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Orthogonally additive mappings on free inner product Z-modules

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Abstract. It is known that, if X is a real inner product space of dimension at least 2 and Y an abelian group, every solution of the conditional Cauchy functional equation $(*)$ (see below) is additive if it is odd and is quadratic if it is even. In this paper the solutions of $(*)$ are determined if X is a special free inner product Z - module. If $\dim_Z X = 2$, Theorem 15 expresses a serious deviation from the situation in the inner product space case while Theorems 13 and 17 show that for $\dim_Z X$ sufficiently large, we have analogies to that case.

1. Introduction

If the sets X and Y are furnished with a binary operation $+$ and X furthermore with a binary relation $\perp$, called orthogonality, then a mapping $f : X \to Y$ is said to be *orthogonally additive* if it satisfies the conditional Cauchy functional equation

$$(*) \qquad f(x+z) = f(x) + f(z) \quad \text{for all } x, y \in X \text{ with } x \perp z.$$

In recent applications, conditional functional equations generally play an increasingly important role.

Orthogonal additivity has one of its roots in inner product spaces, when $\perp$ stems from an inner product; for a brief survey, we refer to the

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second half of the paper [7], where also the orthogonal additivity in the Blaschke–Birkhoff–James sense over normed spaces is mentioned. Several papers, the first being [8], treated orthogonal additivity under regularity conditions. By a complete change of the methods of proof, all the regularity conditions could be avoided a priori, and the following theorem was obtained:

Theorem 1. *If* $(X, \langle \cdot , \cdot \rangle)$ *is a real inner product space with* $\dim_R X \geq 2$, *if* $x \perp z$ *is defined by* $\langle x, z\rangle = 0$ $(x, z \in X)$, *and if* Y *is an abelian group, then* $f : X \to Y$ *is a solution of* $(*)$ *if and only if there exist additive mappings* $l : R \to Y$ *and* $h : X \to Y$ *such that* $f(x) = l(\|x\|^2) + h(x)$ $(\forall x \in X)$ ([10], p. 43, Corollary 10; [14], Theorem 1; for more general versions [11], p. 242, 246).

This result follows from theorems which separately treat the even and the odd case in the framework of an axiomatic theory of orthogonality spaces ([10], p. 38/39, Theorem 5 and 6; p. 41, Corollary 7). Roughly speaking, here and in important other situations, the general even solution of $(*)$ is quadratic and its general odd solution is additive. Pinsker's [8] and other regularity results then follow as corollaries from Theorem 1 ([10], p. 43–46). The following statement is also a consequence of Theorem 1; it extends the already long list of characterizations of Hilbert spaces among inner product spaces:

Corollary 2. *For a real inner product space* $(X, \langle \cdot , \cdot \rangle)$ *with* $\dim_R X \geq 2$, *the following are equivalent:*

(i) *Every orthogonally additive* $f : X \to R$ *which is bounded below attains a minimum.*

(ii) X *is a Hilbert space.* ([10], p. 46, Corollary 15).

Further applications of Theorem 1 are the Boltzmann–Gronwall Theorem in gas dynamics ([1], p. 191–194) and a premium calculation principle in actuarial science ([5], section 3).

One of the actual objectives in the theory of the functional equation $(*)$ is its investigation beyond the general theory developed in [10], [11], [12], [13] on vector spaces. It is the purpose of this paper to present results about orthogonal additivity on a special class of free Z-modules (cf. [6] for a related problem) and to compare them with those of the vector space case.

2. Notation and preliminaries

Throughout the paper, N, N^0, Z, R denote the sets of positive integers, nonnegative integers, integers, real numbers, respectively. We use 0 for the identity element of the groups $(X,+)$ and $(Y,+)$ as well as for the integer zero; it will always be clear from the context what is meant. $\underline{c}$ is the symbol for the constant mapping with value c, and $:=$ means that the right hand side defines the left hand side. Finally, $=_{(\dots)}=$ is used for quoting the earlier result $(\dots)$.

Remark 3. A free Z-module X is, up to isomorphy, a direct sum $Z^{(J)} := \oplus_{j\in J} X_j$ where $X_j = Z$ for all $j \in J$ and where J is an appropriate index set. The elements $e_j := (\delta_{jk})_{k\in J}$ $(j \in J)$ (Kronecker symbols) constitute a basis of the free Z-module $Z^{(J)}$, the so-called canonical basis. Since Z is a commutative ring $\neq \{0\}$, every free Z-module X has a well-defined dimension $\dim_Z X$. Of course, $\dim_Z Z^{(J)} = \operatorname{card} J$. ([3], p. 41, 42, 150, 151).

