Asymptotic behaviour of solutions of a linear unstable delay differential equation

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Dedicated to professor Zoltán Daróczy on his 50th birthday

Abstract. In this paper, we consider the asymptotic behaviour and structure of solutions of a delay differencial equation of the following form:

$$(*) \qquad \dot{x}(t) = p(t)\Big[ax(t) - \int_{t-1}^{t} x(s)\,dg(s-t)\Big], \quad t \geqq 0,$$

where $p(t)$ is a non-negative, continuous function on $[0, \infty)$, a is a positive constant, $g(t)$ is a non-decreasing function of bounded variation on $[-1, 0]$ and

$$\int_{-1}^{0} dg(u) = a.$$

We obtain results which, among others, indicate that under suitable hypotheses the asymptotic forms of all unbounded solutions of equation $(*)$, for large t, have the same dominant term up to linear dependence.

§ 1. Introduction

In [1], ATKINSON and ZHANG discuss the asymptotic behaviour and structure of solutions of the linear delay differential equation

$$(1.1) \qquad \dot{x}(t) = p(t)[x(t) - x(t-1)], \quad t \geqq 0,$$

where $p(t)$ is a continuous function on $[0, \infty)$, and obtain the following result.

Theorem A. *If* $p(t) > 0$ *and* $\int_{t}^{t+1} p(s)\,ds \geqq 1$, *then any solution* $x(t)$ *of equation* (1.1) *can be expressed in the form*

$$(1.2) \qquad x(t) = c_0 x_0(t) + \tilde{x}(t),$$

where c_0 *is a constant,* $x_0(t)$ *is some fixed unbounded solution, and* $\tilde{x}(t)$ *is a bounded solution.*

In [2], this result was extended to a periodic equation. In this paper, we show that under suitable hypotheses the representation (1.2) is valid for the more general equation

$$\dot{x}(t) = p(t)\Big[ax(t) - \int_{t-1}^{t} x(s)\,dg(s-t)\Big], \quad t \geqq 0, \tag{1.3}$$

where $p(t)$ is a non-negative, continuous function on $[0, \infty)$, a is a positive constant, $g(t)$ is a non-decreasing function of bounded variation on $[-1, 0]$ and

$$\int_{-1}^{0} dg(u) = a.$$

§ 2. The main result

Before stating the main result of this paper, we introduce some definitions, notations and lemmas.

Let $C^k[a, b]$ denote the space of real-valued functions with continuous derivative of order k on $[a, b]$. If $k=0$ and $[a, b]=[-1, 0]$, then we use the notation $C=C[-1, 0]$. In addition, we denote by $V[-1, 0]$ the space of real-valued functions of bounded variation on $[-1, 0]$.

Definition 2.1. The function $x(t)$ is called a solution of equation (1.3) if $x(t)\in C[-1, \infty)$, $x(t)\in C^1[0, \infty)$ and $x(t)$ satisfies equation (1.3) for $t\geqq 0$. For given $\varphi(t)\in C$, the function $x(t)$ is called a solution of equation (1.3) corresponding to the initial function $\varphi(t)$ if $x(t)$ is a solution of equation (1.3) and $x(\vartheta)=\varphi(\vartheta)$ for $\vartheta\in[-1, 0]$.

It is known [3, Chapter 2] that if $p(t)\in C[0, \infty)$, $g(t)\in V[-1, 0]$ and $\varphi(t)\in C$, then the following conclusions follow for the solution $x(t)$ of equation (1.3) corresponding to the initial function $\varphi(t)$:

(i) $x(t)$ exists and is unique for $t\geqq 0$;

(ii) the derivative of $x(t)$ is continuous for $t\geqq 0$.

We now state the main result of this paper.

Theorem. *Assume that the following conditions are satisfied:*

(2.1) $g(t)$ *is non-decreasing, continuous from the left on* $[-1, 0]$;

(2.2) $g(-1) = 0, \quad g(0) = a > 0$;

(2.3) $p(t)\in C[0, \infty), \quad p(t) \geqq 0, \quad t \geqq 0$;

(2.4) $\displaystyle\int_{0}^{t} p(s)\,g(s-t)\,ds > 0, \quad 0 < t \leqq 1$;

(2.5) $\displaystyle\int_{t}^{t+1} p(s)\,g(t-s)\,ds > 0, \quad t \geqq 0$;

(2.6) $\displaystyle\int_{t}^{t+1} p(s)\,g(t-s)\,ds \geqq 1, \quad t \geqq T_0 \geqq 0.$

