Group rings of existentially closed locally finite p-groups

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Let K be any field. Assume that the group G is either existentially closed (e.c.) in the class of all groups, or e.c. in the class of all locally finite groups.*) Then a result of K. BONVALLET, B. HARTLEY, D. S. PASSMAN and M. K. SMITH [1] asserts that the group ring KG contains precisely one proper ideal, namely its augmentation ideal $\omega(KG)$. Obviously, a necessary ingredient for this result is the simplicity of G.

Now, consider the e.c. structures in the class $L\mathfrak{F}_p$ of all locally finite p-groups. Up to isomorphism, there exists a unique countable e.c. $L\mathfrak{F}_p$-group E_p [5, Satz 2]. This group cannot be simple, since minimal normal subgroups of $L\mathfrak{F}_p$-groups are cyclic of order p [2, 1.B.8]. However, E_p is characteristically simple [5, Satz 7]. And B. MAIER observed in [5, pp. 124—125] that the characteristical simplicity of E_p together with the main part of the proof of BONVALLET, HARTLEY, PASSMAN and SMITH still yields the following result: If K is a field with $\operatorname{Char} K \neq p$, then $\omega(KE_p)$ is the unique proper ideal in KE_p which is invariant under the basis transformations of KE_p induced by the automorphisms of E_p.

It is the goal of the present note, to extend this result to the modular group rings of E_p.

Theorem. *Let K be any field, and let G be an e.c. $L\mathfrak{F}_p$-group.*

(a) *For every proper ideal I of KG, there exists a nontrivial normal subgroup N in G such that* $\omega(KN)\cdot KG \subseteqq I$.

(b) *If G is countable, then $\omega(KG)$ is the unique proper ideal in KG which is invariant under the basis transformations of KG induced by the automorphisms of G. In particular, KG is either characteristically simple, or $\omega(KG)$ is the unique proper characteristic ideal in KG.*

Clearly, every ideal of $\omega(KG)$ is also an ideal in KG. Hence, if $\omega(KG)$ is the unique proper characteristic ideal in KG, then $\omega(KG)$ is characteristically simple.

Note also, that uncountable e.c. $L\mathfrak{F}_p$-groups are in general not characteristically simple by a result of S. THOMAS [6, Theorem 1]. Therefore, it seems to be unlikely that part (b) of the Theorem does also hold for e.c. $L\mathfrak{F}_p$-groups of arbitrary cardinality.

In the proof of the Theorem, we will use wreath product constructions. Recall that, for any two groups A and B, the unrestricted regular *wreath product* $W = A\,Wr\,B$

* Cf. [3], [5] or [6] for the definition of "e.c.".

is the split extension $W=\Omega \rtimes B$ of its *base group* $\Omega=\{f|f\colon B\to A\}=A^B$ (with componentwise multiplication) by its *top group* B with regard to the action $(b')f^b=(b'b^{-1})f$ for all $f\in\Omega$ and $b, b'\in B$. The *support* of a function $f\in\Omega$ is defined as

$$\operatorname{supp} f = \{b\in B|(b)f \neq 1\}.$$

We will always identify in the natural way B with the top group of W, and A with the *1-component* $\{f\in\Omega|\operatorname{supp} f\subseteqq\{1\}\}$ of W. Note, that

$$\operatorname{supp} a^b \subseteqq \{b\} \quad \text{for all } a\in A,\, b\in B.$$

Proof of the Theorem. Part (b) follows from (a), since every countable e.c. $L\mathfrak{F}_p$-group is characteristically simple [5, Satz 7], and since $\omega(KG)$ has codimension 1 in KG.

For the proof of part (a), suppose we are given a finite elementary abelian p-group A, and a finite subgroup X of G. Then $AWrX=\langle A, X\rangle$ and G are contained in the $L\mathfrak{F}_p$-group $AWrG$. And since G is e.c. in $L\mathfrak{F}_p$, we obtain an embedding

$$\sigma\colon AWrX \to G \quad \text{with} \quad \sigma\restriction X = id.$$

(The group table of $AWrX$ (with the elements from X as constants) can be duplicated in G.) This shows, that we can follow the proof of the theorem of Bonvallet, Hartley, Passman and Smith [1], in order to establish part (a) whenever $\operatorname{Char} K\neq p$, as observed by B. Maier.

