Simulation by ν_1^*-products of automata

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Introduction

In [3] the hierarchy of ν_i-products in comparison with the general product is studied. The aim of this paper is to start similar investigations for the hierarchy of generalized ν_i-products in comparison with the generalized product. Namely, we show that the generalized product is a proper generalization of the generalized ν_1-product from the point of view of homomorphic (isomorphic) simulation. Moreover, we overview some results on ν_1-products.

2. Basic notions

For any finite nonempty set X let X^* denote the *free monoid* of all *words* over X (including the *empty word* λ). Moreover, denote by $X^+(=X^*-\{\lambda\})$ the *free semigroup* of all nonempty words over X. The *length* of a word $p=x_1 \dots x_n \in X^+$ is denoted by $|p|(=n)$. The length of the empty word λ is zero per definitionem. Finally, we put $p^0=\lambda$, $p^n=p^{n-1}p$ $(p\in X^*,\ n>0)$.

By an *automaton* we mean a system $\mathbf{A}=(A, X, \delta)$ where A is the (nonempty finite) *set of states*, X is the (nonempty finite) *set of inputs* and $\delta\colon A\times X\to A$ is the *transition function*. We extend δ to a function $A\times X^*\to A$ as usual, i.e.,

$$\delta(a, \lambda) = a, \quad \delta(a, px) = \delta(\delta(a, p), x) \quad (a\in A,\ p\in X^*,\ x\in X).$$

We can consider an automaton as a special algebraic structure. In this sense we speak about *subautomaton*, furthermore, *homomorphism* and *isomorphism* of automata. We say that an automaton $\mathbf{A}$ *homomorphically (isomorphically) represents* an automaton $\mathbf{B}$ iff $\mathbf{A}$ has a subautomaton which can be mapped homomorphically (isomorphically) onto $\mathbf{B}$.

Let $\mathbf{A}=(A, X, \delta)$ and $\mathbf{B}=(B, Y, \delta')$ be automata. We say that $\mathbf{A}$ *homomorphically simulates* $\mathbf{B}$ if there are $A'\subseteq A$, a surjective mapping $h_1\colon A'\to B$ and a (not necessarily surjective) mapping $h_2\colon Y\to X^*$ with

$$h_1(\delta(a, h_2(y))) = \delta'(h_1(a), y) \quad (a\in A',\ y\in Y).$$

If h_1 is bijective then **A** *isomorphically simulates* **B**. It can be seen easily that the concept of homomorphic (isomorphic) simulation is a natural extension of that of homomorphic (isomorphic) representation.

Let $\mathbf{A}=(A, X, \delta)$ be an automaton. **A** is *monotone* if there is a partial ordering $\leqq$ on its state set A such that $a\leqq\delta(a, x)$ for all $a\in A$ and $x\in X$. An automaton $A=(A, X, \delta)$ is said to be *strongly monotone* if there exists a partial ordering $\leqq$ on A with $a\leqq\delta(a, x)$ $(a\in A, x\in X)$ such that for every pair $a\in A, x\in X$ from $a=\delta(a, x)$ it follows that a is a maximal element with respect to $\leqq$. It is said that **A** *satisfies the semi-Letičevskiĭ condition* if there are $a\in A$, $x, y\in X$, $p\in X^*$ with $\delta(a, x)\neq\delta(a, y)$ and $\delta(a, xp)=a$. Moreover, we say that a class K of automata satisfies the semi-Letičevskiĭ condition if there is an element of K with this property. Finally, we refer to the automaton

$$\mathbf{E} = (\{0, 1\},\ \{x_1, x_2\},\ \delta_{\mathrm{E}}),$$

with

$$\delta_{\mathrm{E}}(0, x_1) = 0, \quad \delta_{\mathrm{E}}(0, x_2) = \delta_{\mathrm{E}}(1, x_1) = \delta_{\mathrm{E}}(1, x_2) = 1$$

as the (two state) *elevator*. Obviously, the elevator is a monotone automaton.