Then, any solution $x(t)$ of equation (1.3) *which corresponds to the initial function $\varphi(t)\in C$ can be written in the form*

$$x(t) = c_0 x_0(t)+\tilde{x}(t), \tag{2.7}$$

where c_0 is a constant, $x_0(t)$ is some fixed unbounded solution of equation (1.3), *and $\tilde{x}(t)$ is a bounded solution of equation* (1.3). *c_0 and $\tilde{x}(t)$ depend on the initial function $\varphi(t)$.*

Remark 2.1. If $\lim_{s\to-1+0} g(s)-g(-1)=a=1$ and $g(t)=\lim_{s\to-1+0} g(s)$ for $t\in(-1, 0]$, then equation (1.3) coincides with equation (1.1). In this case, (2.1)—(2.6) are satisfied provided $p(t)\geqq 0$ for $t\geqq 0$, $\int_0^t p(s)\,ds>0$ for $0<t\leqq 1$, $\int_t^{t+1} p(s)\,ds>0$ for $t\geqq 0$, and $\int_t^{t+1} p(s)\,ds\geqq 1$ for $t\geqq T_0\geqq 0$. Therefore, the Theorem is a generalization of Theorem A.

For simplicity, the proof of the Theorem is reduced to lemmas. Some of them can be verified easily, so their proofs are omitted. Further, unless otherwise stated, we assume that conditions (2.1)—(2.6) of the Theorem are satisfied.

In the proofs, instead of equation (1.3) two equations are used both of which are equivalent to (1.3). This fact is formulated as follows:

Lemma 2.1. *If $\varphi(t)\in C$, then equation* (1.3) *is equivalent to the equation*

$$\dot{x}(t) = p(t) \int_{-1}^{0} [x(t)-x(t+u)]\,dg(u), \quad t \geqq 0, \tag{2.8}$$

and, for $t\geqq 1$, to

$$\dot{x}(t) = p(t) \int_{t-1}^{t} \dot{x}(s) g(s-t)\,ds, \quad t \geqq 1. \tag{2.9}$$

Proof. Introducing the notation $u=s-t$, we get

$$\int_{t-1}^{t} x(s)\,dg(s-t) = \int_{-1}^{0} x(t+u)\,dg(u).$$

In addition, on the basis of (2.2), we can write

$$ax(t) = \int_{-1}^{0} x(t)\,dg(u).$$

By using these equalities, we have

$$ax(t)- \int_{t-1}^{t} x(s)\,dg(s-t) = \int_{-1}^{0} [x(t)-x(t+u)]\,dg(u),$$

i.e. the first part of the lemma is proved.

Because of $x(t)\in C^1[0, \infty)$, the previous equality can be continued for $t\geqq 1$ as follows:

$$\int_{-1}^{0} [x(t)-x(t+u)]\, dg(u) = \int_{-1}^{0}\int_{t+u}^{t} \dot{x}(s)\, ds\, dg(u) = \int_{t-1}^{t} \dot{x}(s) \int_{-1}^{s-t} dg(u)\, ds.$$

From this, using (2.2), we get the second part of the assertion of the lemma.

From a technical point of view, it seems to be practical to introduce a notation for the second factor of the right side of equations (2.8) and (2.9):

$$F(t) = \int_{-1}^{0} [x(t)-x(t+u)]\, dg(u), \quad t \geqq 0, \tag{2.10}$$

and

$$F(t) = \int_{t-1}^{t} \dot{x}(s) g(s-t)\, ds, \quad t \geqq 1. \tag{2.11}$$

If the initial function is continuous, then $F(t)\in C[0, \infty)$.

The first step of our considerations which lead to the assertion of the Theorem is the proof of the existence of an unbounded solution $x_0(t)$. First of all, it can be shown that if the initial function of the solution $x(t)$ satisfies a suitable condition, then $x(t)$ is strictly monotone in a sense on $[0, \infty)$. For simplicity, we write $f(t)\nearrow$ (or $\searrow$) if the function $f(t)$ is monotonically increasing (or decreasing).

Lemma 2.2. *Let $x(t)$ be a solution of equation* (1.3) *corresponding to the initial function $\varphi(t)\in C$. In this case, if*

$$\int_{-1}^{t} [\varphi(t)-\varphi(u)]\, dg(u) \nearrow (\searrow), \not\equiv 0, \quad -1 \leqq t \leqq 0, \tag{2.12}$$

then

$$\int_{-1}^{0} [x(t)-x(t+u)]\, dg(u) > 0 \ (< 0), \quad 0 \leqq t < +\infty.$$

Proof. We prove the assertion for the positive case. (The other case can be proved similarly.)