Now, assume that $\operatorname{Char} K=p$. Let $0\neq\alpha\in I$. Then there exist pairwise distinct elements $x_0, \ldots, x_n\in G$ such that

$$\alpha = \sum_{i=0}^{n} k_i x_i^{-1} \quad \text{with} \quad 0 \neq k_i\in K \quad \text{for} \quad 0 \leqq i \leqq n.$$

Since I is an ideal in KG, we may assume without loss that $x_0=1$ and $k_0=1$. Choose $m\in\mathbf{N}$ such that $p^m\geqq 2^n-1$. Put $X=\langle x_1, \ldots, x_n\rangle$ and

$$Y = \langle y_1\rangle X\ldots X\langle y_n\rangle \quad \text{where each } \langle y_i\rangle \text{ is cyclic of order } p^{m+1}.$$

Arguing as above, we can find an embedding

$$\sigma\colon YWrX \to G \quad \text{with} \quad \sigma\restriction X = id\cdot$$

Therefore, we may assume without loss that $X\leqq YWrX\leqq G$. Put

$$z_i = [y_i, x_i] \quad \text{for} \quad 1 \leqq i \leqq n,$$

and observe that z_i is an element of order p^{m+1} in the base group Θ of $YWrX$.

By inverse induction over $s\in\{n, n-1, \ldots, 0\}$, we will now find elements

$$\alpha_s = \sum_{i=0}^{s} \alpha_{s,i}\cdot x_i^{-1}\in I$$

with

$$\alpha_{s,i}\in K\Theta \quad \text{and} \quad \alpha_{s,0}=(z_n-1)(z_{n-1}-1)\ldots(z_{s+1}-1).$$

To this end, we proceed as follows. Start with $\alpha_n = \alpha$; then $\alpha_{n,i} = k_i \in K$ and $\alpha_{n,0} = k_0 = 1$. Now suppose, that α_s has been found for some $s \geqq 1$. Put

$$\alpha_{s-1} = z_s \cdot \alpha_s - y_s^{-1} \cdot \alpha_s \cdot y_s \in I.$$

Since $K\Theta$ is commutative, we have

$$\alpha_{s-1} = \sum_{i=0}^{s} \alpha_{s,i} \cdot (z_s - [y_s, x_i]) \cdot x_i^{-1}.$$

Obviously,

$$\alpha_{s-1,i} = \alpha_{s,i} \cdot (z_s - [y_s, x_i]) \in K\Theta \quad \text{and}$$

$$\alpha_{s-1,0} = \alpha_{s,0} \cdot (z_s - 1) = (z_n - 1)(z_{n-1} - 1) \dots (z_s - 1).$$

Moreover, $\alpha_{s-1,s} = 0$ by the definition of z_s. Thus, the induction is completed, and we obtain

$$\alpha_0 = \alpha_{0,0} \cdot 1 = (z_n - 1)(z_{n-1} - 1) \dots (z_1 - 1) \in I.$$

Because of supp $z_i = \{1, x_i\}$, we have

$$\langle z_1, \dots, z_n \rangle = \langle z_1 \rangle \times \dots \times \langle z_n \rangle$$

where each $\langle z_i \rangle$ is cyclic of order p^{m+1}.

Put $J = \{1, \dots, n\}$. Define

$$u_S = \prod_{j \in S} z_j^{-1} \quad \text{for every } S \subseteqq J.$$

Then

$$\alpha_0 = \sum_{S \subseteqq J} \varepsilon_S \cdot u_S^{-1} \in I \quad \text{where} \quad \varepsilon_S = (-1)^{|J \setminus S|}.$$

Note, that $\langle u_S \rangle$ is cyclic of order p^{m+1} whenever $S \neq \emptyset$. Let

$$U = \langle u_S \mid S \subseteqq J \rangle = \langle z_1 \rangle \times \dots \times \langle z_n \rangle,$$

and define

$$V = \prod_{\substack{S \subseteqq J \\ S \neq \emptyset}} \langle v_S \rangle \quad \text{where each } \langle v_S \rangle \text{ is cyclic of order } p^{m+2}.$$

If $W = V Wr U$, then every

$$w_S = \Big[\prod_{r=0}^{p^{m+1}-2} v_S^{(u_S^r)}\Big]^{-1} \cdot \Big[v_S^{(u_S^{p^{m+1}-1})}\Big]^{p^{m+1}-1}, \quad \emptyset \neq S \subseteqq J,$$

is contained in the base group Ω of W.

