_i^{p_i}, y_j^{q_j}\colon p_i, q_j = 0, 1, \ i, j = 1, \dots, n\}$$

whose dimension is $2 \cdot 2^n - 1$.

PROOF. An element of $C_n(x, y)$ is of the form

$$\Sigma a_i x_i + \Sigma b_i y_j + \\ + \sum_{i<j} a_{ij} x_i x_j + \dots + a_{1\dots n} x_1 \dots x_n + \\ + \sum_{i<j} b_{ij} y_i y_j + \dots + b_{1\dots n} y_1 \dots y_n + \mathrm{c}$$

where $a_i, b_i, c \in F$ $(i=1, \dots, n)$ from which it follows that

$$\{x_i^{p_i}, y_j^{q_j}\colon p_i, q_j = 0, 1, \ i, j = 1, \dots, n\}$$

is a basis of $C_n(x, y)$, and is of dimension $2 \cdot 2^n - 1$ since

$$2 \sum_{k=1}^{n} \binom{n}{k} - 1 = 2 \cdot 2^n - 1.$$

3. Subalgebras

The $C_k(x, y)$ $(k=1, \dots, n)$ are subalgebras of $C_n(x, y)$. Let

$$C_1(x, y) \equiv C_1(x_1, y_1)$$
$$C_2(x, y) \equiv C_2(x_1, x_2;\ y_1, y_2)$$
$$\vdots$$
$$C_n(x, y) \equiv C_n(x_1, \dots, x_n;\ y_1, \dots, y_n)$$

denote, respectively, the subalgebra generated by

$$\{1, x_1, y_1\}$$
$$\{1, x_1, x_2, y_1, y_2, x_1 x_2, y_1 y_2\}$$
$$\{x_i^{p_i}, y_j^{q_j}\colon p_i, q_j = 0, 1, \ i, j = 1, \dots, n\}$$

The relationship between the NI-algebra $C_n(x, y)$ and the NI-algebra $C_{n+1}(x, y)$ is as follows — wherefrom a certain structural duality is evident.

Theorem 1. *The* NI-*algebra $C_{n+1}(x, y)$ is expressible as a square dot product of the* NI-*algebra $C_n(x, y)$ and*

$$C_{n+1}(x, y) \simeq C_n(x, y) \square \{1, x_{n+1}, y_{n+1}\}$$

the subalgebra of $C_{n+1}(x, y)$ with basis $\{1, x_{n+1}, y_{n+1}\}$.

PROOF. By properties of an NI-algebra the elements of the subalgebra $C_n x_{n+1}$ are the linear combinations of

$$x_{n+1}, x_1 x_{n+1}, \ldots, x_1 x_2 x_{n+1}, \ldots$$

and the elements of the subalgebra $C_n y_{n+1}$ are linear combinations of

$$y_{n+1}, y_1 y_{n+1}, \ldots, y_1 y_2 y_{n+1}, \ldots$$

Therefore

$$C_{n+1}(x, y) \cong C_n(x, y) + C_n x_{n+1} + C_n y_{n+1} \cong C_n(x, y) \square \{1, x_{n+1}, y_{n+1}\}$$

Theorem 2. *The* NI-*algebra* $C_n(x, y)$ *is expressible as a square dot product of the x-subalgebra* $C_n(x)$ *and the y-subalgebra* $C_n(y)$.

$$C_n(x, y) \simeq C_n(x) \square C_n(y)$$

PROOF. The inequality

$$C_n(x) \square C_n(y) \leqq C_n(x, y)$$

is evident, and the reverse inequality follows from the fact that each element of $C_n(x, y)$ is expressible (not necessarily uniquely) as a square dot product of an element of $C_n(x)$ and an element of $C_n(y)$.

Corollary 1.

(a) $C_2(x_1, x_2) \simeq C_1(x_1) \square C_1(x_2)$

(b) $C_3(x_1, x_2, x_3) \simeq C_2(x_1, x_2) \square C_1(x_3)$

(c) $C_4(x_1, x_2, x_3, x_4) \simeq C_2(x_1, x_2) \square C_2(x_3, x_4)$

Corollary 2.

