The law of the iterated logarithm in Banach spaces of type Φ

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Abstract. Some properties of Banach spaces of type Φ are established, where Φ denotes an Orlicz function. The relationship between the law of the iterated logarithm and the type of the Banach space is examined.

1. Introduction

The concept of a Banach space of type p plays an important role in studying the law of large numbers and the central limit theorem in Banach spaces. It turned out that a refinement of this concept was needed to treat the law of the iterated logarithm (LIL) in Banach spaces (see [6]). In [6] the LIL has been examined in Banach spaces of type Φ, where Φ is an Orlicz function. The notion of type Φ has been introduced by LAPRESTÉ [5]. (For a systematic study of Banach spaces of type Φ see e.g. [1] or [2].)

The aim of this paper is to supplement the work of LEDOUX [6]. In [6], LEDOUX used a more restrictive concept of type Φ (property Φ in our terminology, see Definitions 2.4 and 2.7) than ours and considered only those cases for which conditions (2.6) and (2.7) are satisfied. We do not use (2.6) and (2.7) and property Φ.

We prove that in Banach spaces of type Φ_α the LIL holds true under classical moment conditions (Theorem 4.1). This result formally is the same as Theorems 1 and 2 of [6] but our preconditions are weaker than those of [6]. Moreover we show that type Φ_α is really weaker than type 2 and stronger than type p for $p<2$ (Example 3.3). On the other hand we prove that the LIL implies that B is of type Φ for some function Φ (Theorem 4.2).

Section 2 is devoted to the definition of type Φ. From the different possible definitions the weakest and most simple one is adopted (Definition 2.4). In Section 3 the Banach spaces of type Φ are characterized by means of random vectors. Section 4 deals with the LIL.

2. Basic definitions and preliminary results

Let B be a real separable Banach space with norm $|.|$. We suppose that B is equipped with its Borel σ-field. Let (Ω, F, P) be a probability space. A measurable function $X\colon \Omega\to B$ is called a B-valued random variable (r.v.). EX stands for the Bochner integral of X. (ε_n) denotes the sequence of the Rademacher functions ($\varepsilon_1, \varepsilon_2, \ldots$ are independent r.v.'s with $P(\varepsilon_n=\pm 1)=1/2$ for all n). The abbreviation "a.s." means "almost surely". χ_A stands for the indicator of the set A.

Let us introduce the following notation

$$C(B) = \Big\{(x_n)\in B^\infty\colon \sum_{i=1}^{\infty} \varepsilon_i x_i \text{ converges in probability}\Big\},$$

where B^∞ denotes the set of B-valued sequences. Denote by $l_p(B)$ the l_p-space of B-valued sequences.

The Banach space B is said to be of type p $(1\leqq p\leqq 2)$ if

$$l_p(B)\subseteqq C(B). \tag{2.1}$$

It is well-known that B is of type p if and only if there exists a finite positive constant A such that

$$E\Big(\Big|\sum_{i=1}^{n} \varepsilon_i x_i\Big|\Big)^p \leqq A \sum_{i=1}^{n} (|x_i|)^p \tag{2.2}$$

for every n and $x_1, \ldots, x_n\in B$.

We shall extend properties (2.1) and (2.2) with the help of an Orlicz function Φ.

Definition 2.1. A function $\Phi\colon [0,\infty)\to[0,\infty)$ is said to be an *Orlicz function* if it is continuous, convex $\Phi(0)=0$, $\Phi(t)>0$ for $t>0$ and $\lim_{t\to\infty}\Phi(t)=\infty$. An Orlicz function Φ is called a Young function if $\lim_{t\to 0}\Phi(t)/t=0$ and $\lim_{t\to\infty}\Phi(t)/t=\infty$.

We remark that Young functions play an important role in several parts of probability theory, see e.g. [12] and the literature cited there.

An Orlicz function Φ is said to satisfy the Δ_2-condition $(\Phi\sim\Delta_2)$ if there exists a constant $c>0$ such that

$$\Phi(2t)\leqq c\Phi(t) \tag{2.3}$$

for every $t\geqq 0$. We shall say that Φ satisfies the Δ_2^0-condition if there exist constants $c>0$ and $t_0>0$ such that inequality (2.3) holds for every $0\leqq t\leqq t_0$.

