The solution of the word problem in certain groups

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Introduction

In this note we investigate groups G having a presentation

$$(*)\qquad G = \langle x, y \mid x^{n_i} y^{n_i} = y^{n_i} x^{n_i},\ i = 1, 2, \dots, k\rangle, \quad k \geqq 1, \quad n_i \in \mathbf{Z}.$$

The main results are the following.

Theorem A. *Let G be presented by $(*)$. Then G has a solvable word problem.*

The proof of the theorem relies heavily on small cancellation theory. In fact, the theorem below reduces the problem to the case $k \leqq 2$ which is treated mainly by small cancellation theory.

Theorem B. *Let G be presented by $(*)$ and assume that the n_i are pairwise relatively prime. Then G is abelian if and only if $k \geqq 3$. Moreover, if $k \geqq 3$ then G is free abelian of rank 2.*

Remark. With more effort the method of Theorem A solves the conjugacy problem too. This will appear elsewhere.

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1. The Abelian Groups

In this section we prove

Theorem 1. *Let $G = \langle x, y \mid x^{n_i} y^{n_i} = y^{n_i} x^{n_i},\ i = 1, 2, 3\rangle$. If the n_i are pairwise relatively prime, then G is abelian.*

We begin the proof with the following easy observation.

Lemma 1. *Let G be a group generated by x and y and let $S = S(G)$ be the set of integers s for which $x^s y^s = y^s x^s$ holds. Then*

(a) *$s \in S$ implies that $s\mathbf{Z} \subseteq S$ and*

(b) *$a, b, a+b \in S$ implies $a-b \in S$.*

The proof is straightforward, hence we omit it.

Our aim is to show that if G is as defined in Theorem 1 then $S(G)=\mathbf{Z}$. This motivates the following

Lemma 2. *Let S be a subset of $\mathbf{Z}$ which satisfies* (a) *and* (b) *of Lemma* 1. *If $a, b, a+b\in S$ then $a\mathbf{Z}+b\mathbf{Z}\subseteq S$.*

PROOF. We prove first by induction on n that

$$(*) \qquad a\pm nb\in S.$$

For $n=1$ this is clear by condition (b). Assume $(*)$ holds for all $n\leqq k$. We prove it for $n=k+1$. We have $-b\in S$ by condition (a) and by assumption $a+nb$ and $(a+nb)-b$ are in S. Hence by (b), $a+(n+1)b\in S$. Similarly, $a-nb\in S$, $b\in S$ and $(a-nb)+b\in S$ implies $a-(n+1)b\in S$, as required. Thus we have

$$(**) \qquad a+b\mathbf{Z}\in S,$$

and by symmetry

$$(***) \qquad a\mathbf{Z}+b\in S.$$

Finally, we show that $na+mb\in S$ for all $n, m\in\mathbf{Z}$. By $(***)$, $na+b\in S$. Hence by replacing a by na in $(**)$, we get $na+mb\in S$ for all $n, m\in\mathbf{Z}$, as required.

Lemma 3. *Let $a, b, c\in S$, $0<a<b<c$, $(a, b)=(a, c)=(b, c)=1$. Then there exists a $c'\in S$ such that either $c'=1$ or $(a, c')=(b, c')=1$ and $1<c'<c$.*

We divide the proof of the lemma into three steps.

Step 1. Let $a, b, c\in S$ and assume $0<a<b<c$, $(a, b)=(b, c)=(a, c)=1$. Then there exists $c'\in S$ such that either $c'=1$ or $(a, b)=(b, c')=(a, c')=1$ and $0<c'\leqq(a-1)(b-1)$.

PROOF. If $c\leqq(a-1)(b-1)$ we are done. So assume $c>(a-1)(b-1)$, and c is the smallest element of S with this property. Then by [1] or direct calculation, we can write $c=\alpha a+\beta b$ where $\alpha, \beta>0$. But then $\alpha a-\beta b\in S$ by Lemma 2, hence taking $c'=|\alpha a-\beta b|$ we obtain $0<c'<c$ and $(a, b)=(a, c')=(b, c')=1$. Assume $c'\neq 1$. Then by the minimality of c we get $c'\leqq(a-1)(b-1)$ or $c'<b$. However, the second possibility implies the first, i.e. in both cases we get $c'\leqq(a-1)(b-1)$ as required.

