# On metabelian groups of exponent eight

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## 1. Introduction

In an earlier paper [1], we determined the nilpotency class of the 2-generator free metabolian group of exponent $p^e$, for prime $p$ and a positive integer $e$. Here we shall show that the $n$-generator free metabelian group of exponent 8 has class 12 when $n=2$ or 3, and class $3n+1$ when $n\geqq 4$.

The main difficulty is to show that the class is 12 when $n=3$. This is achieved by giving a detailed description of the laws and lower central factors of the 2-generator group. These could be derived from the calculations of HERMANNS [3], but we have found the alternative treatment below more expedient. From the laws, we deduce sufficient information about the 3-generator group to determine its class. Using a version of the nilpotent quotient computer algorithm adapted to the metabelian case, NEWMAN [5] has also obtained, among other interesting results, a complete description of the 3- and 4-generator free metabelian groups of exponent 8: in particular, the former has class 12 and order $2^{450}$, while the latter has class 13 and order $2^{2232}$.

Both LIEBECK [4] and NEWMAN [5] have given examples showing that, for $n\geqq 2$, there is an $n$-generator metabelian group of exponent 8, with nilpotency class at least $3n+1$. We show that $3n+1$ is also an upper bound for the class.

Many of the ideas used here apply, mutatis mutandis, to metabelian groups of exponent $2^n$, and, with slight modifications, to metabelian groups of exponenet $p^3$. We shall report on these later.

## 2. The 2-generator group

*From now on, all groups G considered will be metabelian of exponent 8.* In particular, if $x$ is a commutator, then it commutes with its conjugates, and $x^{ab}=x^{ba}$ for $a, b\in G$. For $x\in G'$ and $a\in G$, the identities $(xa^{-1})^8=a^8=1$ imply that

$$x^{(a+1)(a^2+1)(a^4+1)} = 1.$$

Replacing a by $ab^{-1}$, we deduce that

(C.1) $$x^{(a+b)(a^2+b^2)(a^4+b^4)} = 1,$$

(C.2) $$x^{2(a^2+b^2)(a^4+b^4)} = 1,$$

(C.3) $$x^{4(a^4+b^4)} = 1,$$

for $x \in G'$ and $a, b \in G$. Using the equations $(ab^{-1})^2 = a^2[a, b]b^{-2}$, $(ab^{-1})^4 = \{(ab^{-1})^2\}^2$, $(ab^{-1})^8 = \{(ab^{-1})^4\}^2$ one can also show that

$$(2.1) \qquad [a, b]^{(a^2+b^2)(a^4+b^4)}[a^2, b^2]^{a^4+b^4}[a^4, b^4] = 1,$$

$$(2.2) \qquad \{[a, b^2]^{a^2+b^4}[a^2, b^4]\}^{a^4+1} = 1,$$

$$(2.3) \qquad [a, b^4]^{(a^2+1)(a^4+1)} = 1.$$

Replacing a by $b^{-1}$ a in (2.3), we see that

$$[a, b^4]^{(a^2+b^2)(a^4+b^4)} = 1,$$

and we deduce, using (C.1) and (C.2), that

$$(2.4) \qquad \Gamma_5\langle a, b\rangle^{(a^2+b^2)(a^4+b^4)} = 1.$$

Similarly, replacing a by $b^{-2}$ a in (2.3), we get

$$(2.5) \qquad [a, b^4]^{(a^2+b^4)(a^4+1)} = 1.$$

Commutating (2.2) and $b^2$, and using (2.5) and (C.2), we can show that

$$(2.6) \qquad [a^2, b^4, b^2]^{a^4+1} = 1.$$

Conjugating (2.1) by $1+b^4$, and using (2.4) and (C.2), we find that

$$[a^2, b^2]^{(1+b^4)(a^4+1)} = 1.$$

Combining this with (2.6), we deduce that

$$(2.7) \qquad [a^2, b^2]^{2(a^4+1)} = 1$$

Also (2.3) and (2.5) yield the equations

$$(2.8) \qquad [a, b^4, b^4]^{a^4+1} = [a, b^4]^{2(a^4+1)} = 1.$$

Now (2.7) and (2.8) show that

$$(2.9) \qquad \Gamma_3\langle a, b^2\rangle^{2(a^4+1)} = 1.$$

while (2.3), (2.6) and (2.8) imply that

$$(2.10) \qquad \Gamma_5\langle a, b^2\rangle^{a^4+1} = 1.$$

From (2.9) and (2.10), we conclude that

$$(2.11) \qquad \Gamma_5\langle a, b\rangle^{2(a^4+1)} = \Gamma_9\langle a, b\rangle^{(a^4+1)} = 1.$$