Definition 2.2. For an Orlicz function Φ the *Orlicz space* $l_\Phi(B)$ consists of those B-valued sequences $(u_1, u_2, \ldots)$ for which $\sum_{i=1}^{\infty}\Phi(|u_i|/a)<\infty$ for some $0<a<\infty$. $l_\Phi(B)$ is a Banach space with the Luxemburg norm defined by

$$\|(u_i)_{i=1}^{\infty}\|_\Phi = \inf\Big\{a\colon \sum_{i=1}^{\infty}\Phi(|u_i|/a)\leqq 1\Big\}.$$

If Φ is a Young function then $(L_\Phi(B), \|.\|_{L_\Phi})$ denotes the Orlicz space of B-valued r.v.'s equipped with the Luxemburg norm:

$$\|X\|_{L_\Phi} = \inf\{a\colon E\Phi(|X|/a) \leqq 1\}.$$

The Orlicz space of real-valued sequences (resp. r.v.'s) will be denoted by l_Φ (resp. L_Φ).

Remark 2.3. Let Φ be a Young function. It is well-known that on the spaces l_Φ and L_Φ one can introduce the so-called Orlicz norm which is equivalent to the Luxemburg norm (see [3], [8], [9]).

According to Theorem 10.5 of [3] the Orlicz norm can be calculated by the formula

$$\|X\|_{L_\Phi} = \inf_{k>0} \frac{1}{k}\{1+E\Phi(k|X|)\}.$$

An analogous formula holds in the space l_Φ, too.

Definition 2.4. Let Φ be an Orlicz function. The Banach space B is said to be of *type* Φ if

$$l_\Phi(B) \subset C(B).$$

Remark 2.5. 1. Type Φ depends only on the behaviour of the function Φ in a neighbourhood of the origin.

2. If $l_{\Phi_1} \subseteqq l_{\Phi_2}$ then type Φ_2 is stronger than type Φ_1.

3. It is necessary for the existence of a space of type Φ that $l_\Phi \subseteqq l_2$.

Theorem 2.6. *Let Φ be an Orlicz function with $\Phi \sim \Delta_2^0$. The Banach space B is of type Φ iff there exists a positive constant A such that*

$$E\Big(\Big|\sum_{i=1}^{\infty} \varepsilon_i x_i\Big|\Big) \leqq A\,\|(x_i)_{i=1}^{\infty}\|_\Phi \tag{2.4}$$

for all $(x_i)_{i=1}^{\infty} \in C(B)$.

For the proof see [1]. We remark that it is sufficient to require condition (2.4) only for finite sequences $(x_i)_{i=1}^{n}$.

If we generalize inequality (2.2) with the help of Φ we get a stronger concept than that of type Φ.

Definition 2.7. The Banach space B is said to *satisfy property* Φ if there exists a constant A such that

$$E\Phi\Big(\Big|\sum_{i=1}^{n} \varepsilon_i x_i\Big|\Big) \leqq A \sum_{i=1}^{n} \Phi(|x_i|) \tag{2.5}$$

for every n and $x_1, \ldots, x_n \in B$.

Proposition 2.8. *If B satisfies property Φ, then B is of type Φ.*

Proof. Let $(x_i)_{i=1}^{\infty} \in l_\Phi(B)$. Then there exists $a>0$ such that $\sum_{i=1}^{\infty} \Phi(|x_i|/a) < \infty$. For an arbitrary $\varepsilon>0$ there exists n_ε such that

$$\sum_{i=1}^{m} \Phi\left(\frac{|x_i|}{a}\right) < \frac{\varepsilon^2}{A} \quad \text{for} \quad m > n > n_\varepsilon.$$

From (2.5) we have

$$E\Phi\left(\left|\sum_{i=n}^{m}\varepsilon_i\frac{x_i}{a}\right|\right)\leqq A\frac{\varepsilon^2}{A}=\varepsilon^2 \quad\text{for}\quad m>n>n_\varepsilon.$$

By the Markov inequality

$$P\{|\sum_{i=n}^{m}\varepsilon_i x_i|>a\Phi^{-1}(\varepsilon)\}=P\left\{\Phi\left(\left|\sum_{i=n}^{m}\varepsilon_i\frac{x_i}{a}\right|\right)>\varepsilon\right\}\leqq\frac{\varepsilon^2}{\varepsilon}=\varepsilon.$$

Therefore the series $\sum_{i=1}^{\infty}\varepsilon_i x_i$ is Cauchy in probability so it is convergent in probability.