In the notation $(b_j)_{j\in J}$ for a basis of a free Z-module, we always assume $b_j \neq b_k$ for $j, k \in J$, $j \neq k$.

Definition 4. If $(b_j)_{j\in J}$ is a basis of a free Z-module X, then $\langle\,\cdot\,,\cdot\,\rangle : X \times X \to Z$ defined by $\langle x, z\rangle := \sum_{j\in J} p_j q_j$ $(x = \sum_{j\in J} p_j b_j \in X,\ z = \sum_{j\in J} q_j b_j \in X)$ is called *the standard inner product on X associated with $(b_j)_{j\in J}$*. (Notice that all sums over J with running index j automatically contain only a finite number of nonzero summands). We briefly write $\sum$ for $\sum_{j\in J}$. $(X, (b_j)_{j\in J}, \langle\,\cdot\,,\cdot\,\rangle)$, sometimes more briefly denoted by X, is then said to be a *free standard inner product Z-module* (FSIP Z-module).

(0) $$\text{For } x, z \in X, \text{ we define } x \perp z :\Longleftrightarrow \langle x, z\rangle = 0.$$

Lemma 5. *If $(X, (b_j)_{j\in J}, \langle\,\cdot\,,\cdot\,\rangle)$ is a FSIP Z-module, then we have:*

a) $\langle\,\cdot\,,\cdot\,\rangle$ *is Z-bilinear, symmetric, and positive definite.*

b) $\langle b_j, b_k\rangle = \delta_{jk}$ $(\forall j, k \in J)$, *i.e., $(b_j)_{j\in J}$ is an orthonormal basis.*

c) $x, z \in X;\ x \perp z;\ p, q \in Z \implies px \perp qz$.

d) $j, k \in J;\ p, q \in Z \implies \langle p(b_j + b_k), q(b_j - b_k)\rangle = 0$.

The routine proof is omitted.

Definition 6. a) If X is a FSIP Z-module and $(Y,+)$ an abelian group, a mapping $f : X \to Y$ is called orthogonally additive if

$$(*) \qquad f(x+z) = f(x) + f(z) \quad \text{for all } x, z \in X \text{ with } x \perp z$$

holds.

$\mathrm{Hom}_\perp(X,Y)$ denotes the set of all solutions f of $(*)$,

$(\mathrm{e})\,\mathrm{Hom}_\perp(X,Y) := \{g \in \mathrm{Hom}_\perp(X,Y);\ g \text{ even }\}$,

$(\mathrm{o})\,\mathrm{Hom}_\perp(X,Y) := \{h \in \mathrm{Hom}_\perp(X,Y);\ h \text{ odd }\}$.

b) $\mathrm{Hom}(X,Y) := \mathrm{Hom}_Z(X,Y) = \{f : X \to Y; f(x+z) = f(x) + f(z)(x, z \in X)\}$ is the set of all *additive mappings* $f : X \to Y$.

c) $\mathrm{Quad}(X,Y) := \{f : X \to Y;\ f \text{ satisfies } (JvN)\}$ where (JvN): $f(x+z) + f(x-z) = 2f(x) + 2f(z)(\forall x, z \in X)$ is the set of all *quadratic mappings* $f : X \to Y$.

Definition 7. We say that the abelian group $(Y,+)$ is *uniquely 2-divisible* if the mapping $\omega : Y \to Y$, $\omega(y) := 2y\,(\forall y \in Y)$ is bijective. Then both ω and ω^{-1} are automorphisms of $(Y,+)$, and we write $\frac{1}{2}y$ for $\omega^{-1}(y)$.

Lemma 8. *For a FSIP Z-module X and an abelian group $(Y,+)$ we have:*

a) $\mathrm{Hom}(X,Y) \subset (\mathrm{o})\,\mathrm{Hom}_\perp(X,Y) \subset \mathrm{Hom}_\perp(X,Y) \subset \{f : X \to Y; f(0) = 0\}$.

b) $f \in \mathrm{Hom}_\perp(X,Y)$; $\tilde{f}(x) := f(-x)(\forall x \in X) \implies \tilde{f} \in \mathrm{Hom}_\perp(X,Y)$.

c) $f, g \in \mathrm{Hom}_\perp(X,Y) \implies f+g,\ f-g \in \mathrm{Hom}_\perp(X,Y)$.

d) $(Y,+)$ *uniquely* 2*-divisible,* $f \in \mathrm{Hom}_\perp(X,Y)$, $g(x) := \frac{1}{2}[f(x)+f(-x)]$, $h(x) := \frac{1}{2}[f(x) - f(-x)](\forall x \in X) \implies g \in (\mathrm{e})\,\mathrm{Hom}_\perp(X,Y)$, $h \in (\mathrm{o})\,\mathrm{Hom}_\perp(X,Y)$, $f = g + h$.