Let $\mathbf{A}_t=(A_t, X_t, \delta_t)$ $(t=1, \ldots, k,\ k\geqq 1)$ be automata. Take a finite nonempty set X and the system of *feedback functions*

$$\varphi_t\colon A_1\times\ldots\times A_k\times X \to X_t^* \quad (t = 1, \ldots, k).$$

We let $\mathbf{A}=(A, X, \delta)=\mathbf{A}_1\times\ldots\times\mathbf{A}_k(X, \varphi)$ be the automaton with $A=A_1\times\ldots\times A_k$,

$$\delta((a_1, \ldots, a_k), x) = (\delta_1(a_1, \varphi_1(a_1, \ldots, a_k, x)), \ldots, \delta_k(a_k, \varphi_k(a_1, \ldots, a_k, x)))$$

$((a_1, \ldots, a_k)\in A,\ x\in X)$. The automaton **A** is called the *generalized product* or *g^*-product* of $\mathbf{A}_1, \ldots, \mathbf{A}_k$ (with respect to X and φ). For an arbitrary state $a=(a_1, \ldots, a_n)$ of **A** we use the notation $\pi_i(a)=a_i$ and we say that $\pi_i(a)(=a_i)$ is the *i-th projection* of a. Especially, if φ_t has the form $\varphi_t\colon A_1\times\ldots\times A_k\to X_t$ $(t=1, \ldots, k)$ then we speak about *general product* or *g-product*.

We also use the feedback functions in the following extended sense: For arbitrary $(a_1, \ldots, a_k)\in A$, $p\in X^*$, $x\in X$, $t(=1, \ldots, k)$ let

$$\varphi_t(a_1, \ldots, a_k, \lambda) = \lambda,$$

$$\varphi_t(a_1, \ldots, a_k, px) = \varphi_t(a_1, \ldots, a_k, p)\varphi_t(b_1, \ldots, b_k, x),$$

where

$$b_s = \delta_s(a_s, \varphi_s(a_1, \ldots, a_k, p)) \quad (1 \leqq s \leqq k).$$

Let i be an arbitrary natural number. Moreover, let us given a g^*-product $\mathbf{A}=\mathbf{A}_1\times\ldots\times\mathbf{A}_k(X, \varphi)$ such that for each $t(=1, \ldots, k)$ a set $\gamma(t)\subseteq\{1, \ldots, k\}$ with $|\gamma(t)|\leqq i$ is specified, so that φ_t does not depend on the state variables a_s with $s\notin\gamma(t)$ $(1\leqq s\leqq k)$. Then we write $\mathbf{A}=\mathbf{A}_1\times\ldots\times\mathbf{A}_k(X, \varphi, \gamma)$ and call **A** a *generalized v_i-product* or *v_i^*-product*. Especially, if we have the form $\varphi_t\colon A_1\times\ldots\times A_k\times X\to X_t$ $(t=1, \ldots, k)$ then **A** is a *v_i-product*. (Usually, if $s\notin\gamma(t)$ $(1\leqq s,\ t\leqq k)$ then we omit the s-th argument of φ_t.)

If every component of a product (generalized product) of automata is the same then we speak about a *power (generalized power)* of automata.

By a class K of automata we shall always mean a nonempty class. Let K be a class of automata. We say that K is *isomorphically (homomorphically) S-complete* with respect to the g^*-product (g-product, ν_i^*-product, ν_i-product) if every automaton can be simulated isomorphically (homomorphically) by a g^*-product (g-product, ν_i^*-product, ν_i-product) of automata from K. The following results hold.