Replacing a by ab in (2.11), it is not difficult to establish the identities

$$(2.12) \qquad \Gamma_5\langle a, b\rangle^4 = \Gamma_9\langle a, b\rangle^2 = \Gamma_{13}\langle a, b\rangle = 1.$$

In an earlier paper [1], we have constructed an example showing that the 2-generator free metabelian group of exponent 8 has nilpotency class not less than 12, so (2.12) implies that its class is exactly 12. In order to produce a set of defining relations, we now establish some further structural identities.

Writing (2.2) in the form

(i) $[a, b^2]^{(a^2+1)(a^4+1)}[a, b^4]^{(a+1)(a^4+1)}[a, b^2, b^4]^{(a^4+1)} = 1.$

and replacing a by $b^{-1}$ a, conjugating by $b^6$, and using (2.8), we get the equation

(ii) $[a, b^2]^{(a^2+b^2)(a^4+b^4)}[a, b^4]^{(a+b)(a^4+b^4)b}[a, b^2, b^4]^{(a^4+b^4)} = 1.$

Since

$$1 = [a, b^8] = [a, b^4]^{(1+b^4)} = [a, b^2]^{(1+b^2)(1+b^4)}$$

we see that (ii) is equivalent to the relation

$$[a, b^2, a^2]^{(a^4+b^4)}[a, b^2, a^4]^{(1+b^2)}[a, b^4, a^4]^{(a+b)b}[a, b^4, a^4]^{(b^2-1)} = 1,$$

and, using (2.12) we can deduce that

$$(2.13) \qquad [a, b^2, a^2]^{(a^4+b^4)}[a, b^4, a^4]^{ab} = 1.$$

Similarly, substituting from (i) in (ii), we can derive the identity

$$(2.14) \qquad [a, b^2, b^2]^{(a^4+b^4)}[a, b^2, b^4, a^2][a^2, b^4]^2[a, b^4, a^4, b]^b[a^2, b^4, a^4, b] = 1.$$

In particular, (2.13) and (2.14) imply that

$$\Gamma_5\langle a, b\rangle^2 = 1 \bmod \Gamma_9\langle a, b\rangle.$$

Further, using (2.4) and (2.12), we can show from (ii) that

$$(2.15) \quad [a, b^4, a^4, a, a]^{(a+b)} = [a, b^4, a^4, a, b]^{(a+b)} = [a, b^4, a^4, b, b]^{(a+b)} = 1.$$

Replacing a by $[a, b]$ in (2.14), we get

$$[a, b, b^2, b^2]^{(1+b^4)} = 1,$$

while (2.7) yields

$$[a, b^2, b^2, a]^{(1+b^4)} = [a, b^4, a]^{(1+b^4)} = 1$$

Using these facts, and commutating (2.14) by $b$ and by $a$, we obtain the relations

$$(2.16) \qquad [a, b^2, b, b^2, a^4][a, b^2, b, b^4, a^2][a, b^4, b, b^4, a][a, b^4, b, b, a^4] = 1$$

$$(2.17) \qquad [a, b^2, a, b^2, a^4][a, b^2, a, b^4, a^2][a, b^4, a, b^4, a][a, b^4, a, b, a^4] = 1.$$

Finally, if $\alpha, \beta$ denote commutation by $a, b$ respectively, then the equations (2.13) and (2.14) can be written succinctly and summetrically using additive notation:

$$(2.18) \qquad \Gamma_5[2+(\alpha^4+\alpha^2\beta^2+\beta^4)+(\alpha^4\beta+\alpha\beta^4)+(\alpha^6+\alpha^3\beta^3+\beta^6)] = 0.$$

Similarly, (2.16) and (2.17) combine to give

$$(2.19) \qquad \Gamma_4[(\alpha^2\beta^4+\alpha^4\beta^2)+(\alpha\beta^6+\alpha^6\beta)] = 0.$$

while (2.15) yields

$$(2.20) \qquad \Gamma_9[\alpha\beta^2+\alpha^2\beta] = 0.$$

Moreover, (2.1) becomes

$$(2.21) \quad \Gamma_2[4+2(\alpha^2+\alpha\beta+\beta^2)+(\alpha^6+\alpha^5\beta+\alpha^4\beta^2+\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5+\beta^k)] = 0.$$

These 4 relations, together with the equations $a^8=b^8=1$ and $\Gamma_{13}\langle a, b\rangle=1$, define the free metabelian group of exponent 8 with 2 generators $a$ and $b$. It is now a relatively easy matter to compute the orders of the lower central factors. In particular $|\Gamma_{13}|=2^3$ and $|\Gamma_{11}/\Gamma_{12}|=|\Gamma_{10}/\Gamma_{11}|=2^6$. Furthermore, the group itself has order $2^{63}$.