Proof. a) The first two inclusions are evident. By (0), $0 \perp 0$, so $f(0) = f(0+0) = f(0)+f(0)$, i.e., $f(0) = 0$. — b) If $x, z \in X$, $x \perp z$, then by (0) $(-x) \perp (-z)$, and then $\tilde{f}(x+z) = f(-x-z) = f(-x) + f(-z) = \tilde{f}(x) + \tilde{f}(z)$. — c) is straightforward, and d) follows from b) and c).

3. Main results

Throughout this section we suppose that $(X, (b_j)_{j\in J}, \langle\cdot,\cdot\rangle)$ be a FSIP Z-module and $(Y,+)$ an abelian group.

Remark 9. We first separate the case $\dim_Z X \le 1$ from the rest of the theory.

a) If $\dim_Z X = 0$, then by Lemma 8a) $\mathrm{Hom}_\perp(X,Y) = \{\underline{0}\}$.

b) If $\dim_Z X = 1$, $\{b\}$ a basis of X, then we have for $x, z \in X$, $x = pb$, $z = qb$ by Definition 4 $\langle x, z\rangle = pq$, therefore by (0)

$$x, z \in X \implies [x \perp z \Longleftrightarrow x = 0 \text{ and/or } z = 0]. \tag{1}$$

Let be $f : X \to Y$, $f(0) = 0$, and $x, z \in X$, $x \perp z$. By (1) $x = 0$ and/or $z = 0$, say $z = 0$. $f(x+z) = f(x) = f(x) + f(0) = f(x) + f(z)$. So $f \in \mathrm{Hom}_\perp(X,Y)$, and together with Lemma 8a) we get

$$\mathrm{Hom}_\perp(X,Y) = \{f : X \to Y; f(0) = 0\}. \tag{2}$$

Hence in the case $\dim_Z X \le 1$, the determination of $\mathrm{Hom}_\perp(X,Y)$ is completely settled.

Example 10. Let be $\dim_Z X = 1, \{b\}$ a basis of X, $Y \ne \{0\}$, $a \in Y\backslash\{0\}$. Define $h : X \to Y$ by

$$h(pb) = \begin{cases} a & (p > 0) \\ 0 & (p = 0) \\ -a & (p < 0) \end{cases} \qquad (p \in Z).$$

Then $h(2b) = a$, $h(b) = a$, $2h(b) = 2a$, so $h(2b) \ne 2h(b)$. This shows that $h \notin \mathrm{Hom}(X,Y)$. But by (2), $h \in \mathrm{Hom}_\perp(X,Y)$ and h is odd, i.e.,

$$\text{(o)}\, \mathrm{Hom}_\perp(X,Y) \not\subset \mathrm{Hom}(X,Y),$$

which means that the first inclusion in Lemma 8a) may be strict.

Example 11. Let be $\dim_Z X = 1, \{b\}$ a basis of X, and assume that there exists $a \in Y$ such that $a \ne 4a$. Define $g : X \to Y$ by

$$g(pb) = \begin{cases} a & (p \in Z\backslash\{0\}) \\ 0 & (p = 0) \end{cases}.$$

Then $g(2b) = a$, $g(b) = a$, $4g(b) = 4a$, so $g(2b) \neq 4g(b)$. Assume that $g \in \mathrm{Quad}(X,Y)$. Then put $x = z = b$ in (JvN) to obtain $g(2b) + g(0) = 2g(b) + 2g(b)$, i.e., $g(2b) = 4g(b)$, contradiction. So $g \notin \mathrm{Quad}(X,Y)$, but by (2) $g \in \mathrm{Hom}_\perp(X,Y)$, and g is even, therefore

$$\text{(e)}\,\mathrm{Hom}_\perp(X,Y) \not\subset \mathrm{Quad}(X,Y).$$

We now turn to the case $\dim_Z X \geq 2$ where we shall find situations contrasting with those in Examples 10 and 11.