(a) $C_2(y_1, y_2) \simeq C_1(y_1) \square C_2(y_2)$

(b) $C_3(y_1, y_2, y_3) \simeq C_2(y_1, y_2) \square C_1(y_3)$

(c) $C_4(y_1, y_2, y_3, y_4) \simeq C_2(y_1, y_2) \square C_2(y_3, y_4)$.

An NI-algebra of any dimension is reducible to a square dot product of lower dimensional ones.

Theorem 3. *The* NI-*algebra* $C_n(x, y)$ *for* $n = p + q$ *is expressible as a square dot product*

$$C_n(x, y) \simeq C_{p+q}(x, y) \simeq$$

$$\simeq C_p(x_1, \ldots, x_p;\ y_1, \ldots, y_p) \square C_{p+1}(x_{p+1}, \ldots, x_n;\ y_{p+1}, \ldots, y_n)$$

PROOF. Without loss of generality letting $n=4$, then

$$C_4(x_1, \ldots, x_4; y_1, \ldots, y_4) \simeq C_4(x_1, \ldots, x_4)\Box C_4(y_1, \ldots, y_4) \simeq$$
$$\simeq (C_2(x_1, x_2)\Box C_2(x_3, x_4))\Box(C_2(y_1, y_2)\Box C_2(y_3, y_4)) \simeq$$
$$\simeq (C_2(x_1, x_2)\Box C_2(y_1, y_2))\Box(C_2(x_3, x_4)\Box C_2(y_3, y_4)) \simeq$$
$$\simeq C_2(x_1, x_2;\ y_1, y_2)\Box C_2(x_3, x_4;\ y_3, y_4)$$

This completes the proof.

4. *Structure*

Any NI-algebra has a direct sum decomposition.

Proposition 4. *The* NI-*algebra* $C_n(x, y)$ *has* 2^n *pairwise idempotents.*

PROOF. Now

$$1/2+1/2y_i = 1/2(1+y_i)$$

and

$$(1/2+1/2y_i)(1/2+1/2y_i) = 1/4(1+2y_i+1) = 1/2+1/2y_i$$

and similarly $1/2-1/2y_i$ is an idempotent.

Now

$$\prod_{i=1}^{n} (1/2+1/2y_i) =$$
$$= 2^{-n}(1+y_1+\ldots+y_n+y_1y_2+y_1y_3+\ldots+y_1\ldots y_n).$$

By the result just obtained, it follows that $1/2+1/2y_i$ is idempotent, and this continues to be true if y_i is replaced by $-y_i$ for some i of $\{1, \ldots, n\}$.

Proposition 5. *The* NI-*algebra has pairwise idempotents* e_i, $i=1, \ldots, 2^n$, *such that*

$$\sum_{i=1}^{2^n} e_i = 1$$

and

$$e_ie_j = 0, \quad i \neq j$$

where

$$e_1 = 2^{-n}(1+y_1+\ldots+y_n+y_1y_2+y_1y_3+\ldots+y_1\ldots y_n)$$

and $e_j, j=2, \ldots, 2^n$, *is obtained by replacing some* y_i *by* $-y_i$ *for some* i *of* $\{1, \ldots, n\}$.

PROOF. Let

$$I(y_1, \ldots, y_n) = 2^{-n}(1+y_1+\ldots+y_n+y_1y_2+\ldots+y_1\ldots y_n).$$

Then it is obvious from the above factorization that the e_i are pairwise orthogonal because

$$(1/2+1/2y_i)(1/2-1/2y_i) = 0$$

and that

$$I(y_1, \ldots, y_n, y_{n+1})+I(y_1, \ldots, y_n, -y_{n+1}) = I(y_1, \ldots, y_n).$$

From this equation it follows by induction that

$$\sum_{i=1}^{2^n} e_i = 1.$$

Proposition 6. *The subalgebra* X_i, $1 \leqq i \leqq n$, *of the* NI-*algebra* $C_n(x, y)$ *with basis*

$$\{x_i, x_i x_j, x_i x_j x_k, \ldots, x_1 \ldots x_i \ldots x_n\}$$

is a nilpotent two-sided ideal of $C_n(x, y)$.