Generally speaking the concept of type Φ is weaker than that of property Φ as the following example shows.

Example 2.9. Let

$$\Phi(x)=\begin{cases}x, & \text{for}\quad 0\leqq x\leqq 1;\\ x^2, & \text{for}\quad 1\leqq x.\end{cases}$$

Then $(x_n)_{n=1}^{\infty}\in l_\Phi(B)$ implies that $\sum_{i=1}^{\infty}|x_i|<\infty$. Therefore $\sum_{i=1}^{\infty}\varepsilon_i x_i$ converges in probability because every Banach space is of type 1. So every Banach space is of type Φ.

On the other hand, let B satisfy property Φ. Let $x_1, \dots, x_n\in B$ and let $K>0$ such that $\min_i |Kx_i|>1$. Then

$$\Phi(E|\sum_{i=1}^{n}\varepsilon_i Kx_i|)\leqq E\Phi(|\sum_{i=1}^{n}\varepsilon_i Kx_i|)\leqq A\sum_{i=1}^{n}\Phi(|Kx_i|).$$

Therefore

$$KE|\sum_{i=1}^{n}\varepsilon_i x_i|\leqq\Phi^{-1}(A\sum_{i=1}^{n}\Phi(|Kx_i|))=\sqrt{A}K(\sum_{i=1}^{n}(|x_i|)^2)^{1/2},$$

that is B is of type 2.

If type Φ implied property Φ, then every Banach space would be of type 2.

Remark 2.10. In [6] it is proved that the notion of type Φ and that of property Φ are equivalent if the following conditions are satisfied. There exist constants γ_1 and γ_2 such that

$$\gamma_1 E\Phi(|X|)\leqq\Phi(\|X\|_{L_\Phi})\leqq\gamma_2 E\Phi(|X|) \tag{2.6}$$

for every $X\in L_\Phi(B)$ and

$$\gamma_1\sum_{i=1}^{\infty}\Phi(|x_i|)\leqq\Phi(\|x\|_\Phi)\leqq\gamma_2\sum_{i=1}^{\infty}\Phi(|x_i|) \tag{2.7}$$

for every $x=(x_i)_{i=1}^{\infty}\in l_\Phi(B)$.

However, it is an open question which functions Φ satisfy conditions (2.6) and (2.7). It seems that this question is related to the definition of the norm in Orlicz spaces. It is known that Orlicz spaces cannot be normed analogously to L^p spaces

[11]. Therefore one can conjecture that conditions (2.6) and (2.7) are satisfied only for a narrow class of functions Φ. In particular, (2.6) and (2.7) are not fulfilled for functions $\tilde{\Phi}_\alpha$ which play an important role in the LIL.

Example 2.11. Let $\alpha>0$ and let $\tilde{\Phi}_\alpha$ be an Orlicz function for which

$$\tilde{\Phi}_\alpha(t)=\begin{cases} t^2[L_2(1/t)]^\alpha, & \text{in a neighbourhood of the origin;}\\ t^2/[L_2(t)]^\alpha, & \text{in a neighbourhood of the infinity.}\end{cases}$$

Here $L_2(t)=L(L(t))$ and $L(t)=\log t$ for $t\geqq e$ and $L(t)=1$ for $0\leqq t\leqq e$.

Then (2.6) and (2.7) are not fulfilled even in the case $B=R$ (=the set of real numbers). To see this one can put $X=r\chi_A$ in (2.6) and $x=(s,s,\dots,s,0,0,\dots)$ in (2.7).

In the following we shall not use the concept of property Φ and conditions (2.6) and (2.7).

3. Properties of spaces of type Φ

First we describe the type of the Orlicz space l_Φ.

Proposition 3.1. *Let Φ be a Young function for which $\Phi(\sqrt{x})$ is concave and $\Phi(xy)\leqq c\Phi(x)\Phi(y)$ for $x,y>0$. Then l_Φ is of type Φ.*

Proposition 3.2. *Let Φ_1 and Φ_2 be Orlicz functions. If $l_{\Phi_2}\subsetneqq l_{\Phi_1}$, then l_{Φ_1} is not of type Φ_2.*

For the proof see [2].