By way of introduction, we note that (2.12) is equivalent to the following two conditions:

$$\int_{-1}^{0} [\varphi(0)-\varphi(u)]\, dg(u) > 0, \tag{2.13}$$

$$\varphi(t)\nearrow, \quad -r \leqq t \leqq 0, \tag{2.14}$$

where

$$r = \inf\{s\in[0, 1] | g(-1) = g(-s) = 0\}. \tag{2.15}$$

By the conditions (2.1) and (2.2), we have $r\in(0, 1]$.

To verify the equivalence, suppose (2.13) and (2.14) hold and let $-1\leqq t_1<$ $<t_2\leqq 0$. Then

$$\int_{-1}^{t_2}[\varphi(t_2)-\varphi(u)]\,dg(u)-\int_{-1}^{t_1}[\varphi(t_1)-\varphi(u)]\,dg(u)=$$

$$=\int_{t_1}^{t_2}[\varphi(t_2)-\varphi(u)]\,dg(u)+[\varphi(t_2)-\varphi(t_1)]\int_{-1}^{t_1}dg(u)\geqq 0,$$

while satisfaction of the condition "$\not\equiv$" is ensured by (2.13).

Conversely, if (2.12) is true, then (2.13) holds because of the continuity of the integral function. Suppose there exist $-1<\bar{t}_1<\bar{t}_2\leqq 0$ such that $\varphi(\bar{t}_1)>\varphi(\bar{t}_2)$ and $g(\bar{t}_1)>0$. Define the numbers $t_1, t_2\in[\bar{t}_1, \bar{t}_2]$ as follows:

$$t_1=\max\Big\{t\in[\bar{t}_1, \bar{t}_2]\,|\,\varphi(t)=\max_{s\in[\bar{t}_1,\bar{t}_2]}\varphi(s)\Big\},$$

$$t_2=\min\Big\{t\in[t_1, \bar{t}_2]\,|\,\varphi(t)=\min_{s\in[t_1,\bar{t}_2]}\varphi(s)\Big\}.$$

Then, $t_1<t_2$ and $\varphi(t_1)>\varphi(t)>\varphi(t_2)$ for $t_1<t<t_2$. However,

$$0\leqq\int_{-1}^{t_2}[\varphi(t_2)-\varphi(u)]\,dg(u)-\int_{-1}^{t_1}[\varphi(t_1)-\varphi(u)]\,dg(u)=$$

$$=\int_{t_1}^{t_2}[\varphi(t_1)-\varphi(u)]\,dg(u)+[\varphi(t_2)-\varphi(t_1)]\int_{-1}^{t_2}dg(u)<0,$$

since the absolute value of the first integrand is less than the absolute value of the quotient of the second integral, except at the point t_2, and it follows from the assumption that $g(t_1)>0$. Thus, we have reached a contradiction.

We next turn to the proof of the assertion. By using the notation (2.10), it can be formulated in such a way that if (2.12) (or (2.13) and (2.14)) holds, then

$$F(t)>0,\quad t\geqq 0.$$

First of all, it may be noted that, because of (2.13), $F(0)>0$ is satisfied. Suppose there exists a point $t_1>0$ such that $F(t_1)=0$, and let

$$t_0=\inf\{t_1>0\,|\,F(t_1)=0\}.$$

It is clear that $t_0>0$ and $F(t_0)=0$ since $F(t)$ is continuous.

If $t_0\geqq 1$, then, using (2.11) and (2.9), we have

$$0=F(t_0)=\int_{t_0-1}^{t_0}\dot{x}(s)g(s-t_0)\,ds=\int_{t_0-1}^{t_0}F(s)p(s)g(s-t_0)\,ds.$$

Hence, according to the assumptions on the functions $p(t)$ and $g(t)$, we get

$$(2.16)\qquad p(s)g(s-t_0)=0,\quad s\in[t_0-1, t_0],$$

because the integrand is non-negative. However, considering the definition (2.15), the inequality

$$\int_{t}^{t+1} p(s)\,g(t-s)\,ds = \int_{t}^{t+r} p(s)\,g(t-s)\,ds > 0, \quad t \geqq 0,$$

follows from (2.5), so we have

$$\int_{t}^{t+r} p(s)\,ds > 0, \quad t \geqq 0.$$

Then, we can deduce from this and (2.15) that

$$\int_{t-1}^{t} p(s)\,g(s-t)\,ds = \int_{t-r}^{t} p(s)\,g(s-t)\,ds > 0, \quad t \geqq 1.$$

Thus, the inequality

$$\int_{t_0-1}^{t_0} p(s)\,g(s-t_0)\,ds > 0$$

holds, which contradicts (2.16).