Lemma 12. *If $f, g \in \mathrm{Hom}_\perp(X,Y)$, then we have:*

$$f(x) = \sum f(p_j b_j) \quad \text{for } x = \sum p_j b_j \in X. \tag{3}$$

$$g \text{ even}; \ \ j,k \in J,\ j \neq k; \tag{4}$$
$$p \in Z \text{ even} \implies g(pb_j) = 2g\left(\frac{p}{2}b_j\right) + 2g\left(\frac{p}{2}b_k\right).$$

Proof. (3): By Lemma 8a), $f(0) = 0$, so a zero summand $p_j b_j$ of $\sum p_j b_j$ produces a zero summand of $\sum f(p_j b_j)$. (3) is a matter of finite sums and is established by induction on the number of nonzero summands starting from Lemma 5a), b), c) and from $(*)$. — (4): $g(pb_j) = g[\frac{p}{2}(b_j + b_k) + \frac{p}{2}(b_j - b_k)] =_{\mathrm{L.5d}}= g[\frac{p}{2}(b_j + b_k)] + g[\frac{p}{2}(b_j - b_k)] =_{(3)}= g\left(\frac{p}{2}b_j\right) + g\left(\frac{p}{2}b_k\right) + g\left(\frac{p}{2}b_j\right) + g\left(\frac{p}{2}b_k\right) = 2g\left(\frac{p}{2}b_j\right) + 2g\left(\frac{p}{2}b_k\right)$.

Theorem 13. *If $\dim_Z X \geq 2$, then* $\text{(o)}\,\mathrm{Hom}_\perp(X,Y) = \mathrm{Hom}(X,Y)$.

Proof. i) $\mathrm{Hom}(X,Y) \subset \text{(o)}\,\mathrm{Hom}_\perp(X,Y)$ follows from Lemma 8a). ii) Assume that $h \in \text{(o)}\,\mathrm{Hom}_\perp(X,Y)$. By partially using a method in [4], p. 4.74/4.75, we show that

$$(H,j,p):\ \ h(pb_j) = ph(b_j) \qquad \text{holds for all } j \in J \text{ and all } p \in Z. \tag{5}$$

$p = 0$: By Lemma 8a) $h(0) = 0$, so $(H,j,0)$ holds for all $j \in J$.
$p = 1$: $(H,j,1)$ trivially holds for all $j \in J$.

Let be $n \in N$, $n \geq 2$, and assume that (H,j,p) holds for all $p \in N^0$ such that $p \leq n-1$ and for all $j \in J$. Let be $j \in J$ arbitrary and choose

$k \in J\backslash\{j\}$ arbitrary (notice that $\dim_Z X \geq 2$). Then we get

$$
\begin{aligned}
& h[nb_j + (n-2)b_k] = h[(n-1)(b_j + b_k) + (b_j - b_k)] \\
(6) \qquad & =_{\text{L.5d}}= h[(n-1)(b_j + b_k)] + h(b_j - b_k) =_{(3)} \\
& = h[(n-1)b_j] + h[(n-1)b_k] + h(b_j) - h(b_k) \\
& =_{(H,j,n-1),(H,k,n-1)}= (n-1)h(b_j) + (n-1)h(b_k) + h(b_j) - h(b_k) \\
& = nh(b_j) + (n-2)h(b_k).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(7) \qquad & h[nb_j + (n-2)b_k] =_{(3)}= h(nb_j) + h[(n-2)b_k] =_{(H,k,n-2)} \\
& = h(nb_j) + (n-2)h(b_k).
\end{aligned}
$$

From (6) and (7) we obtain $h(nb_j) = nh(b_j)$. Analogously, with j, k interchanged, $h(nb_k) = nh(b_k)$, so (H, j, n) holds for all $n \in N^0$ and all $j \in J$, i.e.,

$$
(8) \qquad h(nb_j) = nh(b_j) (\forall n \in N^0, \forall j \in J).
$$

Let be $p \in Z$, $p < 0$, and $j \in J$ arbitrary. Then $h(pb_j) = h(-(-p)b_j) = -h((-p)b_j) =_{(8)}= ph(b_j)$. This and (8) imply (5).

Let be $x, z \in X$ arbitrary, say $x = \sum p_j b_j$, $z = \sum q_j b_j$, so $x + z = \sum (p_j + q_j)(b_j)$, and we have $h(x+z) =_{(3)}= \sum h[(p_j+q_j)b_j] =_{(5)}= \sum (p_j + q_j)h(b_j) = \sum p_j h(b_j) + \sum q_j h(b_j) =_{(5)}= \sum h(p_j b_j) + \sum h(q_j b_j) =_{(3)}= h\left(\sum p_j b_j\right) + h\left(\sum q_j b_j\right) = h(x) + h(z)$. Therefore $h \in \text{Hom}(X, Y)$, and (o) $\text{Hom}_\perp(X, Y) \subset \text{Hom}(X, Y)$ is established, which completes the proof.