Theorem 2.1. (GÉCSEG [4], [5].) *Let K be a class of automata. K is isomorphically (or homomorphically) S-complete with respect to the g^*-product iff K contains a nonmonotone automaton.*

Theorem 2.2. (DÖMÖSI—IMREH [1], DÖMÖSI—ÉSIK [2].) *Let K be a class of automata. K is isomorphically (or homomorphically) S-complete with respect to the ν_1^*-product iff K contains a nonmonotone automaton.*

Now let K be a class of automata again. We define the following classes.

$P_g(K)$:=all g-products of automata from K;
$P_g^*(K)$:=all g^*-products of automata from K;
$P_{\nu_i}(K)$:=all ν_i-products of automata from K;
$P_{\nu_i}^*(K)$:=all ν_i^*-products of automata from K;
$IS(K)$:=all automata which can be represented isomorphically by automata from K;
$HS(K)$:=all automata which can be represented homomorphically by automata from K;
$IS^*(K)$:=all automata which can be simulated isomorphically by automata from K;
$HS^*(K)$:=all automata which can be simulated homomorphically by automata from K.

Let O_1 and O_2 be one of the operators IS, HS, IS^*, HS^* and P_g, P_g^*, P_{ν_i}, $P_{\nu_i}^*$ $(i=1, 2, \ldots)$, respectively. For every class K of automata we define $O_1O_2(K)$ as the class $O_1(O_2(K))$. We shall use the following consequence of results in [4] and [5].

Theorem 2.3. (GÉCSEG [4], [5].)

$$HS^*P_g^*(\{\mathbf{E}\}) = IS^*P_g^*(\{\mathbf{E}\})$$

is the class of all monotone automata (where **E** *denotes the elevator).*

Consider any class K of automata with the following properties. For arbitrary integer $k \geqq 0$, there exist an automaton $\mathbf{A}=(A, X, \delta)\in K$, a state $a\in A$, an input word $p\in X^*$ with $|p|=k$, and a pair $x_1, x_2 \in X$ of inputs such that $\delta(a, px_1) \neq \neq \delta(a, px_2)$. It is shown in [5] that *metrically complete classes of automata for the general product* are exactly such classes K. We have as follows.

Theorem 2.4. (GÉCSEG and IMREH [7].) *If K is a metrically noncomplete class of automata then*

$$HSP_g(K) = HSP_{\nu_1}(K).$$

We now prove briefly the following result.

Theorem 2.5. *If K is a class of strongly monotone automata then*

$$HSP_g(K) = HSP_{v_1}(K).$$

PROOF. Since the class of strongly monotone automata does not satisfy the semi-Letičevskiĭ condition, Theorem 2.5 directly follows from Theorem 4.2 of GÉCSEG and JÜRGENSEN in [8]. □

3. v_1^*-product

Consider the automaton $\mathbf{A}=(\{a_1, a_2, a_3, a_4\}, \{x_1, x_2\}, \delta)$, where

$$\delta(a_1, x_1) = a_1, \quad \delta(a_1, x_2) = a_2, \quad \delta(a_2, x_1) = a_3, \quad \delta(a_2, x_2) = a_4,$$

$$\delta(a_3, x_1) = \delta(a_3, x_2) = a_3, \quad \delta(a_4, x_1) = \delta(a_4, x_2) = a_4.$$

Suppose that $\mathbf{A}$ can be simulated homomorphically by a v_1^*-power

$$\mathbf{M} = (M, X, \delta_{\mathbf{M}}) = \mathbf{E}^m(X, \varphi, \gamma)$$

under the subset $M'\subseteq M$ and mappings $h_1: M' \to \{a_1, a_2, a_3, a_4\}$ and h_2: $\{x_1, x_2\} \to X^*$. Let $h_2(x_1)=p_1$, $h_2(x_2)=p_2$ and let m_1 be a counter image of a_1. Without loss of generality, we may suppose that m_1 is choosen in such a way that $\delta_{\mathbf{M}}(m_1, p_1)=m_1$.