If $t_0 < 1$, then, making use of (2.10), we have

$$0 = F(t_0) = \int_{-1}^{0} [x(t_0) - x(t_0+u)]\,dg(u) = \int_{-1}^{-t_0} [x(0) - x(t_0+u)]\,dg(u) +$$

$$+ \int_{-1}^{-t_0} [x(t_0) - x(0)]\,dg(u) + \int_{-t_0}^{0} [x(t_0) - x(t_0+u)]\,dg(u).$$

Because of $x(t) \in C^1[0, \infty)$, the sum of the second and third terms is equivalent to

$$\int_{-1}^{-t_0} \int_{0}^{t_0} \dot{x}(s)\,ds\,dg(u) + \int_{-t_0}^{0} \int_{t_0+u}^{t_0} \dot{x}(s)\,ds\,dg(u).$$

Reversing the order of integration, we can derive the equality

$$\int_{0}^{t_0} \dot{x}(s) \int_{-1}^{-t_0} dg(u)\,ds + \int_{0}^{t_0} \dot{x}(s) \int_{-t_0}^{s-t_0} dg(u)\,ds = \int_{0}^{t_0} \dot{x}(s)\,g(s-t_0)\,ds.$$

We can summarize the above results as follows:

$$0 = \int_{-1}^{-t_0} [\varphi(0) - \varphi(t_0+u)]\,dg(u) + \int_{0}^{t_0} F(s)\,p(s)\,g(s-t_0)\,ds.$$

Since the integrands are non-negative, both integrals are equal to zero. However, in this case we get

$$p(s)\,g(s-t_0) = 0, \quad s \in [0, t_0],$$

which contradicts (2.4).

This completes the proof of the lemma.

By using (2.8)—(2.11) and (2.3), from the above assertion we obtain the following:

Corollary 2.1. *Under the assumptions of Lemma* 2.2,

(*i*) $\dot{x}(t) \geqq 0(\leqq 0)$ *for* $t \geqq 0$;

(*ii*) $\int\limits_{t-1}^{t} \dot{x}(s)\,ds > 0(<0)$ *for* $t \geqq 1$.

We can now prove the result which guarantees the existence of an unbounded solution.

Lemma 2.3. *Let* $x(t)$ *be a solution of equation* (1.3) *corresponding to the initial function* $\varphi(t) \in C$. *In this case, if* (2.12) *holds, then*

$$x(t) \to +\infty(-\infty) \quad as \quad t \to +\infty.$$

Proof. We prove the assertion for $+\infty$. (The other case can be proved similarly.)

Let T_1 be such that $T_1 \geqq T_0$, $T_1 \geqq 1$ and $p(T_1) > 0$. Integrating equation (2.9) from T_1 to T, we get

$$\int\limits_{T_1}^{T} \dot{x}(s)\,ds = \int\limits_{T_1}^{T} p(s) \int\limits_{s-1}^{s} \dot{x}(u)\,g(u-s)\,du\,ds.$$

From this, we have

$$\int\limits_{T_1}^{T} \dot{x}(s)\,ds = \int\limits_{T-1}^{T} \dot{x}(u) \int\limits_{u}^{T} p(s)\,g(u-s)\,ds\,du +$$

$$+ \int\limits_{T_1}^{T-1} \dot{x}(u) \int\limits_{u}^{u+1} p(s)\,g(u-s)\,ds\,du + \int\limits_{T_1-1}^{T_1} \dot{x}(u) \int\limits_{T_1}^{u+1} p(s)\,g(u-s)\,ds\,du.$$

If we omit the first term, use (2.6) for the second one and again reverse the order of integration in the third one, we get

$$\int\limits_{T_1}^{T} \dot{x}(s)\,ds \geqq \int\limits_{T_1}^{T-1} \dot{x}(u)\,du + \int\limits_{T_1}^{T_1+1} p(s) \int\limits_{s-1}^{T} \dot{x}(u)\,g(u-s)\,du\,ds.$$

Denote by I the value of the second integral on the right side. It is positive, since $p(T_1) > 0$ because of the choice of T_1, $F(T_1) > 0$ according to Lemma 2.2, and both functions are non-negative and continuous. Assume that $\int\limits_{T_1}^{\infty} \dot{x}(s)\,ds < +\infty$. Then, letting $T \to +\infty$, we have

$$\int\limits_{T_1}^{\infty} \dot{x}(s)\,ds \geqq \int\limits_{T_1}^{\infty} \dot{x}(u)\,du + I,$$

which is a contradiction. Thus, $\int_{T_1}^{\infty} \dot{x}(s)\,ds = +\infty$, which implies

$$x(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Remark 2.2. 1. The previous two lemmas are also valid for any $[t_0-1, t_0]$, where $t_0 \geqq 0$.