Example 3.3. Let $\alpha\in(0,2)$. There exists a Young function Φ_α for which

$$\Phi_\alpha(x)=\begin{cases} x^2[L_2(1/x)]^\alpha, & \text{for}\quad 0<x<a;\\ x^2, & \text{for}\quad x>b\end{cases}$$

(where a and b are suitable constants) and which satisfies the conditions of Proposition 3.1. Therefore the Orlicz space l_{Φ_α} is of type Φ_α. But, by Proposition 3.2, it is not of type 2 and not of type Φ_β for $\beta<\alpha$.

So type 2 is "better" than type Φ_β which is "better" than type Φ_α for $\beta<\alpha$. This example plays an important role in the LIL.

In the sequel we shall deal with independent r.v.'s in spaces of type Φ.

Theorem 3.4. *Let Φ be an Orlicz function, $\Phi\sim\Delta_2^0$. B is of type Φ iff there exists a constant $K>0$ such that*

$$E\Big|\sum_{i=1}^n X_i\Big|\leqq KE\,\|(X_i)_{i=1}^n\|_\Phi$$

for every n and every independent B-valued r.v.'s *$X_1,\dots,X_n$ with $EX_i=0$ $(i=1,\dots,n)$.*

For the proof see [1].

The following lemma is a generalization of Proposition 5.1 of PISIER [10].

Lemma 3.5. *Let Φ be a Young function. Suppose that*

$$E\left|\sum_{i=1}^{N} Y_i\right| \leqq \alpha \|Y_1\|_{L_\Phi}$$

for any independent identically distributed (i.i.d.) symmetric bounded B-valued r.v.'s $Y_1, \ldots, Y_N$, where $1 \leqq \alpha < \infty$. Then for every $x_1, \ldots, x_N \in B$

$$E(\left|\sum_{i=1}^{N} \varepsilon_i x_i\right|) \leqq 2\alpha \|(x_i)_{i=1}^{N}\|_{\Phi_N}$$

where $\Phi_N(x) = \dfrac{1}{N}\Phi(x)$.

PROOF. This result can be derived in the same way as the above mentioned proposition of Pisier. We notice only that in the first step of the proof one has to use the following equality

$$\|Y_1\|_{L_\Phi} = \|(x_i)_{i=1}^{N}\|_{\Phi_N}$$

if the distribution of Y_1 is $\dfrac{1}{2N}\sum_{i=1}^{N}(\delta_{x_i} + \delta_{-x_i})$.

Banach spaces of type Φ can be characterized with the help of i.i.d. r.v.'s.

Theorem 3.6. *Let Φ be a Young function, $\Phi \sim \Delta_2^0$. B is of type Φ iff there exists a constant A such that*

$$E\left|\sum_{i=1}^{N} Y_i\right| \leqq A \|Y_1\|_{L_{\Phi^N}} \tag{3.1}$$

for every N and every i.i.d. B-valued r.v.'s $Y_1, \ldots, Y_N$ with zero mean, where $\Phi^N(x) = N\Phi(x)$.

PROOF. a) Let B be a space of type Φ. According to Remark 2.3

$$\|Y_1\|_{L_{\Phi^N}} = \inf_{k>0} \frac{1}{k}\{1 + E\Phi^N(k|Y_1|)\}.$$

Therefore for an arbitrary $\varepsilon > 0$ there exists a k_ε such that

$$\|Y_1\|_{L_{\Phi^N}} \geqq \frac{1}{k_\varepsilon}\{1 + E\Phi^N(k_\varepsilon|Y_1|)\} - \varepsilon. \tag{3.2}$$

On the other hand

$$\|Y_i(\omega)_{i=1}^{N}\|_\Phi = \inf_{k>0} \frac{1}{k}\left\{1 + \sum_{i=1}^{N} \Phi(k|Y_i(\omega)|)\right\} \leqq \tag{3.3}$$

$$\leqq \frac{1}{k_\varepsilon}\left\{1 + \sum_{i=1}^{N} \Phi(k_\varepsilon|Y_i(\omega)|)\right\}.$$

Using Theorem 3.4 and inequalities (3.3) and (3.2) we get

$$E\left|\sum_{i=1}^{N} Y_i\right| \leqq AE\,\|(Y_i)_{i=1}^{N}\|_{\Phi} \leqq A\,\frac{1}{k_\varepsilon}\left\{1+\sum_{i=1}^{N} E\Phi(k_\varepsilon|Y_i|)\right\} =$$

$$= A\frac{1}{k_\varepsilon}\{1+NE\Phi(k_\varepsilon|Y_1|)\} \leqq A(\|Y_1\|_{L_{\Phi^N}}+\varepsilon).$$

Therefore inequality (3.1) holds.

b) Suppose that inequality (3.1) is satisfied. The Lemma 3.5 implies

$$E\left(\left|\sum_{i=1}^{N} \varepsilon_i x_i\right|\right) \leqq 2A\,\|(x_i)_{i=1}^{N}\|_{\Phi}$$

for every N and $x_1, \ldots, x_N \in B$. So B is of type Φ.