Remark 14. A more explicit form of $h \in Hom(X, Y)$ is of course

$$
(9) \qquad h(x) = \sum p_j c_j \text{ for } x = \sum p_j b_j \in X \text{ and } c_j := h(b_j) \quad (\forall j \in J).
$$

Theorem 15. *If* $\dim_Z X = 2$ *and the basis* $(b_j)_{j \in J}$ *is denoted by* (b_1, b_2), *then* $g : X \to Y$ *belongs to* (e) $\text{Hom}_\perp(X, Y)$ *if and only if there* *exist a_1, $a_2 \in Y$ such that*

$$g(p_1b_1+p_2b_2)=g(p_1b_1)+g(p_2b_2) \qquad (\forall p_1,p_2\in Z), \tag{10}$$

$$g(pb_1)=g(pb_2)=\frac{1}{2}p^2(a_1+a_2) \qquad (p\in Z \textit{ even}), \tag{11}$$

$$g(pb_1)=\frac{1}{2}(p^2+1)a_1+\frac{1}{2}(p^2-1)a_2 \qquad (p\in Z \textit{ odd}), \tag{12}$$

$$g(pb_2)=\frac{1}{2}(p^2-1)a_1+\frac{1}{2}(p^2+1)a_2 \qquad (p\in Z \textit{ odd}). \tag{13}$$

Proof. i) Assume that $g\in (\mathrm{e})\operatorname{Hom}_\perp(X,Y)$. (10) is implied by (3), and (4) leads to

$$g(pb_1)=g(pb_2)=2g\left(\frac{p}{2}b_1\right)+2g\left(\frac{p}{2}b_2\right) \qquad (p\in Z \text{ even}). \tag{14}$$

We put

$$a_1:=g(b_1), \quad a_2:=g(b_2) \tag{15}$$

and prove (11), (12), (13) by induction on p; evenness of g implies that we need only consider $p\in N^0$. $g(0)=0$ (from Lemma 8a)) and (15) guarantee that (11), (12), (13) hold for $p=0$, $p=1$, respectively.

Let be $p\in N$ odd and assume that (11), (12), (13) hold for $g(rb_1)$, $g(sb_2)$ where $0\le r,s\le p$. Since $p+1$ is even, (14) yields

$$g[(p+1)b_1]=g[(p+1)b_2]=2g\left(\frac{p+1}{2}b_1\right)+2g\left(\frac{p+1}{2}b_2\right). \tag{16}$$

From $1\le p$ we conclude $\frac{p+1}{2}\le p$ so that (11), (12), (13) can be applied to the right hand side of (16).

Case 1: $\frac{p+1}{2}$ is even. By (11) for $\frac{p+1}{2}$ we get

$$g\left(\frac{p+1}{2}b_1\right)=g\left(\frac{p+1}{2}b_2\right)=\frac{1}{2}\left(\frac{p+1}{2}\right)^2(a_1+a_2), \tag{17}$$

so $g[(p+1)b_1]=g[(p+1)b_2]=_{(16),(17)}=2\left(\frac{p+1}{2}\right)^2(a_1+a_2)=\frac{1}{2}(p+1)^2(a_1+a_2)$, which means that $(11)_{p+1}$ holds here.

Case 2: $\frac{p+1}{2}$ is odd. By (12), (13) for $\frac{p+1}{2}$ we get

$$g\Big(\frac{p+1}{2}b_1\Big)=\frac{1}{2}\Big[\Big(\frac{p+1}{2}\Big)^2+1\Big]a_1+\frac{1}{2}\Big[\Big(\frac{p+1}{2}\Big)^2-1\Big]a_2, \tag{18}$$

$$g\Big(\frac{p+1}{2}b_2\Big)=\frac{1}{2}\Big[\Big(\frac{p+1}{2}\Big)^2-1\Big]a_1+\frac{1}{2}\Big[\Big(\frac{p+1}{2}\Big)^2+1\Big]a_2. \tag{19}$$

So $g[(p+1)b_1]=g[(p+1)b_2]=_{(16),(18),(19)}=\left[\left(\frac{p+1}{2}\right)^2+1\right]a_1+\left[\left(\frac{p+1}{2}\right)^2-1\right]\cdot a_2+\left[\left(\frac{p+1}{2}\right)^2-1\right]a_1+\left[\left(\frac{p+1}{2}\right)^2+1\right]a_2=2\left(\frac{p+1}{2}\right)^2a_1+2\left(\frac{p+1}{2}\right)^2a_2=\frac{1}{2}(p+1)^2\cdot(a_1+a_2)$, and $(11)_{p+1}$ holds again. In the total, $(11)_{p+1}$ holds in every case.