2. If we suppose that $r=1$, where r is the constant defined by (2.15), then the assumption concerning the initial function $\varphi(t)$ and others also become simpler. Indeed, if

$$g(t) > 0, \quad -1 < t \leqq 0,$$

then instead of (2.12) the condition

$$\int_{-1}^{t} [\varphi(t)-\varphi(u)]\,du \nearrow (\searrow), \not\equiv 0, \quad -1 \leqq t \leqq 0,$$

or, by virtue of (2.13), (2.14), the conditions

$$\int_{-1}^{0} [\varphi(0)-\varphi(u)]\,du > 0 (<0)$$

and

$$\varphi(t) \nearrow (\searrow), \quad -1 \leqq t \leqq 0,$$

can be used, whereas instead of (2.4) and (2.5) the assumptions

$$\int_{0}^{t} p(s)\,ds > 0, \quad 0 < t \leqq 1,$$

and

$$\int_{t}^{t+1} p(s)\,ds > 0, \quad t \geqq 0,$$

can be applied.

Next, we consider the existence of the form (2.7). Let $\varphi_1(t), \varphi_2(t) \in C$ be such that

$$\int_{-1}^{t} [\varphi_1(t)-\varphi_1(u)]\,dg(u) \nearrow, \not\equiv 0, \quad -1 \leqq t \leqq 0, \tag{2.17}$$

$$\int_{-1}^{t} [\varphi_2(t)-\varphi_2(u)]\,dg(u) \searrow, \not\equiv 0, \quad -1 \leqq t \leqq 0. \tag{2.18}$$

If $x_1(t)$, $x_2(t)$ are the solutions corresponding to them, then the function

$$x(t, \tau) = (1-\tau)x_1(t) + \tau x_2(t), \quad 0 \leqq \tau \leqq 1, \quad -1 \leqq t < \infty, \tag{2.19}$$

is also a solution of equation (1.3) because of linearity. Define the subsets S_1 and

S_2 of the interval $[0, 1]$ as follows:

$$(2.20)\quad S_1 = \left\{\tau\in[0, 1] \middle| \exists T_1(\tau) \geqq 0\colon \int_{-1}^{0} [x(t, \tau)-x(t+u, \tau)]\, dg(u) > 0 \text{ for } t \geqq T_1(\tau)\right\},$$

$$(2.21)\quad S_2 = \left\{\tau\in[0, 1] \middle| \exists T_2(\tau) \geqq 0\colon \int_{-1}^{0} [x(t, \tau)-x(t+u, \tau)]\, dg(u) < 0 \text{ for } t \geqq T_2(\tau)\right\}.$$

We claim the following:

Lemma 2.4. *S_1 and S_2 are non-empty, half-open intervals and have empty intersection.*

PROOF. Lemma 2.2, (2.17) and (2.18) imply that $\tau=0\in S_1$, whereas $\tau=1\in S_2$. Hence, both S_1 and S_2 are non-empty.

From definitions (2.20) and (2.21) for S_1 and S_2, it is clear that S_1 and S_2 have empty intersection.

Finally, we claim that S_1 is a half-open interval on the right. (Similarly, we can show that S_2 is a half-open interval on the left.) Suppose $\tau\in S_1\setminus\{0\}$. By (2.20), there exists some $T=T(\tau)$ such that

$$\int_{-1}^{0} [x(t, \tau)-x(t+u, \tau)]\, dg(u) > 0 \quad \text{for} \quad t \geqq T.$$

Because of the continuity of the integral in (t, τ), we can choose $\varepsilon>0$ so that

$$\int_{-1}^{0} [x(t, \tau')-x(t+u, \tau')]\, dg(u) > 0, \quad T \leqq t \leqq T+1,$$

provided $|\tau'-\tau|<\varepsilon$ and $\tau'\in[0, 1]$. Since $x(t, \tau')$ is a solution, then, according to (2.8), we have $\dot{x}(t, \tau')\geqq 0$ for $T\leqq t\leqq T+1$. Then, Lemma 2.2 implies that

$$\int_{-1}^{0} [x(t, \tau')-x(t+u, \tau')]\, dg(u) > 0 \quad \text{for} \quad t \geqq T,$$

i.e. $\tau'\in S_1$. This means that $S_1\setminus\{0\}$ is open.