4. The LIL in Banach spaces of type Φ

In this section $X, X_1, X_2, \ldots$ are i.i.d. B-valued r.v.'s and as usual $S_n = X_1+\ldots+X_n$. Let $a_n=\sqrt{2nL_2 n}$, that is a_n is the classical normalizing constant in the LIL. We shall say that X satisfies the bounded LIL ($X\in$BLIL) if

$$P\left\{\limsup_n |S_n|/a_n < \infty\right\} = 1.$$

We shall say that X satisfies the compact LIL ($X\in$CLIL) if there exists a non-random compact set $D\subseteqq B$ such that

$$P\{d(S_n/a_n, D)\to 0\} = 1$$

and

$$P\{D = C(S_n/a_n)\} = 1,$$

where $d(x, A)$ denotes the distance of the point x from the set A and $C(x_n)$ denotes the set of limit points of the sequence x_n.

In the following theorem let Φ_α be the Young function defined in Example 3.3.

Theorem 4.1. *Let $E|X|^2<\infty$ and $EX=0$.*

1. *If B is of type Φ_1, then $X\in$BLIL.*
2. *If B is of type Φ_α for an $\alpha<1$, then $X\in$CLIL.*

Remark. Ledoux [6] proved this theorem for Banach spaces satisfying property Φ_1 (resp. Φ_α).

Proof. According to Kuelbs [4] it is sufficient to prove that in the first case $E|S_n|/a_n$ is bounded and in the second case $E(|S_n|/a_n)\to 0$ as $n\to\infty$. Because B is of type Φ_α we have

$$E|S_n| \leqq AE\|(X)_{i=1}^{n}\|_{\Phi_\alpha}.$$

By Remark 2.3 we have

$$\|(X)_{i=1}^n\|_{\Phi_\alpha} = \inf_{k>0}\left\{\frac{1}{k}\left(1+\sum_{i=1}^n \Phi_\alpha(k|X_i|)\right)\right\}. \tag{4.1}$$

Putting $k=\big(n(L_2n)^\alpha\big)^{-1/2}$ into (4.1) we get

$$E|S_n| \leqq A\big(n(L_2n)^\alpha\big)^{1/2}\big\{1+nE\Phi_\alpha\big(|X|/(n(L_2n)^\alpha)^{1/2}\big)\big\}. \tag{4.2}$$

For the sake of brevity let $u=|X|/\big(n(L_2n)^\alpha\big)^{1/2}$. Then

$$E\Phi_\alpha(u) = E_{\chi\{u\geqq b\}}\Phi_\alpha(u)+E_{\chi\{b>u>a\}}\Phi_\alpha(u)+E_{\chi\{u\leqq a\}}\Phi_\alpha(u) \leqq \tag{4.3}$$

$$\leqq Eu^2+\Phi_\alpha(b)P(b>u>a)+E_{\chi\{u\leqq a\}}\Phi_\alpha(u) \leqq Eu^2+\Phi_\alpha(b)E(u^2/a^2)+E_{\chi\{u\leqq a\}}\Phi_\alpha(u).$$

The last term of expression (4.3) is

$$E_{\chi\{u\leqq a\}}\Phi_\alpha(u) \leqq E\left\{\chi\{u\leqq a\}\frac{|X|^2}{n(L_2n)^\alpha}L_2\left(\frac{\big(n(L_2n)^\alpha\big)^{1/2}}{|X|}\right)^\alpha\right\}.$$

This expression is bounded from above by $\dfrac{E|X|^2}{n}$ if $n>\dfrac{\big(n(L_2n)^\alpha\big)^{1/2}}{|X|}=\dfrac{1}{u}$. Otherwise $u<\dfrac{1}{n}$ so $E\Phi_\alpha(u)\leqq\Phi_\alpha\left(\dfrac{1}{n}\right)=\dfrac{1}{n^2}(L_2n)^\alpha$.