We now turn to $(12)_{p+2}$, $(13)_{p+2}$; notice that $p+2$ is odd. By (3) and evenness of g we obtain

$$\begin{aligned} g[(p+1)b_1+b_2]&=g[(p+1)b_1]+g(b_2),\\ g[b_1-(p+1)b_2]&=g(b_1)+g[(p+1)b_2]. \end{aligned}\tag{20}$$

Taking into account $(p+1)b_1+b_2\perp b_1-(p+1)b_2$, we infer $g[(p+2)b_1]+g(pb_2)=_{(3)}=g[(p+2)b_1-pb_2]=g[(p+1)b_1+b_2+b_1-(p+1)b_2]=_{(*)}=g[(p+1)b_1+b_2]+g[b_1-(p+1)b_2]=_{(20)}=g[(p+1)b_1]+g(b_2)+g(b_1)+g[(p+1)b_2]$, i.e., by $(13)_p$, (15), $(11)_{p+1}$: $g[(p+2)b_1]+\frac{1}{2}(p^2-1)a_1+\frac{1}{2}(p^2+1)a_2=(p+1)^2(a_1+a_2)+a_1+a_2$, i.e., $g[(p+2)b_1]=[(p+1)^2+1-\frac{1}{2}(p^2-1)]a_1+[(p+1)^2+1-\frac{1}{2}(p^2+1)]a_2=\frac{1}{2}[(p+2)^2+1]a_1+\frac{1}{2}[(p+2)^2-1]a_2$, which means that $(12)_{p+2}$ holds. $(13)_{p+2}$ is obtained in a similar way. Therefore, (10), (11), (12), (13) do hold.

ii) Conversely, assume that $a_1,a_2\in Y$ and $g:X\to Y$ are such that (10), (11), (12), (13) hold. Then g obviously is even. $p=1$ in (12), (13) gives

$$g(b_1)=a_1,\quad g(b_2)=a_2. \tag{21}$$

Let be $x,z\in X$ arbitrary, but fixed for the moment, $x\perp z$, say $x=p_1b_1+p_2b_2,: z=q_1b_1+q_2b_2$. By Definition 4

$$p_1q_1+p_2q_2=\langle x,z\rangle=0, \tag{22}$$

hence

$$(p_1+q_1)^2+(p_2+q_2)^2=p_1^2+q_1^2+p_2^2+q_2^2. \tag{23}$$

By (22) the following cases of parity constellations for p_1, q_1, p_2, q_2 are excluded: Three numbers odd, one number even (4 cases); p_1 and q_1 odd, p_2 and q_2 even, or conversely (2 cases). On the basis of (23), we find by inspection in the ten remaining cases that always

$$g[(p_1+q_1)b_1]+g[(p_2+q_2)b_2]=g(p_1b_1)+g(p_2b_2)+g(q_1b_1)+g(q_2b_2)$$

is valid, so by (10) $g(x+z)=g(x)+g(z)$. As $x,z\in X$ with $x\perp z$ were arbitrary, we got $g\in \mathrm{Hom}_\perp(X,Y)$, and in the total $g\in \mathrm{(e)}\,\mathrm{Hom}_\perp(X,Y)$.

Remark 16. a) The procedure in part i) of the proof of Theorem 15 is different from that in the vector space case. There we have the conclusion

$$g\in \mathrm{(e)}\,\mathrm{Hom}_\perp(X,Y);\ x,z\in X;\ x+z\perp x-z \implies g(x)=g(z) \tag{24}$$

([10], p. 39, Theorem 6, iii). Here (24) is not available in general since the choice $a_1\neq a_2$, possible by Theorem 15 if Y allows it, leads to $g(b_1)\neq g(b_2)$ although $b_1+b_2\perp b_1-b_2$ (Lemma 5d)). On the other hand, (11) is a weak substitute of (24), but (11) is restricted to p even, and the difference has its origin in the missing 2-divisibility of the free Z-module X.

b) In the vector space case we have $\mathrm{(e)}\,\mathrm{Hom}_\perp(X,Y)\subset \mathrm{Quad}(X,Y)$ for X at least 2-dimensional ([10], p. 39, Theorem 6). In the context of the present Theorem 15, a mapping $g\in \mathrm{(e)}\,\mathrm{Hom}_\perp(X,Y)$ is quadratic if and only if $2a_1=2a_2$, i.e., if and only if $\omega(a_1)=\omega(a_2)$; if ω is injective, this is equivalent to $a_1=a_2$. In fact: i) Let $g\in \mathrm{Quad}(X,Y)$. By (11), $g(0)=0$, and $x=z$ in (JvN) gives $g(2x)=4g(x)$ $(\forall x\in X)$. So $2(a_1+a_2)=_{(11)}=g(2b_1)=4g(b_1)=_{(12)}=4a_1$, i.e., $2a_2=2a_1$. — ii) Let be $2a_1=2a_2$. If p is even, then $4\mid p^2$, so $\frac{1}{2}p^2(a_1+a_2)=\frac{1}{4}p^2(2a_1+2a_2)=p^2a_1=p^2a_2$. If p is odd, then $4\mid(p^2-1)$, and $\frac{1}{2}(p^2+1)a_1+\frac{1}{2}(p^2-1)a_2=\frac{1}{2}(p^2-1)(a_1+a_2)+a_1=\frac{1}{4}(p^2-1)(2a_1+2a_2)+a_1=(p^2-1)a_1+a_1=p^2a_1$, and in an analogous way we obtain $\frac{1}{2}(p^2-1)a_1+\frac{1}{2}(p^2+1)a_2=p^2a_2$. So by (10), (11), (12); (13) $g(p_1b_1+p_2b_2)=g(p_1b_1)+g(p_2b_2)=p_1^2a_1+p_2^2a_2(\forall p_1,p_2\in Z)$, consequently $g\in \mathrm{Quad}(X,Y)$.