In order for S_1 to be an interval, it is sufficient to show that if $\tau_1, \tau_2\in S_1, \tau_1<\tau_2$, then $\tau\in S_1$ for any $\tau_1<\tau<\tau_2$. By definition (2.19), we can write

$$\int_{-1}^{0} [x(t, \tau)-x(t+u, \tau)]\, dg(u) =$$

$$= (1-\tau) \int_{-1}^{0} [x_1(t)-x_1(t+u)]\, dg(u)+\tau \int_{-1}^{0} [x_2(t)-x_2(t+u)]\, dg(u).$$

We know from the assumptions and from Lemma 2.2 that the integrals on the right side have fixed, positive and negative signs on $[0, \infty)$, respectively. Hence, the integral on the left side is strictly monotonically decreasing in τ on $[0, 1]$ for fixed t. Thus, for any $\tau\in(\tau_1, \tau_2)$, since $x(t, \tau)$ is a solution, by using the above facts and (2.8), we have

$$\int_{-1}^{0} [x(t,\tau)-x(t+u\tau)]\,dg(u) > 0 \quad \text{for} \quad t \geqq T(\tau),$$

where $T(\tau)=\max\{T(\tau_1), T(\tau_2)\}$, i.e. $\tau\in S_1$.

As we have established earlier, $0\in S_1$, so S_1 is closed from the left.

This completes the proof of the lemma.

The following result is a simple consequence of the lemma.

Corollary 2.2. *There must be some* $\tau\in(0,1)$ *such that the function* $x(t,\tau)$ *satisfies one of the following conditions:*

(i) $x(t,\tau)=\text{const.}$ *for* $t\geqq T(\tau)$;

(ii) $x(t,\tau)$ *is not monotone on any interval* $[T,\infty)$.

Proof. It follows from the conditions and the properties of the sets S_1 and S_2 that $S_1\cap S_2=\emptyset$, $S_1\cup S_2\subset[0,1]$ and $S_1\cup S_2\neq[0,1]$.

Suppose $\tau\in[0,1]\setminus(S_1\cup S_2)$ and there exists no $T(\tau)\geqq 0$ such that $x(t,\tau)=$ $=\text{const.}$ for $t\geqq T(\tau)$. If there exists a $T(\tau)\geqq 0$ such that $x(t,\tau)\nearrow$ for $t\geqq T(\tau)$ (the other case can be continued similarly), then there must be a $t_0\geqq T(\tau)+1$ such that $\dot{x}(t_0,\tau)>0$. By using notation (2.11), this implies $F(t_0)>0$. However, in this case it follows from Lemma 2.2 that $\tau\in S_1$, which is a contradiction.

Although it is not quite obvious, the function $x(t,\tau)$ is bounded in the case of condition (ii) as well, since we can prove the following:

Lemma 2.5. *If some solution* $x(t)$ *of equation* (1.3) *is not monotone on any interval* $[T,\infty)$, *then* $x(t)$ *must be bounded.*

Proof. Assume that solution $x(t)$ is not monotone on any $[T,\infty)$ and is not bounded from above. (The other case can be verified similarly.) Then, there exist numbers $0<t_2<t_3$, $t_3-t_2<1$, with the following properties:

(i) $x(t) < x(t_2), \quad -1 \leqq t < t_2,$

(ii) $\dot{x}(t) = 0, \quad t_2 \leqq t \leqq t_3,$

(iii) $F(t_2) > 0,$

(iv) $F(t) \geqq 0, \quad t_2 \leqq t \leqq t_2,$

(v) $F(t_3) = 0.$

Indeed, one can easily construct a number $t_1>0$ such that

$$\begin{aligned} &\dot{x}(t_1) = 0,\\ &x(t) < x(t_1), \quad -1 \leqq t < t_1. \end{aligned} \tag{2.22}$$