Using these facts, (4.2) and (4.3) imply that

$$E|S_n| \leqq A\big(n(L_2n)^\alpha\big)^{1/2}\left\{1+n\left(\frac{E|X|^2}{n}+\frac{\Phi_\alpha(b)}{a^2}\frac{E|X|^2}{n}+\frac{E|X|^2}{n}+\frac{1}{n^2}(L_2n)^\alpha\right)\right\} \leqq$$

$$\leqq A\big(n(L_2n)^\alpha\big)^{1/2}K.$$

So $E(|S_n|/a_n)$ converges to zero if $\alpha<1$ and it is bounded if $\alpha=1$.

Remark 4.2. In [7], Ledoux and Talagrand have proved the BLIL under the conditions $E|X|^2/L_2(|X|)<\infty$, $Ef(X)=0$ and $E|f(X)|^2<\infty$ for every $f\in B^*$ (=the dual space of B). It is an interesting question whether these moment conditions imply the BLIL in Banach spaces of type Φ for some Orlicz function Φ.

We shall prove that the BLIL implies that B is of type Φ for some Orlicz function Φ. Let Ψ be a Young function, $\Psi\sim\Delta_2$. Suppose that $L_\Psi\subseteq L_2$. Introduce the notation

$$L_\Psi^{0;2}(B) = \{X\in L_\Psi(B):\ EX=0,\ E|f(X)|^2<\infty\ \ \forall f\in B^*\}.$$

Then $(L_\Psi^{0;2}(B), \|\ \|_{L_\Psi})$ is a Banach space.

Theorem 4.3. *Let Ψ be a Young function such that $\Psi\sim\Delta_2$ and $L_\Psi\subseteq L_2$. If every $X\in L_\Psi^{0;2}(B)$ satisfies the* BLIL, *then B is of type Φ, where $\Phi(x)=x^2|\log x|^{2+\delta}$ in a neighbourhood of the origin, where $\delta>0$.*

Proof. Let

$$N(X) = \sup_n \frac{1}{n}E|S_n|$$

and

$$M = \{X:\ N(X)<\infty\}.$$

Then (M, N) is a Banach space. According to Proposition 2.2 of [10] if $|S_n|/a_n < \infty$ a.s., then $E(\sup_n |S_n|/a_n) < \infty$. Therefore the assumptions of our theorem imply that $L_\Psi^{0,2}(B) \subseteq M$. By the closed graph theorem

$$\sup_n \frac{1}{n} E|S_n| \leqq A \|X\|_{L_\Psi}.$$

By Lemma 3.5

$$E\Big|\sum_{i=1}^n \varepsilon_i x_i\Big| \leqq 2Aa_n \|(x_i)_{i=1}^n\|_{\Psi_n} \tag{4.4}$$

for every $x_1, \dots, x_n \in B$. Let b_k be an increasing sequence and let

$$A_k^n = \left\{ j \in \{1, 2, \dots, n\} \colon \Phi^{-1}\left(\frac{\sum_{i=1}^n \Phi(|x_i|)}{b_{k+1}}\right) \leqq |x_j| < \Phi^{-1}\left(\frac{\sum_{i=1}^n \Phi(|x_i|)}{b_k}\right)\right\}.$$

Denote by $|A_k^n|$ the cardinality of A_k^n. Then

$$\sum_{i=1}^n \Phi(|x_i|) \geqq \sum_{k=0}^\infty \sum_{j \in A_k^n} \Phi(|x_j|) \geqq \sum_{k=0}^\infty \frac{|A_k^n|}{b_{k+1}} \sum_{i=1}^n \Phi(|x_i|),$$

which implies $|A_k^n| \leqq b_{k+1}$.

From (4.4) we get

$$\begin{aligned} E\Big|\sum_{i=1}^n \varepsilon_i x_i\Big| &\leqq \sum_{k=0}^\infty E\Big|\sum_{j \in A_k^n} \varepsilon_j x_j\Big| \leqq 2A \sum_{k=0}^\infty a_{|A_k^n|} \|(x_i)_{i \in A_k^n}\|_{\Psi_{|A_k^n|}} \leqq \\ &\leqq 2A \sum_{k=0}^\infty a_{|A_k^n|} \frac{1}{\Psi^{-1}(1)} \max_{i \in A_k^n} |x_i| \leqq \tilde{A} \sum_{k=0}^\infty \sqrt{b_{k+1} L_2(b_{k+1})}\, \Phi^{-1}\left(\frac{\sum_{i=1}^n \Phi(|x_i|)}{b_k}\right). \end{aligned} \tag{4.5}$$

In the fourth step we replaced all of the $|x_i|$-s by their maximal value and calculated the Luxemburg norm in this special case.