Corollary. *Let $a, b, c\in S$ as in Step* 1. *Then we may assume that $c\leqq(a-1)(b-1)$.*

From now on we shall assume that a, b and c are positive integers which satisfy

$$\text{(i)} \qquad a, b, c\in S, a<b<c,$$

$$\text{(ii)} \qquad (a, b)=(a, c)=(b, c)=1,$$

$$\text{(iii)} \qquad c\leqq(a-1)(b-1).$$

Step 2. There exist $\alpha, \beta\in\mathbf{Z}$ with $|\alpha|<b$ such that $c=\alpha a+\beta b$. Similarly, there are $\gamma, \delta\in\mathbf{Z}$ with $|\delta|<a$ such that $c=\gamma a+\delta b$.

PROOF. Let $c=qa+r$, $0<r<a$. Then $q<b$. Since $(a,b)=1$, there exists a natural number $t<b$ such that $t(b-a)\equiv r \bmod b$, i.e., $r=t(b-a)-ub$, $u\in \mathbf{Z}$. Thus $c=q-t)a+(t-u)b$. Clearly $|q-t|<b$.

Step 3. Let $c=\alpha a+\beta b$ with $|\alpha|<b$ and let $d=(\alpha a, \beta b)$. Then $(d,a)=(d,b)=$ $=(a,b)=1$ and $0<d<c$.

PROOF. Cleraly $d|c$. If $d=c$ then $c|\alpha a, \beta b$. As $(a,b)=1$ we must have $c|\alpha$ and $c|\beta$. But then $0<c\leqq|\alpha|<b$, contradicting $c>b$. This proves Step 3.
We turn now to the proof of the lemma.

By Step 1 we may assume $c<(a-1)(b-1)$. Hence by Step 2 we may assume $c=\alpha a+\beta b$ with $|\alpha|<b$. Consequently by Step 3 with $c'=d$ we get the result.
We turn now to the proof of Theorem 1. Let G be the group of Theorem 1 and let $S=S(G)$. We may assume $n_1<n_2<n_3$. Denote by $r(S)$ the set of all the triplets (a,b,c) such that $a,b,c\in S$, $a<b<c$ and $(a,b)=(a,c)=(b,c)=1$. Define $|a,b,c|=a+b+c$ and assume that (a,b,c) is a minimal element of $r(S)$ with respect to $|\ ,\ ,\ |$. Since $(n_1, n_2, n_2)\in r(S)$ such a minimal element exists. However, by Lemma 3 there exists a c' such that either $c'=1$ or (a,b,c') or (a,c',b) or (c',a,b) belongs to $r(S)$ and $a+b+c'<a+b+c$. Consequently $c'=1$ and $1\in S$. But then $S=\mathbf{Z}$ and G is abelian. *This completes the proof of Theorem* 1.

2. The Non-Abelian Case

In this section we prove the following

Theorem 2. *Let* $G=\langle x,y|x^ny^n=y^nx^n,\ \ x^my^m=y^mx^m\rangle$. *Then G has a solvable word problem.*

The proof is by small cancellation theory. Let us fix some notation. For unexplained terms see |2|.
Let $F=\langle x,y\rangle$ and let $R_1=x^ny^nx^{-n}y^{-n}$ and $R_2=x^my^mx^{-m}y^{-m}$.
Let $w\in F$ be a reduced word. We shall denote by (w) the length i.e. the number of letters in w. F also has the free product structure $F=\langle x\rangle*\langle y\rangle$ and the corresponding free product normal form. We shall denote by $\|w\|$ the free product length of w. Thus, if $w=x^{\alpha_1}y^{\beta_1}x^{\alpha_2}\dots y^{\beta_n}$ then $|w|=\sum_{i=1}^{n}(\alpha_i+\beta_i)$ while $\|w\|=2n$. In these terms we have

Lemma 4. *Let G be as in Theorem 2 and let $\mathcal{R}_0$ be the symmetric closure of R_1 and R_2. Then there exists a symmetrical subset $\mathcal{R}$ of F such that*