By using the definition of t_1 and (2.10), we have $F(t_1)>0$. Because of the continuity, this holds in a neighbourhood of t_1, too. Hence, we find that $\dot{x}(t)\geqq 0$ there. Define t_3 as follows:

$$t_3 = \inf\{t \geqq t_1 | \dot{x}(t) < 0\}.$$

Since $x(t)$ is not monotone, this number exists and, by virtue of continuity, $\dot{x}(t_3)=0$. It is obvious that $t_1<t_3$. Furthermore, considering the definitions of the numbers t_1, t_3 and (2.8), one can see that $F(t_3)=0$. Now let t_2 be the following number:

$$t_2 = \inf\{\bar{t}\in[t_1, t_3] | \dot{x}(t) = 0,\ t\in[\bar{t}, t_3]\}.$$

If $t_1=t_2$, we are ready. If $t_1<t_2$, then, since $\dot{x}(t)\geqq 0$ on $[t_1, t_3]$, for arbitrary $\varepsilon>0$ there exists some $\hat{t}=\hat{t}(\varepsilon)$ such that

$$t_1 < \hat{t} < t_2, \quad |\hat{t}-t_2| < \varepsilon \quad \text{and} \quad \dot{x}(\hat{t}) > 0.$$

This means that t_2 satisfies (2.22) and the above t_3 belongs to it as well, and thus $t_2<t_3$. The condition $t_3-t_2<1$ must hold because of the uniqueness.

Then, using (v) and (ii), we can write

$$0 = F(t_3) = \int_{-1}^{0} [x(t_3)-x(t_3+u)]\, dg(u) = \int_{-1}^{t_2-t_3} [x(t_3)-x(t_3+u)]\, dg(u).$$

Since the integrand is positive except at the point t_2-t_3,

$$0 = g(-1) = g(t_2-t_3). \tag{2.23}$$

Considering the continuity of $p(t)$ and the monotony of $g(t)$, it follows from the above equality and (2.5) that

$$\int_{t_2}^{t_2+1} p(s)\, g(t_2-s)\, ds = \int_{t_2}^{t_3} p(s)\, g(t_2-s)\, ds > 0.$$

Hence, it follows from this equality that there must exist some $\bar{t}\in(t_2, t_3)$ such that

$$p(\bar{t})\, g(t_2-\bar{t}) > 0 \tag{2.24}$$

holds, because the integral is positive. By virtue of (ii) and (iii), $p(t_2)=0$, and if t_3 were suitable then one could find a $\bar{t}$ which would be less than it. Then, by using (ii), (2.8), (2 23) and again (ii), respectively, we have

$$0 = \dot{x}(\bar{t}) = p(\bar{t}) \int_{-1}^{0} [x(\bar{t})-x(\bar{t}+u)]\, dg(u) =$$

$$= p(\bar{t}) \int_{t_2-t_3}^{0} [x(\bar{t})-x(\bar{t}+u)]\, dg(u) = p(\bar{t}) \int_{t_2-t_3}^{t_2-\bar{t}} [x(\bar{t})-x(\bar{t}+u)]\, dg(u).$$

Since $p(\bar{t})>0$ because of (2 24), repeating the earlier consideration, we get

$$0 = g(-1) = g(t_2-\bar{t}),$$

which contradicts (2.24).

This completes the proof of the lemma

Before showing how we can give the form (2.7) by means of the above assertions, we introduce a result which guarantees the uniqueness of this form.

Lemma 2.6. *The τ in Corollary 2.2 is unique.*

PROOF. Suppose there exist τ_1 and τ_2, $\tau_1 \neq \tau_2$ such that both

$$x(t, \tau_1) = (1-\tau_1)x_1(t) + \tau_1 x_2(t) \tag{2.25}$$

and

$$x(t, \tau_2) = (1-\tau_2)x_1(t) + \tau_2 x_2(t) \tag{2.26}$$

satisfy one of the conditions of Corollary 2.2. Then, according to Lemma 2.5, both functions are bounded. Eliminating $x_2(t)$ from (2.25) and (2.26), we get

$$x_1(t) = \frac{\tau_2}{\tau_2 - \tau_1} x(t, \tau_1) - \frac{\tau_1}{\tau_2 - \tau_1} x(t, \tau_2).$$

Since $\tau_1 \neq \tau_2$ and $x(t, \tau_1)$, $x(t, \tau_2)$ are bounded, it follows that $x_1(t)$ is bounded. However, by Lemma 2.3, we know that $x_1(t) \to +\infty$ as $t \to +\infty$.

Hence, we have reached a contradiction.