Let $(x_i) \in l_\Phi(B)$. Then $\Phi \sim \Delta_2$ implies that for every $\varepsilon > 0$ there exists an N_ε such that

$$\sum_{i=m}^n \Phi(|x_i|) \leqq \varepsilon \quad \text{for} \quad n > m > N_\varepsilon.$$

Therefore, by (4.5),

$$E\Big|\sum_{i=m}^n \varepsilon_i x_i\Big| \leqq \tilde{A} \sum_{k=0}^\infty \sqrt{b_{k+1} L_2(b_{k+1})}\, \Phi^{-1}\left(\frac{\varepsilon}{b_k}\right) \tag{4.6}$$

for $n > m > N_\varepsilon$. Put $b_k = \varepsilon/(a^{2k} |\log a^k|^{2+\delta})$, where $\delta > 0$ and $a = e^{-1}$. Then $\Phi^{-1}(\varepsilon/b_k) = a^k$, so it can easily be seen that the right hand side of (4.6) is not greater than $\sqrt{\varepsilon} K$, where K is a constant not depending on ε.

So $(x_i) \in l_\Phi(B)$ implies that the series $\sum_{i=1}^\infty \varepsilon_i x_i$ is convergent in L_1, therefore B is of type Φ.

Remark 4.4. a) Theorem 4.3 remains true also for those Young functions Ψ for which $L_\Psi \supseteqq L_2$ (e.g. $\Psi(x) = x^2/L_2(x)$ in a neighbourhood of infinity).

b) It is an interesting problem whether the BLIL (or CLIL) imply a better type of the underlying Banach space than the type given in Theorem 4.3.

References

[1] I. FAZEKAS, A strong law of large numbers in Banach spaces of type Φ. *In Mathematical Statistics and Probability Theory*, Vol. A (ed. Puri, Révész and Wertz) pp. 89—97. Reidel Publishing Company, Dordrecht, 1987.

[2] I. FAZEKAS, On Banach spaces of type Φ. In Function Spaces (Proc. Internat. Conf., Poznań, 1986, ed. J. Musielak) *Teubner Texte zur Math.* Vol. **103**, pp. 20—25, Leipzig, 1988.

[3] M. A. KRASNONSELSKIĬ and YA. B. RUTICKIĬ, Convex Functions and Orlicz Spaces. *Noordhoff, Groningen*, 1961.

[4] J. KUELBS, The law of the iterated logarithm for Banach space valued random variables. *Lect. Notes Math.* **860**, pp. 268—276, Springer-Verlag, *Berlin*, 1980.

[5] J. T. LAPRESTÉ, Type, cotype et measures de Lévy sur les espaces de Banach. Seminaire sur la géométrie des espaces de Banach, *Paris*, 1978.

[6] M. LEDOUX, Loi du logarithme itéré et types Φ d'espaces de Banach. *Ann. Sci. Univ. Clermont-Ferrand II, Math.* **19** (1981), pp. 207—220.

[7] M. LEDOUX and M. TALAGRAND, La loi du logarithme itéré dans les espaces de Banach. *C. R. Acad. Sc. Paris*, **303**, *série I* (1986), pp. 57—60.

[8] J. LINDENSTRAUSS and L. TZAFRIRI, Classical Banach Spaces. Vol. I—II, *Springer-Verlag, Berlin*, 1977 and 1979.

[9] J. NEVEU, Discrete-Parameter Martingales. *North-Holland Publishing Company, Amsterdam*, 1975.

[10] G. PISIER, Le théorème de la limite centrale et la loi du logarithme itéré dans les espaces de Banach. Seminaire Maurey—Schwartz, 1975—1976, École Politechnique, *Paris*, exposés 3 et 4.

[11] W. WNUK, Orlicz spaces cannot be normed analogously to L^p spaces. *Indagationes Mathematicae, Vol.* **46** (1984), pp. 357—359.

[12] J. MOGYORÓDI and T. F. MÓRI, Necessary and sufficient condition for the maximal inequality of convex Young functions. *Acta Sci. Math. Szeged*, Vol. **45** (1983), pp. 325—332.

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