On the inequalities of O. Szász type

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Summary. In his paper [3] 0. **Szász** gave a refinement of Hadamard' s celebrated inequality for determinants with positive definite Hermite-symmetric matrices. A very simple and new proof for the Szász's inequality was given by L. **Mirsky** ([2]). The author disccovered that the procedure of Mirsky can be used to prove more general inequalities, which are common source of some other inequalities. As special cases, inequalities, which ard related to matrices, can be derived.

1. Definitions and theorems

In this section we give the necessary definition, and the general theorems with proofs.

Definition 1.1. The set of numbers

$$a(i_1, \ldots, i_k) > 0, \quad 1 \leqq i_1 < \ldots < i_k \leqq n, \quad n \geqq 2, \quad k = 1, \ldots, n \tag{1.1}$$

is said to be convex system if the inequalities

$$\prod_{j_1 < \ldots < j_{k-1}} a(j_1, \ldots, j_{k-1}) \leqq a^l(i_1, \ldots, i_k) \tag{1.2}$$

hold for $k=2, \ldots, n$, where $(j_1, \ldots, j_{k-1})$ runs over all combination of order $k-1$ without repetition of elements $i_1, \ldots, i_k$, and $l=k-1$.

If (1.2) holds with reserve inequality sign, then the set of numbers $a(i_1, \ldots, i_k)$ is said to be *concave system.*

If (1.2) (or the reserve inequality) holds with $l=k$, then the set of numbers $a(i_1, \ldots, i_k)$ is said to be *quasi-convex (quasi-concave) system.*

Let $1 \leqq i_1 < \ldots < i_n$ be integers. Let us introduce the following notation:

$$P_k(i_1, \ldots, i_n) = \sum_{j_1 < \ldots < j_k} a(j_1, \ldots, j_k) \quad (k = 1, \ldots, n),$$

where $(j_1, \ldots, j_k)$ runs over all combination of order k without repetition of the elements $i_1, \ldots, i_n$.

Let

$$P_k = P_k(1, \ldots, n) \quad (k = 1, \ldots, n). \tag{1.3}$$

The main results of the paper are the following two theorems.

Theorem 1.1. *If the set of numbers* $a(i_1, ..., i_k)$ *is a convex (concave) system, then the sequence*

$$\{P_k^{\frac{1}{\binom{n-1}{k-1}}}\}_1^n$$

is increasing (decreasing).

Theorem 1.2. *If the set of numbers* $a(i_1, ..., i_k)$ *is a quasi-convex (quasi-concave) system, then the sequence*

$$\{P_k^{\frac{1}{\binom{n}{k}}}\}_1^n$$

is increasing (decreasing).

If we replace the numbers (1.1) with their reciprocals in the inequality (1.2), we get an inequality with reserve inequality sign. Thus it is enough to prove Theorems 1.1 and 1.2 only in the convex and quasi-convex cases, respectively.

In this two cases Theorems 1.1 and 1.2 will be proved at the same time. The proofs are similar those of MIRSKY theorem.

By making use of (1.2) in the convex and quasi-convex cases, it can be show that the statement of Theorems 1.1 and 1.2 are valid for $n=2$. Assume that the theorems are valid for all integers through $n-1 \geqq 2$. i.e. inequalities

$$(1.4) \qquad \begin{aligned} P_k^{n-k-1}(i_1, ..., i_{n-1}) &\leqq P_{k+1}^k(i_1, ..., i_{n-1}), \\ P_k^{n-k-1}(i_1, ..., i_{n-1}) &\leqq P_{k+1}^{k+1}(i_1, ..., i_{n-1}) \end{aligned}$$

hold for

$$1 \leqq i_1 < ... < i_{n-1} \leqq n, \quad k = 1, 2, ..., n-2.$$

By (1.4) we get

$$\prod_{1 \leqq i_1 < ... < i_{n-1} \leqq n} P_k^{n-k-1}(i_1, ..., i_{n-1}) \leqq \prod_{1 \leqq i_1 < ... < i_{n-1} \leqq n} P_{k+1}^k(i_1, ..., i_{n-1}),$$

$$\prod_{1 \leqq i_1 < ... < i_{n-1} \leqq n} P_k^{n-k-1}(i_1, ..., i_{n-1}) \leqq \prod_{1 \leqq i_1 < ... < i_{n-1} \leqq n} P_{k+1}^{k+1}(i_1, ..., i_{n-1}),$$

respectively, for $k=1, ..., n-2$. Regarding (1.3) we obtain the following inequalities:

$$[P_k(1, ..., n)]^{(n-k-1)(n-k)} \leqq [P_{k+1}(1, ..., n)]^{k(n-k-1)},$$

$$[P_k(1, ..., n)]^{(n-k-1)(n-k)} \leqq [P_{k+1}(1, ..., n)]^{(k+1)(n-k-1)}$$

for $k=1, ..., n-2$. Consequently

$$(1.5) \qquad \begin{aligned} P_k^{n-k} &\leqq P_{k+1}^k, \\ P_k^{n-k} &\leqq P_{k+1}^{k+1} \end{aligned}$$

for $k=1, ..., n-2$. But inequalities (1.5) are valid for $k=n-1$ by conditions (1.2) in the convex and quasi-convex cases, respectively.