(a) $\langle x,y|\mathcal{R}\rangle=G$;
(b) $\mathcal{R}\supseteq\mathcal{R}_0$ *and* $\mathcal{R}$ *is recursive;*
(c) *For every* $R\in\mathcal{R}$ *we have* $\|R\|\geqq 4$;
(d) *Let M be an $\mathcal{R}$-diagram with labeling function Φ and let D be a region in M.*
  (i) *If μ is a boundary path on ∂D which is a piece, then $\|\Phi(\mu)\|=1$ and μ is a proper subpath of an edge e of ∂D with $\|\Phi(e)\|=1$.*
  (ii) *M satisfies* $C(8)$.
  (iii) *If v_1, v_2, v_3 are consecutive pieces on ∂D then $|v_1v_2v_3|<\frac{1}{2}|\partial D|$ and if θ is the complement of $v_1v_2v_3$ to ∂D then $\|\Phi(\theta)\|\geqq 3$.*

Note that (b) and (d) (ii) solve the word problem (see |2|).

In the construction of $\mathscr{R}$ we apply a basic technique developed by E. Rips in his fundamental work [3]. Let us recall this construction in a way most convenient for us.

Let M be an $\mathscr{R}_0$-diagram and assume that there is defined some equivalence relation on the regions of M. For every equivalence class $\mathscr{E}$ let $\mathscr{E}'$ be the interior of the closure of the union of the elements of $\mathscr{E}$. If for every equivalence class $\mathscr{E}$ we have that $\mathscr{E}'$ is connected and simply connected, then the set of the $\mathscr{E}'$ where $\mathscr{E}$ ranges over all the equivalence classes of M gives rise to a diagram over a symmetrical subset $\mathscr{R}$ of F which contains $\mathscr{R}_0$. We call this diagram a *derived diagram* and the $\mathscr{E}'$ the *derived region.*

We turn now to the construction of the desired derived diagram. From now on we set $F=\langle x, y\rangle$, $\mathscr{R}_0=$the symmetric closure of $R_1=x^ny^nx^{-n}y^{-n}$ and $R_2=x^my^m\cdot\cdot x^{-m}y^{-m}$. We assume $m>n$. Let M be a connected and simply connected $\mathscr{R}_0$-diagram.

Definitions.

1) Let M be an $\mathscr{R}_0$-diagram, let D be a region in M and let $\partial D=v_0e_0v_1e_1v_2e_2v_3e_3v_0$ such that $\Phi(e_1)=\Phi(e_3)=x^k$ and $\Phi(e_2)=\Phi(e_3)=y^k$, where Φ is the labeling function and $k\in\{\pm n, \pm m\}$. We call the vertices v_0, v_1, v_2 and v_3 *separating vertices.*

2) Let e be an edge in M. We call e a *standard edge* if

(i) both endpoints of e are separating vertices, and

(ii) $\Phi(e)=x^{\pm l}$ or $\Phi(e)=y^{\pm l}$ where $l\in\{1, 2, \ldots, m\}$.

A *standard piece* is a standard edge which is a piece.

Figure 1 below shows standard pieces, while Figure 2 shows a a non-standard piece.