Indeed, this can be shown in the following way. Since $\mathbf{M}$ is finite and $\delta(a_1, x_1)=a_1$, there exist a state $m_1 \in M'$ with $h_1(m_1)=a_1$ and a positive integer t such that $\delta_{\mathbf{M}}(m_1, p^t)=m_1$ which, by the special structure of $\mathbf{M}$, implies $\delta_{\mathbf{M}}(m_1, p_1)=m_1$.

In the following two Lemmas and in Statement 3.3 we use the above automata $\mathbf{A}$, $\mathbf{M}$, subset M', mappings h_1, h_2, words p_1, p_2 and state m_1. Moreover, let $m_2 = \delta_{\mathbf{M}}(m_1, p_2)$, Z_1 the set of all letters occurring in p_1 and Z_2 the corresponding set for p_2. Finally, set $Z=Z_1 \cup Z_2$.

Lemma 3.1. *Let i, j $(1 \leqq i, j \leqq m)$ be arbitrary integers with $\gamma(i)=\{j\}$. If $\pi_i(m_2)=\pi_j(m_2)=0$, then $\varphi_i(0, z)=x_1$ for all $z \in Z$.*

PROOF. First of all, observe that

$$\pi_i(m_2) = \pi_j(m_2) = 0$$

implies

$$\pi_i(m_1) = \pi_j(m_1) = 0.$$

Since $\delta_{\mathbf{M}}(m_1, p_1)=m_1$, for every subword p of p_1 we have

$$\pi_i(\delta_{\mathbf{M}}(m_1, p)) = \pi_j(\delta_{\mathbf{M}}(m_1, p)) = 0.$$

Therefore, $\varphi_i(0, z)=x_1$ for all $z \in Z_1$. Similarly, $\delta_{\mathbf{M}}(m_1, p_2)=m_2$ implies that for all subwords p of p_2 the equality

$$\pi_i(\delta_{\mathbf{M}}(m_1, p)) = \pi_j(\delta_{\mathbf{M}}(m_1, p)) = 0.$$

Thus $\varphi_i(0, z)=x_1$ for all $z \in Z_2$. □

Lemma 3.2. *Let* $i_1, \ldots, i_j$ $(1 \leqq i_1, \ldots, i_j \leqq m)$ *be a sequence of positive integers such that*

$$\gamma(i_t) = \{i_{t-1}\} \quad (1 < t \leqq j), \quad \pi_{i_1}(m_2) = 1$$

and

$$\pi_{i_2}(m_2) = \ldots = \pi_{i_j}(m_2) = 0.$$

Then the following statements hold.

(i) *For arbitrary* t $(1 < t \leqq j)$ *and* $n \geqq t-1$ *we have*

$$\pi_{i_t}(\delta_{\mathrm{M}}(m_2, (p_1p_2)^n)) = 1$$

iff

$$\varphi_{i_2}(1, z_1) = \ldots = \varphi_{i_t}(1, z_{t-1}) = x_2$$

under some $z_1, \ldots, z_{t-1} \in Z$.

(ii) *For arbitrary* t $(1 < t \leqq j)$ *and* $n \geqq t-1$ *we have*

$$\pi_i(\delta_{\mathrm{M}}(m_2, (p_2p_1)^n)) = 1$$

iff

$$\varphi_{i_2}(1, z_1) = \ldots = \varphi_{i_t}(1, z_{t-1}) = x_2$$

under some $z_1, \ldots, z_{t-1} \in Z$.

(iii) *For arbitrary* t $(1 \leqq t \leqq j)$ *and* $n \geqq j-1$ *the equality*

$$\pi_{i_t}(\delta_{\mathrm{M}}(m_2, (p_1p_2)^n)) = \pi_{i_t}(\delta_{\mathrm{M}}(m_2, (p_2p_1)^n))$$

holds.