2. Convex (quasi-convex) and concave (quasi-concave) matrices with respect to a functional

Let $n \geqq 2$ be an integer, and let $A=(a_{jk})_1^n$ be a $n \times n$ matrix with real or complex entries. As it is usual, matrices

$$A(i_1, \ldots, i_k) = \begin{pmatrix} a_{i_1 i_1} \ldots a_{i_1 i_k} \\ \cdot \quad \ldots \quad \cdot \\ a_{i_k i_1} \ldots a_{i_k i_k} \end{pmatrix}, \quad 1 \leqq i_1 < \ldots < i_k \leqq n$$

are said to be principal k rowed matrices, where $k=1, \ldots, n$. It is obvious that $A(1, \ldots, n)=A$. For brevity let us denote

$$A(i_1, \ldots, i_{n-1}) = A_j \quad (j = 1, \ldots, n),$$

where $\{i_1, \ldots, i_{n-1}, j\}$ is equal to the set $\{1, \ldots, n\}$.

Let F be a real-valued functional defined on all square matrices with real or complex entries.

Definition 2.1. The $n \times n$ matrix A is said to be convex, concave, quasi-convex and quasi-concave, respectively, with respect to the functional F, if conditions of the convex, concave, quasi-convex and quasi-concave systems are satisfied by

$$a(i_1, \ldots, i_k) = FA(i_1, \ldots, i_k) > 0$$

for

$$1 \leqq i_1 < \ldots < i_k \leqq n, \quad k = 1, \ldots, n.$$

In the following it will be shown that there are convex, concave, quasi-convex and quasi-concave matrices with respect to certain functionals, implying that there are convex, concave, quasi-convex and quasi-concave systems, respectively.

Examples for convex matrices with respect to functionals.

Denote by E the unit matrix, and by M the matrix with all entries equal to one.

Theorem 2.1. *Matrix* $A=xE+M$ *is convex with respect to the functional* $FA=\operatorname{Per} A$ *for* $x \geqq 0$.

PROOF. It is trivial that consition (1.1) is satisfied.

It can easily be seen that

$$\operatorname{Per} A = n! S_n,$$

where

$$S_n = S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \tag{2.1}$$

and hence

$$\operatorname{Per} A_j = (n-1)! S_{n-1} \quad (j = 1, \ldots, n).$$

The statement of the Theorem holds, if we show that

$$1 \leqq \frac{S_{n-1}^n}{S_n^{n-1}} < \frac{n^n}{n!}, \quad x \geqq 0. \tag{2.2}$$

Equality on the left hand side holds if and only if $x=0$, and on the right hand side the boundary can'nt be refined.

To prove this statement the following Lemma is needed.

Lemma 2.1. *The inequality*

$$1+\frac{1}{n-1} > \frac{S_{n-1}^2}{S_{n-2}S_n} \geqq 1 \tag{2.3}$$

holds for $x\geqq 0$ *with equality on the right hand side if and only if* $x=0$.

PROOF. Let

$$S_{n-2}S_n = \sum_{k=0}^{2(n-1)} \frac{a_k}{k!}x^k, \quad S_{n-1}^2 = \sum_{k=0}^{2(n-1)} \frac{b_k}{k!}x^k.$$

After a short calculation we get

$$\begin{gathered} a_k = 2^k \quad (k=0,1,\ldots,n-2), \\ a_{n-1} = 2^{n-1}-1, \\ a_{n+k} = \sum_{\nu=k}^{n-2}\binom{n+k}{\nu} \quad (k=0,1,\ldots,n-2), \end{gathered} \tag{2.4}$$

further more

$$\begin{gathered} b_k = 2^k \quad (k=0,1,\ldots,n-1), \\ b_{n+k} = \sum_{\nu=k+1}^{n-1}\binom{n+k}{\nu} \quad (k=0,1,\ldots,n-2). \end{gathered} \tag{2.5}$$