### 3. The 3-generator group

Let $G=\langle a, b, c\rangle$ be the 3-generator free metabelian group of exponent 8. We prove now that the identities (2.12) are also valid for $G$. We may assume that $\Gamma_{14}(G)=1$. We begin by showing that many commutators of weight 10 in $G$ are congruent to each other modulo $\Gamma_{11}$.

Raplacing a by $ac^2$ in (2.2), we obtain the congruence

$$[c^2, b^2]^{(a^2+1)(a^4+1)} \equiv 1 \bmod \Gamma_{11}(G). \tag{3.1}$$

Similarly, replacing a by ax in (2.2), where $x\in G'$, we find that

$$[x, b^2]^{(a^2+1)(a^4+1)} \equiv 1 \bmod \Gamma_{11}(G) \tag{3.2}$$

while, commutating (2.2) by $c$, we see that

$$[a, b^2, c]^{(a^2+1)(a^4+1)} \equiv 1 \bmod \Gamma_{11}(G). \tag{3.3}$$

We may, of course, interchange $b$ and $c$ in (3.2) and (3.3), and replace $c$ by $cb$ in (3.1). From these calculations, together with (C.1) and (C.2), it follows that

$$\Gamma_4(G)^{(a^2+1)(a^4+1)} \equiv 1 \bmod \Gamma_{11}(G) \tag{3.4}$$

In (3.4), we replace a by $ab$, and combine the result with (3.4), to obtain

$$\Gamma_4(G)^{(1-b^2)+a^2(1-b^4)+a^4(1-b^6)} \equiv 1 \bmod \Gamma_{11}(G). \tag{3.5}$$

A similar procedure, replacing a by $ac$ in (3.5), shows that

$$[y, c^2, b^4] = [y, c^4, b^2] \bmod \Gamma_{11}(G) \tag{3.6}$$

for $y\in\Gamma_4(G)$. Clearly, we may replace $c$ by $b$ or a in (3.6).

If $x\in\Gamma_2(G)$, then by substituting $ax$ for a in (2.14), we find that

$$[x, b^2, b^2, a^4][x, b^2, b^4, a^2][x, b^4, a]^2[x, b^4, a^4, b]^b[x, b^4, a^4, b, a] = 1. \tag{3.7}$$

Replacing a by $ac^2$ in (3.7) we can show that

$$[x, b^4, b^4, a, c^2][x, b^4, a^4, b, c^2] = 1 \tag{3.8}$$

for $x\in G'$. The effect of (3.6) and (3.8) is that many of the basic commutators in $\Gamma_{13}(G)$ are equal. It remains to show that they are all trivial. We shall denote by $[i, j, k]$ a commutator of the form

$$[x, \underbrace{y, y, \ldots, y}_{j}, z, \underbrace{z, \ldots, z}_{k}, x, \underbrace{x, \ldots, x}_{i-1}]$$

in which the first entry appears i times, the second $j$ times, and the third $k$ times. We have to show that $[i, j, k]=1$ in $G$ when $i+j+k=13$.

In the group $\langle b, c\rangle$, the commutator [2, 11], with 2 $b$'s and 11 $c$'s, is trivial. Replacing $b$ by $ab$, and using the Jacobi identity, we can deduce that $[1, 1, 11]=1$ in $G=\langle a, b, c\rangle$. From the equalities

$$[2, 10] = [3, 9] = \dots = [10, 2]$$

in $\langle b, c\rangle$, on replacing $b$ by $[a, b]\, b$ and by $[a, c]b$, we find that

$$[1, 2, 10] = [1, 3, 9] = \dots = [1, 11, 1]. \tag{3.9}$$

Writing the equation $[2, 10]=[4, 8]$ in the form

$$[b^2, {}_{10}c] = [b^2, {}_8c, b^2],$$

and replacing $b$ by $ba^{-2}$, so that $b^2$ becomes $b^2[b, a^2]a^4$, we see that $[b, a^2, {}_{10}c]= =[b, a^2, {}_8c, b^2]$, whence