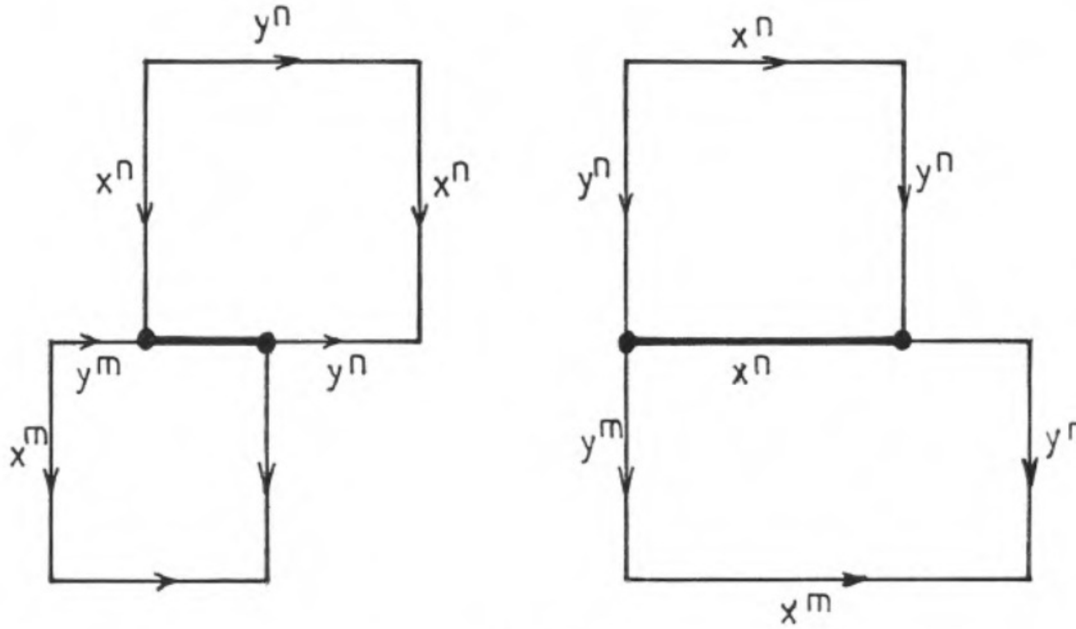

Figure 1.

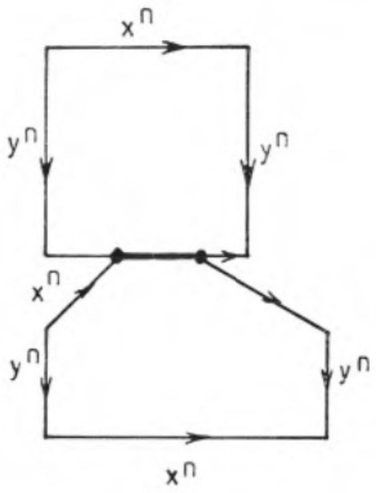

Figure 2.

We are now ready to construct $\mathscr{R}$. Thus let M be a connected and simply connected $\mathscr{R}_0$-diagram. Say that two regions D_1 and D_2 in M with a common edge are *weakly equivalent* if every component of their common boundary is a standard piece. Let "$\approx$" be the transitive closure of the weak equivalence defined above. Then "$\approx$" is an equivalence relation on the regions of M. Let $D_1', \dots, D_r'$ be the derived regions with respect to "$\approx$" as described above and let M' be the corresponding derived diagram. We have to show that

1) D_i' is simply connected for every i, $i=1, \dots, r$,

2) conditions (a)-(d) of Lemma 4 hold.

The next lemma is useful in showing that (2) follows from (1).

Lemma 5. *Let D' be a derived region of M.*

(a) *If e is a boundary edge of D' having endpoints with valency $\geqq 3$ (in M) and such that $\|\Phi(e)\|=1$ then e is a standard edge;*

(b) *If D' is simply connected then $\|\Phi(\partial D')\| \geqq 4$.*

The lemma follows by an immediate induction on the number of regions of D' (as a subdiagram of M) and the fact that the sum of the exponents of x and y in $\Phi(\partial D)$ is zero. We omit it.

Assume now that D_i' is simply connected for $i=1, \dots, r$. We prove (a)-(d). Let $\mathscr{R}$ be the set of the boundary labels of all the possible derived regions in M with respect to "$\approx$", where M runs over all the simply connected $\mathscr{R}_0$-diagrams which have connected interior. Then (a) and the first part of (b) are immediate. The second part of (b) follows from Lemma 6 below.

Lemma 6. *Let M be a simply connected $\mathscr{R}_0$-diagram with connected interior. Let M_0 be a simply connected subdiagram of M with connected interior. Let $b(M_0)$ be the number of regions in M_0 and let $t=\min\{m^2-n^2, n^2\}$.* If all the pieces of M_0 *are standard, then* $b(M_0) \leqq \dfrac{3}{4t}\,|\Phi(\partial M_0)|^2$.

PROOF. Since every piece in M_0 is standard, all vertices have valency not more than 4. Moreover, we may represent M_0 as the union of 3 kinds of basic plane figures described below in such a way that every side of a plane figure is either horizontal or vertical. The basic plane figures are as follows.