Let T_S be a complement to $\langle u_S \rangle$ in U. Then

$$\langle w_S^t \mid t \in T_S \rangle = \prod_{t \in T_S} \langle w_S^t \rangle,$$

since supp $w_S^t \subseteqq \langle u_S \rangle \cdot t$. Further, $\langle w_S^t \mid t \in T_S \rangle \subseteqq \langle v_S^U \rangle$, while V is the direct product of the $\langle v_S \rangle$, $\emptyset \neq S \subseteqq J$. Hence,

$$\Omega_0 = \langle w_S^t \mid t \in T_S, \emptyset \neq S \subseteqq J \rangle = \prod_{\substack{S \subseteqq J \\ S \neq \emptyset}} \prod_{t \in T_S} \langle w_S^t \rangle.$$

Define

$$L = \langle (w_{S_1}^p)^{t_1} \cdot (w_{S_2}^{-p})^{t_2} \mid t_1 \in T_{S_1},\, t_2 \in T_{S_2},\, \emptyset \neq S_1, S_2 \subseteqq J \rangle.$$

Now, $p \cdot (p^{m+1}-1) \equiv -p \bmod p^{m+2}$, and therefore

$$w_S^p = \Big[\prod_{r=0}^{p^{m+1}-1} v_S^{(u_S^r)}\Big]^{-p}.$$

Thus, u_S centralizes w_S^p. And since $U = \langle u_S \rangle \times T_S$, the set $\{(w_S^p)^t \mid t \in T_S\}$ is U-invariant for every non-empty $S \subseteqq J$. Hence, L is a normal subgroup of W. Note, that Ω_0/L is the central product of the groups $\langle w_S^t \rangle$, $t \in T_S$, $\emptyset \neq S \subseteqq J$, with amalgamations $(w_{S_1}^p)^{t_1} = (w_{S_2}^p)^{t_2}$.

Let $\theta\colon W \to W/L$ be the canonical epimorphism. Because of $U \cap L \leqq U \cap \Omega = 1$, the restriction $\theta \upharpoonright U$ is an embedding. If

$$\Sigma\colon\ 1 = U_0\theta < U_1\theta < \ldots < U_l\theta = U\theta$$

is any chief series in $U\theta$, then the terms of Σ are exactly the intersections of $U\theta$ with the terms of the normal series

$$1 < \Omega/L < \Omega U_1/L < \ldots < \Omega U_l/L = W/L$$

in W/L. Therefore, [3, Theorem 2.1] ensures the existence of an $L\mathfrak{F}_p$-group $H \geqq G$ and an embedding $\tau\colon W/L \to H$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{id} & G \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle id} \\ W/L & \xrightarrow{\tau} & H \end{array}$$

commutes. Since G is e.c. in $L\mathfrak{F}_p$, there does already exist an embedding

$$\hat{\tau}\colon W/L \to G \quad \text{with} \quad (\theta\hat{\tau})\upharpoonright U = id.$$

Identifying U via θ with $U\theta$, and suppressing $\hat{\tau}$, we may assume without loss that $U = U\theta \leqq W/L \leqq G$. In the sequel, epimorphic images modulo L will be denoted by bars.

Let $\pi\colon \{0, \ldots, 2^n-1\} \to \{S \mid S \subseteqq J\}$ be a bijection with $0\pi = \emptyset$. By inverse induction over $s \in \{2^n-1, 2^n-2, \ldots, 0\}$, we will find now elements

$$\beta_s = \sum_{i=0}^{s} \beta_{s,i} \cdot \bar{u}_{i\pi}^{-1} \in I \quad \text{with}$$

$$\beta_{s,i} \in K\bar{\Omega} \quad \text{and} \quad \beta_{s,0} = \varepsilon_\emptyset \cdot \prod_{j=s+1}^{2^n-1} (\bar{w}_{j\pi} - 1).$$

To this end, we proceed as follows. Start with $\beta_{2^n-1} = \alpha_0$; then $\beta_{2^n-1,i} = \varepsilon_{i\pi} \in K$ and $\beta_{2^n-1,0} = \varepsilon_\emptyset$. Now suppose, that β_s has been found for some $s \geqq 1$. Put

$$\gamma_s = \prod_{r=0}^{p^{m+1}-2} \big[v_{s\pi}^{(u_{s\pi}^r)}\big]^{r+1} \in \Omega,$$

and let

$$\beta_{s-1} = \bar{w}_{s\pi} \cdot \beta_s - \bar{\gamma}_s^{-1} \cdot \beta_s \cdot \bar{\gamma}_s \in I.$$

Since $K\bar{\Omega}$ is commutative, we have

$$\beta_{s-1} = \sum_{i=0}^{s} \beta_{s,i} \cdot (\bar{w}_{s\pi} - [\bar{\gamma}_s, \bar{u}_{i\pi}]) \cdot \bar{u}_{i\pi}^{-1}.$$