Proof. Since

$$(a_i x_i + a_{ij} x_i x_j + \ldots)(b_i x_i + b_{ij} x_i x_j + \ldots) = 0$$

then it follows that

$$X_i^2 = 0$$

that is, X_i is a nilpotent subalgebra. By the properties of the NI-algebra $C_n(x, y)$ it follows that

$$C_n(x, y) X_i \leqq X_i$$

and

$$X_i C_n(x, y) \leqq X_i.$$

Therefore X_i, $1 \leqq i \leqq n$, is a nilpotent two-sided ideal of $C_n(x, y)$.

Proposition 7. *The sum*

$$N = \sum_{i=1}^{n} X_i$$

$(1 \leqq i \leqq n)$ *is the nilpotent radical of* $C_n(x, y)$, *and the difference algebra*

$$\bar{C}_n(x, y) = C_n(x, y)/N$$

is semisimple.

Proof. It is clear that N is a nilpotent two-sided ideal. There remains to show that N is maximal.

Suppose to the contrary that M is a nilpotent two-sided ideal of $C_n(x, y)$ and $N < M$. Then there is at least one element $z \in M$ such that $z \notin N$. Then z is of the form

$$z = c + \Sigma a_i x_i + \Sigma b_i y_i + \Sigma a_{ij} x_i x_j + \Sigma b_{ij} y_i y_j + \ldots$$

But constant terms and the coefficient of terms with y are not all zero, and, furthermore, none of the powers of such elements can be zero. Therefore z is not a nilpotent element, but this contradicts the assumption that M is nilpotent.

It is obvious that $C_n(x, y)/N$ is semisimple.

Theorem 4. *The semisimple algebra*

$$\bar{C}_n(x, y) = C_n(x, y)/N$$

has pairwise orthogonal idempotents $\bar{e}_i$, $i = 1, \ldots, 2^n$, *that is,*

$$\sum_{i=1}^{2^n} \bar{e}_i = \bar{1}$$

and

$$\bar{e}_i\bar{e}_j = \bar{0}, \quad i \neq j$$

and the semisimple algebra $\bar{C}_n(x, y)$ has a direct sum decomposition

$$\bar{C}_n(x, y) \simeq \bar{e}_1\bar{C}_n(x, y)\bar{e}_1 \oplus \ldots \oplus \bar{e}_{2^n}\bar{C}_n(x, y)\bar{e}_{2^n} \simeq$$

$$\simeq \bar{C}_n(x, y)\bar{e}_1 \oplus \ldots \oplus \bar{C}_n(x, y)\bar{e}_{2^n} \simeq F\bar{e}_1 \oplus \ldots \oplus F\bar{e}_{2^n}$$

PROOF. The first part of the theorem follows from Proposition 5, and the second part follows from the structure theorem on semisimple algebra. Since $\bar{C}_n(x, y) \simeq C_n(y)$, $\bar{C}_n(x, y)$ is a commutative, associative and semi-simple algebra, and hence the direct sum of fields.

The following is the main structure theorem for an NI-algebra.

Theorem 5. *The* NI-*algebra $C_n(x, y)$ has pairwise orthogonal idempotents e_i, $i=1, \ldots, 2^n$, that is,*

$$\sum_{i=1}^{2^n} e_i = 1$$

and

$$e_i e_j = 0, \quad i \neq j$$

and $C_n(x, y)$ has a direct sum decomposition

$$C_n(x, y) \simeq C_n(x, y)e_1 \oplus \ldots \oplus C_n(x, y)e_{2^n} \simeq$$

$$\simeq (e_1) \oplus \ldots \oplus (e_{2^n})$$

where (e_i) denotés the ideal generated by e_i, $i=1, \ldots, 2^n$.

PROOF. The first part of the theorem follows from Proposition 5.

For the second part of the theorem, since Theorem 4 yields the 2^n direct-sum components of

$$\bar{C}_n(x, y) = C_n(x, y)/N$$

then the proof follows by a result of Deuring (cf. M. DEURING, Algebra, p. 16).

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