We now pick some fixed solution $x_0(t)$, which corresponds to the following initial condition:

$$x_0(t) = \varphi_0(t), \quad \int_{-1}^{t} [\varphi_0(t) - \varphi_0(u)]\, dg(u) \nearrow, \not\equiv 0, \quad -1 \leqq t \leqq 0, \tag{2.27}$$

whereas $x(t)$ may be any solution corresponding to

$$x(t) = \varphi(t), \quad \int_{-1}^{t} [\varphi(t) - \varphi(u)]\, dg(u) \nearrow, \not\equiv 0, \quad -1 \leqq t \leqq 0, \tag{2.28}$$

where $\varphi_0(t), \varphi(t) \in C$.

We consider $x_0(t)$ as $x_1(t)$, and $-x(t)$ as $x_2(t)$. It follows from Lemma 2.6 that there must be a unique $\tau \in (0, 1)$ such that

$$x(t, \tau) = (1-\tau)x_0(t) + \tau\big(-x(t)\big)$$

is bounded. Then, we have

$$x(t) = c_0 x_0(t) + \tilde{x}(t), \tag{2.29}$$

where

$$c_0 = \frac{1-\tau}{\tau},$$

$$\tilde{x}(t) = -\frac{1}{\tau} x(t, \tau).$$

Clearly, $0 < c_0 < +\infty$ and $\tilde{x}(t)$ is bounded. (If $\varphi(t) \equiv \varphi_0(t)$, $-r \leqq t \leqq 0$, then $c_0 = 1$ and $\tilde{x}(t) \equiv 0$ for $t \geqq -r$.)

Similarly, any solution $x(t)$ satisfying

$$x(t) = \varphi(t), \quad \int_{-1}^{t} [\varphi(t) - \varphi(u)]\, dg(u) \searrow, \not\equiv 0, \quad -1 \leqq t \leqq 0, \tag{2.30}$$

can be uniquely written as (2.29), where

$$c_0 = \frac{\tau-1}{\tau},$$

$$\tilde{x}(t) = +\frac{1}{\tau}\,x(t, \tau).$$

Clearly, $-\infty < c_0 < 0$ and $\tilde{x}(t)$ is bounded. (If $\varphi(t) \equiv -\varphi_0(t)$, $-r \leqq t \leqq 0$, then $c_0 = -1$ and $\tilde{x}(t) \equiv 0$ for $t \geqq -r$.)

In general, it can happen that neither (2.28) nor (2.30) holds, i.e. the initial function $\varphi(t)$ is not monotone (on $[-r, 0]$). However, making use of the existence and uniqueness of the solutions of equation (1.3), one can give the above form in this case, too.

Let $x(t)$, $t \in [-1, \infty)$, be the solution which corresponds to the initial function $\varphi(t) \in C$. Since $x(t) \in C^1[0, \infty)$, we have

$$x(t) = x(0) + \int_0^t \dot{x}(s)\,ds = x(0) + \int_0^t [\dot{x}(s)]_+\,ds - \int_0^t [\dot{x}(s)]_-\,ds, \quad t \geqq 0,$$

where $[a]_+$ and $[a]_-$ denote the positive and negative parts of any real number a, respectively. Hence, we can write the solution in the form

$$x(t) = x^+(t) - x^-(t), \quad t \geqq 0,$$

where the functions

$$x^+(t) = x(0) + \int_0^t [\dot{x}(s)]_+\,ds,$$

$$x^-(t) = \int_0^t [\dot{x}(s)]_-\,ds,$$

are monotonically increasing on $[0, \infty)$. Both $x^+(t)$ and $x^-(t)$ satisfy the assumptions of Lemma 2.2 on $[0, 1]$. Therefore, applying the above conclusions, we get

$$x(t) = (c_0^+ - c_0^-)x_0(t) + \big(\tilde{x}^+(t) - \tilde{x}^-(t)\big), \quad t \geqq 0,$$

where $c_0^+ - c_0^-$ is a constant, $x_0(t)$ is some fixed unbounded solution corresponding to the initial condition (2.27), and $\tilde{x}^+(t) - \tilde{x}^-(t)$ is a bounded solution. The constants and bounded solutions depend on the initial function $\varphi(t)$ by the mediation of the values taken by $x(t)$ on $[0, 1]$.

Finally, considering the uniqueness of the solutions, we have

$$x(t) = c_0 x_0(t) + \tilde{x}(t), \quad t \geqq -1,$$

where

$$c_0 = c_0^+ - c_0^-,$$

$$\tilde{x}(t) = \begin{cases} x(t) - (c_0^+ - c_0^-)x_0(t) & \text{for} \quad -1 \leqq t \leqq 0, \\ \tilde{x}^+(t) - \tilde{x}^-(t) & \text{for} \quad t \geqq 0. \end{cases}$$

This completes the proof of the Theorem.

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