Proof. Statements (i) and (ii) easily follow from Lemma 3.1 by induction on t. If $t=1$ then Statement (iii) follows from

$$\pi_{i_1}(\delta_{\mathrm{M}}(m_2, p)) = 1 \quad (p \in X^*).$$

If $1 < t \leqq j$ then Statement (iii) is a direct consequence of (i) and (ii). □

Statement 3.3. *For some integer* $k > 0$, $\delta_{\mathrm{M}}(m_2, (p_1p_2)^k) = \delta_{\mathrm{M}}(m_2, (p_2p_1)^k)$.

Proof. Let $k \geqq m-1$. It is enough to show that for arbitrary i $(1 \leqq i \leqq m)$,

$$\pi_i(\delta_{\mathrm{M}}(m_2, (p_1p_2)^k)) = 1 \quad \text{iff} \quad \pi_i(\delta_{\mathrm{M}}(m_2, (p_2p_1)^k)) = 1.$$

It is also clear that we may restrict ourselves to the case when $\pi_i(m_2) = 0$.

Let $i_1, \ldots, i_j$ be a chain such that

(1) $i_j = i$ and $\gamma(i_t) = \{i_{t-1}\}$ $(1 < t \leqq j)$,

(2a) $\pi_{i_2}(m_2) = \ldots = \pi_{i_j}(m_2) = 0$ and $\pi_{i_1}(m_2) = 1$ or

(2b) $\pi_{i_1}(m_2) = \ldots = \pi_{i_j}(m_2) = 0$ and $\gamma(i_1) \subseteq \{i_1, \ldots, i_j\}$.

By Lemma 3.1, in case (2b), $\pi_i(\delta_{\mathrm{M}}(m_2, (p_1p_2)^t)) = 0$ and

$$\pi_i(\delta_{\mathrm{M}}(m_2, (p_2p_1))^t) = 0$$

for arbitrary $t \geqq 0$ and $i \in \{i_1, \ldots, i_j\}$. In case (2a), by (iii) of Lemma 3.2,

$$\pi_{i_t}(\delta_{\mathrm{M}}(m_2, (p_1p_2)^n)) = \pi_{i_t}(\delta_{\mathrm{M}}(m_2, (p_2p_1)^n))$$

for arbitrary $t (=1, \ldots, j)$ and $n \geqq j-1$.

Since for ν_1^*-products the length of any chain satisfying (1) and (2a) is less than or equal to m, the conclusion of the Statement holds for arbitrary k with $k \geqq m-1$. □

Observe that for every $k \geqq 1$, $\delta(a_2, (x_1 x_2)^k) = a_3$ and $\delta(a_2, (x_2 x_1)^k) = a_4$. Therefore,

$$\delta_{\mathrm{M}}(m_2, (p_1 p_2)^k) \neq \delta_{\mathrm{M}}(m_2, (p_2 p_1)^k)$$

which contradicts Statement 3.3. This contradiction arised from the assumption that there is a ν_1^*-power **M** of **E** such that **M** homomorphically simulates **A**. Moreover, it is clear that **A** is a monotone automaton. Thus we get the following result.

Theorem 3.4. *$HS^*P_{\nu_1}^*(\{\mathbf{E}\})$ does not contain all monotone automata.*

It is clear that $HS^*P_{\nu_1}^*(\{\mathbf{E}\}) \subseteq HS^*P_g^*(\{\mathbf{E}\})$. Thus, by Theorems 2.3 and 3.4

$$HS^*P_{\nu_1}^*(\{\mathbf{E}\}) \subset HS^*P_g^*(\{\mathbf{E}\}) = IS^*P_g^*(\{\mathbf{E}\}).$$

Consequently, we obtain our main result.

Theorem 3.5. *There exists a class K of automata with*

$$HS^*P_{\nu_1}^*(K) \subset HS^*P_g^*(K) = IS^*P_g^*(K).$$

In other words, the generalized product is a proper generalization of the generalized ν_1-product from the point of view of homomorphic (and isomorphic) simulation.

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