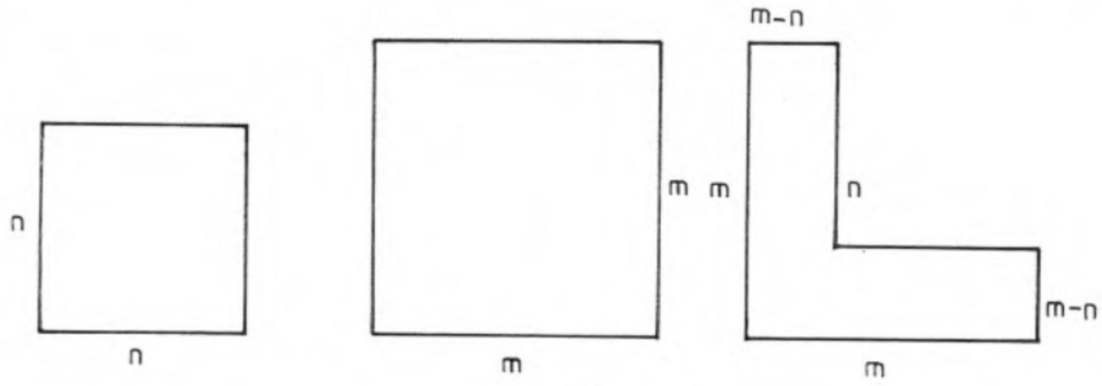

Figure 3.

Clearly, all figures have area $\leqq t$. If M_0 contains k basic figures, then

(1) $$k \leqq b(M_0) \leqq 3k.$$

Denote by $S(M_0)$ the area of M_0 as represented above. Since all basic figures have area $\geqq t$ we get

(2) $$kt \leqq S(M_0).$$

Let $T = T(M_0)$ be the minimal rectangle which insribes M_0 such that the sides of T contain an edge of M_0 represented as above. Then

(3) $$S(T) \geqq S(M_0).$$

On the other hand, if $l(T)$ is the length of the boundary of T then

(4) $$|\Phi(\partial M_0)| \geqq l(T).$$

Combining (2) and (3) we get

(5) $$kt \leqq S(T).$$

Since $S(T) \leqq \frac{1}{4} l(T)^2$ we get from (4) and (5) that

(6) $$kt \leqq \frac{1}{4} |\Phi(\partial M_0)|^2.$$

Finally, combining (6) with (1) yields

$$b(M_0) \leqq \frac{3}{4t} |\Phi(\partial M_0)|^2,$$

as required.

Also (c) follows from part (b) of Lemma 5 and (d) (i) follows from part (a) of Lemma 5 and the definition of "$\approx$". d(ii) and d(iii) are now immediate by standard arguments from small cancellation theory.

We still have to prove that D_i', $i=1, \dots, r$, are simply connected. Assume D' is not simply connected and let D' be a minimal derived region with this property. Then D' has a "hole" H which is filled in with derived regions which are already simply connected. Consequently, conditions (a)-(d) of Lemma 4 are satisfied by H. But then H has a boundary path e with $\|\Phi(e)\|=1$, guaranteed by part (b) of Lemma 5, which by part (a) of the same lemma has an endpoint v with valency 2. Clearly v is necessarily a separating vertex. Consequently H has a common standard piece with D' contradicting the definition of "$\approx$". Thus D_i', $i=1, \dots, r$, are simply connected and Lemma 4 together with Theorem 2 is proved. Theorem 2 and hence Theorem A now follow by parts (b) and d(ii) of Lemma 4.

Finally, we prove Theorem B. We only have to show that the group in Theorem 2 is not abelian. This can be shown directly by mapping G on $C_n * C_m$* but it also follows from the above results, for it is immediate from Lemma 4(d) that either M' contains one region in which case by Lemma 5, $|\Phi(\partial M)| \geqq 3n \geqq 6$ or M' contains more than one region in which case $\|\partial M\| \geqq 6$, by Lemma 4(c) and (d)(iii). *This proves Theorem B.*

References

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[3] E. Rips, Generalized Small Cancellation Theory and Applications, *Isr. J. of Math.*, **41** (1982), pp 1—146.

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