Thus,

$$\beta_{s-1,i} = \beta_{s,i} \cdot (\bar{w}_{s\pi} - [\bar{\gamma}_s, \bar{u}_{i\pi}]) \in K\bar{\Omega}.$$

Further,

$$[\gamma_s, u_{s\pi}] = \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^r)}]^{-r-1} \cdot \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^{r+1})}]^{r+1} =$$

$$= \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^r)}]^{-r-1} \cdot \prod_{r=1}^{p^{m+1}-1} [v_{s\pi}^{(u_{s\pi}^r)}]^{r} =$$

$$= \prod_{r=0}^{p^{m+1}-2} [v_{s\pi}^{(u_{s\pi}^r)}]^{-1} \cdot [v_{s\pi}^{(u_{s\pi}^{p^{m+1}-1})}]^{p^{m+1}-1} = w_{s\pi}.$$

Therefore, $\beta_{s-1,s}=0$. Moreover, $u_{0\pi}=u_{\emptyset}=1$, and thus

$$\beta_{s-1,0} = \beta_{s,0} \cdot (\bar{w}_{s\pi} - 1) = \varepsilon_{\emptyset} \cdot \prod_{j=s}^{2^n-1} (\bar{w}_{j\pi} - 1).$$

This completes the induction, and we obtain

$$\delta = \varepsilon_{\emptyset} \cdot \beta_0 = \varepsilon_{\emptyset} \cdot \beta_{0,0} = \prod_{\substack{S \subseteq J \\ S \neq \emptyset}} (\bar{w}_S - 1) \in I.$$

From the definition of L, we have $\bar{w}_{S_1}^p = \bar{w}_{S_2}^p$ for any two nonempty subsets S_1, S_2 of J. Thus, there exists $h \in G$ with $h = \bar{w}_S^p$ for $\emptyset \neq S \subseteq J$. Because of Char $K = p$, it follows that

$$\delta^p = \prod_{\substack{S \subseteq J \\ S \neq \emptyset}} (\bar{w}_S^p - 1) = (h-1)^{2^n-1} \in I.$$

Since $p^m \geqq 2^n - 1$, successive multiplication with $(h-1)$ eventually yields

$$(h-1)^{p^m} = (h^{p^m} - 1) \in I.$$

But h is an element of order p^{m+1} in $\Omega_0/L \leqq G$. Therefore, $N = \{g \in G | (g-1) \in I\}$ is a non-trivial normal subgroup of G with $\omega(KN) \cdot KG \subseteq I$. □

Finally note, that the normal subgroups of every e.c. $L\mathfrak{F}_p$-group are totally ordered by inclusion [3, Theorem 2.3], and that the embeddings of finite p-groups into e.c. $L\mathfrak{F}_p$-groups are quite well understood (see [3, § 3]). In particular, in the proof of part (a) of the Theorem, if $f \in G$ is any fixed element with $\langle f^G \rangle \cap X = 1$, then a combination of [3, Theorem 3.1] and [4, Theorem 3.3] yields that the embeddings σ and $\hat{\tau}$ can be so chosen that, in addition,

$$\langle f^G \rangle \cap \operatorname{Im} \sigma = 1 \quad \text{and} \quad \langle f^G \rangle \cap \operatorname{Im} \hat{\tau} = 1.$$

But then, $f \in \langle f^G \rangle \leqq \langle h^G \rangle \leqq N = \{g \in G | (g-1) \in I\}$. This shows, that the following stronger version of part (a) of the Theorem holds.

Proposition. *Let K be any field, and let G be an e.c. $L\mathfrak{F}_p$-group. Assume that I is a proper ideal of KG, and that*

$$0 \neq \alpha = \sum_{i=0}^{n} k_i x_i \in I \quad \text{with} \quad 0 \neq k_i \in K$$

and pairwise distinct elements $1 = x_0, x_1, \ldots, x_n \in G$. *If* $X = \langle x_1, \ldots, x_n \rangle$, *and if* $\tilde{M}/\tilde{N}$ *is the (unique) chief factor in G with* $1 = \tilde{N} \cap X \subsetneqq \tilde{M} \cap X$, *then* $\tilde{N} \leqq N = \{g \in G \mid (g-1) \in I\} \trianglelefteq G$ *and* $\omega(K\tilde{N}) \cdot KG \subseteq I$.

However, in the above situation, it remains open whether for every proper ideal I of KG there exists $N \ntrianglelefteq G$ such that

$$I \cap \omega(KG) = \omega(KN) \cdot KG,$$

i.e., whether the $\omega(KN) \cdot KG$, $N \trianglelefteq G$, are the only ideals of $\omega(KG)$.

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