c) Part b) shows how to construct non-quadratic $g \in \text{(e)}\operatorname{Hom}_\perp(Z^2, Y)$. Such a g cannot be extended to a $\widehat{g} \in \text{(e)}\operatorname{Hom}_\perp(R^2, Y)$, R^2 being equipped with the standard inner product, for g would have to be quadratic as a restriction of a quadratic mapping $\widehat{g}$.

d) It turns out that Theorem 15 describes an exceptional case because we have:

Theorem 17. *If $\dim_Z X \geq 3$, then $g : X \to Y$ belongs to (e) $\operatorname{Hom}_\perp(X, Y)$ if and only if there exist elements $a_j \in Y$ $(j \in J)$ with $2a_j = 2a_k$ for all $j, k \in J$ and $g(x) = \sum p_j^2 a_j$ for all $x = \sum p_j b_j \in X$.*

Proof. i) Let be $g \in \text{(e)}\operatorname{Hom}_\perp(X, Y)$. Put

$$a_j := g(b_j) \quad (\forall j \in J). \tag{25}$$

Let be $j, k \in J$ with $j \neq k$. Since $\dim_Z X \geq 3$, there exists $l \in J$ such that $l \neq j, l \neq k$. By (4), $g(2b_l) = 2g(b_l) + 2g(b_j)$ as well as $g(2b_l) = 2g(b_l) + 2g(b_k)$, so by (25)

$$2a_j = 2g(b_j) = 2g(b_k) = 2a_k, \tag{26}$$

and this trivially holds also for $j = k$. Now we show by induction on p that

$$(Q, j, p): \quad g(pb_j) = p^2 a_j \text{ holds for all } j \in J \text{ and all } p \in Z. \tag{27}$$

Evenness of g implies that we need only consider $p \in N^0$. $g(0) = 0$ (ensured by Lemma 8a)) and (25) guarantee that $(Q, j, 0)$ and $(Q, j, 1)$ hold for all $j \in J$.

Let be $n \in N, n \geq 2$, and assume that (Q, j, p) holds for all $p \in N^0$ such that $p \leq n - 1$ and for all $j \in J$. Let be $j, k \in J$, $j \neq k$, arbitrary.

$$\begin{aligned} g[nb_j + (n-2)b_k] &= g[(n-1)(b_j + b_k) + (b_j - b_k)] =_{\text{L.5d}} \\ &= g[(n-1)(b_j + b_k)] + g(b_j - b_k) =_{(3)} \\ &= g[(n-1)b_j] + g[(n-1)b_k] + g(b_j) + g(b_k) \\ =_{(Q,j,n-1),(Q,k,n-1)} &= (n-1)^2 g(b_j) + (n-1)^2 g(b_k) + g(b_j) + g(b_k) =_{(25)} \\ &= [(n-1)^2 + 1](a_j + a_k). \end{aligned} \tag{28}$$

On the other hand,

$$(29)\qquad \begin{aligned} &g[nb_j + (n-2)b_k] =_{(3)}= g(nb_j) + g[(n-2)b_k] =_{(Q,k,n-2)} \\ &\quad = g(nb_j) + (n-2)^2 g(b_k) =_{(25)}= g(nb_j) + (n-2)^2 a_k. \end{aligned}$$

From (28) and (29) we obtain $g(nb_j) = [(n-1)^2 + 1]a_j + [(n-1)^2 + 1 - (n-2)^2]\, a_k = (n^2 - 2n + 2)a_j + (2n-2)a_k =_{(26)}= (n^2 - 2n + 2)a_j + (2n-2)a_j = n^2 a_j$. Analogously, with j, k interchanged, we obtain also $g(nb_k) = n^2 a_k$. So (Q, j, n) holds for all $j \in J$, and by induction, (27) is established.