$$[1, 2, 10] = [3, 2, 8]. \tag{3.10}$$

Replacing $b$ by $ba$, we conclude that $[b, a^2, {}_8c, a^2]=1$ or equivalently

$$[1, 4, 8] = 1. \tag{3.11}$$

Therefore all the commutators in (3.9) are trivial. Since commutators of weight 13 are all involutions, the relations $[1, 2, 10]=[10, 1, 2]=1$, together with the Jacobi identity, imply that

$$[2, 10, 1] = 1.$$

From (3.10), we get $[3, 2, 8]=1$. Taking $x=[c, a]$ in (3.8), we find that $[3, 2, 8]\cdot \cdot[3, 5, 5]=1$, so

$$[3, 5, 5] = 1 \tag{3.12}$$

Further, taking $x=[a, b]$ in (3.8), we see that $[2, 9, 2][5, 6, 2]=1$. But (3.6) implies that $[2, 9, 2]=[2, 3, 8]=1$, and consequently

$$[5, 6, 2] = 1. \tag{3.13}$$

It is now easy to check, using (3.6), (3.11), (3.12) and (3.13) that $[i, j, k]=1$ whenever $i+j+k=13$ and so $\Gamma_{13}\langle a, b, c\rangle=1$. The identities $\Gamma_9^2\langle a, b, c\rangle=\Gamma_5^4\langle a, b, c\rangle=1$ follow immediately.

## 4. The $n$-generator group for $n\geqq 4$

Let $G$ be a metabelian group of exponent 8. As we have seen, for $x\in\Gamma_2(G)$, the equations $(xa^{-1})=a^8=1$ imply that

$$x^{1+a+a^2+\cdots+a^7} = 1.$$

Combining this relation with the equation obtained by substituting $ab$ for $a$, we can show that

$$[x, b][x, b^2]^a[x, b^3]^{a^2}[x, b^4]^{a^3}[x, b^5]^{a^4}[x, b^6]^{a^5}[x, b^7]^{a^6} = 1.$$

Using a similar argument, substituting $ac$ for $a$, we find that

$$[x, b^2, c][x, b^3, c^2]^a[x, b^4, c^3]^{a^2}[x, b^5, c^4]^{a^2}[x, b^6, c^5]^{a^4}[x, b^7, c^6]^{a^5} = 1.$$

Contibuing this process as in [2, Lemma 1], we conclude that

$$[x, b^7, c^6, d^5, e^4, f^3, g^2, h] = 1$$

whence

(4.1) $$[x, b, c^2, d, e^4, f, g^2, h] = 1$$

for $x \in \Gamma_2(G)$ and $b, c, \ldots, h \in G$. Replacing $a, b$ by $cy, g$ in (2.13), where $y \in \Gamma_2(G)$ we see that

$$[y, c^2, g^2]^{(c^4+g^4)}[y, c^4, g^4]^{cg} = 1.$$

Taking $y=[x, b, d, f, h]$, where $x \in \Gamma_2(G)$, and using (4.1) we deduce that

(4.2) $$[x, b, c^2, d, f, g^2, h]^2 = 1.$$

Also, replacing $b$ by $ba^{-1}$ in the relation $[b^2, c, d^2]^4=1$, we find that

(4.3) $$[a, b, c, d^2]^4 = 1.$$

Using (4.2) and (4.3), we obtain the identities

(4.4) $$[x, b, c, c, d, f, g, g, h]^2 = [x, b, c, c, d, e, e, e, e, f, g, g, h] = 1$$

for $x \in \Gamma_2(G)$ and $b, c, \ldots, h \in G$.

Now it can be seen [2, Lemma 2] that for a fixed number $n$ of generators, if a basic simple commutator is of weight $3n+2$, then one entry must appear four times, and in addition two entries (which maybe equal to each other or to the previous entry) must appear twice each. It follows that (4.4), together with the Jacobi identity, gives $3n+1$ as an upper bound for the nilpotency class of $G$. The examples of LIEBECK [4] and NEWMAN [5] show that $3n+1$ is also a lower bound.

Finally, we remark that our identity (4.3) is not the best possible. Newman [5] has found the relation $[b^2, c, d, e]^4=1$ in the 4-generator group, and has deduced, replacing $b$ by $ba^{-1}$, that

(4.5) $$[a, b, c, d, e]^4 = 1.$$

Thus $\Gamma_5^4(G)=1$ for every metabelian group $G$ of exponent 8, regardless of the number of generators.

## References

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