If $x = \sum p_j b_j \in X$ is arbitrary, then $g(x) =_{(3)}= \sum g(p_j b_j) =_{(27)}= \sum p_j^2 a_j$, i.e., g has the form required.

ii) Assume that there exist $a_j \in Y\,(j \in J)$ such that $2a_j{=}2a_k\ (\forall j, k{\in}J)$ and $g(x) = \sum p_j^2 a_j$ for all $x = \sum p_j b_j \in X$. Let be $x, z \in X$, $x \perp z$, say $x = \sum p_j b_j$, $z = \sum q_j b_j$, so $x + z = \sum (p_j + q_j) b_j$. By Definition 4

$$(30)\qquad \sum p_j q_j = \langle x, z \rangle = 0$$

Let d be the common value of all elements $2a_j (j \in J)$. Then $g(x+z) = \sum (p_j + q_j)^2 a_j = \sum (p_j^2 + 2p_j q_j + q_j^2) a_j = \sum p_j^2 a_j + \sum p_j q_j \cdot 2a_j + \sum q_j^2 a_j = g(x) + \sum p_j q_j \cdot d + g(z) =_{(30)}= g(x) + g(z)$. So $g \in \mathrm{Hom}_\perp(X, Y)$, and obviously g is even.

Remark 18. a) The mappings $g : X \to Y$ of the form $g(x) = \sum p_j^2 a_j$ for all $x = \sum p_j b_j \in X$, as occurring in Theorem 17, are quadratic no matter whether $2a_j = 2a_k (\forall j, k \in J)$ or not. In fact: If also $z = \sum q_j b_j \in X$, then $g(x{+}z){+}g(x{-}z) = \sum (p_j{+}q_j)^2 a_j {+} \sum (p_j{-}q_j)^2 a_j = \sum [(p_j{+}q_j)^2 + (p_j - q_j)^2] a_j = \sum (2p_j^2 + 2q_j^2) a_j = 2 \sum p_j^2 a_j + 2 \sum q_j^2 a_j = 2g(x) + 2g(z)$.

b) A quadratic mapping $g : X \to Z$ satisfying $2g(b_j) = 2g(b_k) (\forall j, k {\in} J)$ need not be in $(\mathrm{e})\,\mathrm{Hom}_\perp(X, Z)$. This shows that the quadratic mappings occurring in Theorem 17 form a very special class in $\mathrm{Quad}(X, Y)$ In fact: $g(x) := (\sum p_j)^2$ for $x = \sum p_j b_j \in X$. For $x, z \in X$ we have $g(x{+}z){+}g(x{-}z) = (\sum (p_j{+}q_j))^2 {+} (\sum (p_j{-}q_j))^2 = (\sum p_j {+} \sum q_j)^2 {+} (\sum p_j - \sum q_j)^2 = 2(\sum p_j)^2 + 2(\sum q_j)^2 = 2g(x) + 2g(z)$, i.e., $g \in \mathrm{Quad}(X, Z)$. Furthermore $g(b_j) = 1\,(\forall j \in J)$. If $\dim_Z X \geq 2$ and $b_j \perp b_k$, then $g(b_j + b_k) = 2^2 \neq 2 = g(b_j) + g(b_k)$, i.e., $g \notin (\mathrm{e})\,\mathrm{Hom}_\perp(X, Z)$.

Remark 19. Theorems 13 and 17 and Remark 18a) show that for sufficiently large dimension of X,

$$\text{(31)} \qquad \text{(o)}\,\mathrm{Hom}_\perp(X,Y) = \mathrm{Hom}(X,Y),$$

$$\text{(32)} \qquad \text{(e)}\,\mathrm{Hom}_\perp(X,Y) \subset \mathrm{Quad}(X,Y),$$

and Remark 18b) shows that the inclusion in (32) may be strict. Examples 10 and 11 demonstrate that (31), (32) do not hold for $\dim_Z X = 1$. Theorem 15 exhibits the exceptional case for (e) $\mathrm{Hom}_\perp(X,Y)$ when $\dim_Z X = 2$ where (32) still can be violated for suitable groups Y. The proof of Theorem 17 shows in what way the hypothesis $\dim_Z X \geq 3$ enforces the condition (26) $2a_j = 2a_k (\forall j,k \in J)$ and then the quadratic character of the mapping g considered there (for the 2-dimensional situation cf. Remark 16b).

There are many situations in connection with inner product spaces for which dimension 2 is exceptional. The most prominent one might be the characterization of inner product spaces among normed spaces where certain conditions are effective only for $\dim X \geq 3$ ([2], p. 97–156). For a much simpler question concerning a characterization of the inner product on real and complex vector spaces cf. [9].

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