Hence we get the inequalities

$$na_k-(n-1)b_k = 2^k > 0 \quad (k=0,1,\ldots,n-2), \tag{2.6}$$

and

$$na_{n-1}-(n-1)b_{n-1} = 2^{n-1}-n \geqq 0 \tag{2.7}$$

with equality if and only if $n=1, 2$. Further for $k=0, 1, \ldots, n-2$ we obtain

$$na_{n+k}-(n-1)b_{n+k} = n\binom{n+k}{k}-(n-1)\binom{n+k}{n-1}+\sum_{\nu=1}^{n-k-2}\binom{n+k}{k+\nu}.$$

By making use of the identity

$$n\binom{n+k}{k}-(n-1)\binom{n+k}{n-1} = (k-n+2)\binom{n+k}{k+1},$$

we can derive the following formula:

$$na_{n+k}-(n-1)b_{n+k} = \sum_{\nu=1}^{n-k-2}\left[\binom{n+k}{k+\nu}-\binom{n+k}{k+1}\right].$$

It is easy to see that

$$\binom{n+k}{k+\nu}-\binom{n+k}{k+1} \geqq 0 \quad (\nu=1,\ldots,n-k-2).$$

Thus

$$na_{n+k}-(n-1)\,b_{n+k} \geqq 0 \quad (k=1, \dots, n-2). \tag{2.8}$$

The relations (2.6), (2.7) and (2.8) prove the left hand side inequality of (2.3).

On the basis of (2.4) and (2.5)

$$a_k - b_k = 0 \quad (k = 0, 1, \dots, n-2),$$

$$a_{n-1} - b_{n-1} = -1,$$

$$a_{n+k} - b_{n+k} = \binom{n+k}{k} - \binom{n+k}{n-1} < 0 \quad (k = 0, 1, \dots, n-2).$$

Therefore inequality

$$S_{n-2}(x)\,S_n(x) \leqq S_{n-1}^2(x), \quad x \geqq 0 \tag{2.9}$$

holds with equality if and only if $x=0$, i.e. the right hand side inequality of (2.3) is justified too. This completes the proof of the Lemma.

Returning to the proof of Theorem 2.1, let

$$S(x) = \frac{S_{n-1}^n(x)}{S_n^{n-1}(x)} = \frac{S_{n-1}^n}{S_n^{n-1}}, \quad x \geqq 0.$$

By Lemma 2.1

$$S'(x) = \frac{S_{n-1}^{n-1}}{S_n^n}\,[nS_{n-2}S_n-(n-1)\,S_{n-1}^2] > 0$$

for $x \geqq 0$. Thus $S(x)$ is strictly increasing for $x \geqq 0$. This result along with

$$\lim_{x \to \infty} S(x) = \frac{n^n}{n!}$$

give us the proof of inequality (2.2). Thus Theorem 2.1 is proved.

Using the terminology of G. Szegő ([4], Chapter 3) the right hand side of the inequality (2.3) can be formulated in the following way.

Theorem 2.2. *The sequence* $\{S_n(x)\}_0^\infty$ *of polynomials is a polynomial system of the Turán type for* $x \geqq 0$.

By (2.9) we get that

$$\frac{S_{n-2}}{S_{n-1}} \leqq \frac{S_{n-1}}{S_n}$$

for $x \geqq 0$, with equality if and only if $x=0$. Thus sequence

$$\left\{\frac{S_{n-1}}{S_n}\right\}_1^\infty$$

is increasing with the same limit 1 for $x \geqq 0$.

Theorem 2.3. *Let the diagonal elements of the matrix* $A=(a_{jk})_1^n$ *be positive numbers, and let the remaining entries be not smaler than* 1. *Then* A *is a convex matrix with respect to the functional*

$$P(A) = \prod_{j,k=1}^{m} a_{jk}.$$

PROOF. We shall use the following identity. If the entries of matrix $A=(a_{jk})_1^n$ are real or complex numbers different from zero, then

$$\prod_{1\le i_1<\dots<i_k\le n} P(A(i_1, \dots, i_k)) = (a_{11}\dots a_{nn})^{\binom{n-2}{k-1}} (P(A))^{\binom{n-2}{k-2}}. \tag{2.10}$$

Returning to the proof of the theorem, we get under the condition of the theorem, using the identity (2.10), that

$$\prod_{j=1}^{n} P(A_j) = a_{11}\dots a_{nn}(P(A))^{n-2} \leqq P(A)^{n-1},$$

with equality if and only if the non-diagonal entries are equal to 1. Thus we proved the theorem.

Now we formulate two conjectures, which are suggested partly by Theorem 2.1 partly by Theorems 2.4 and 2.5.

It is known ([1], p. 85, Def. 4) that the $n\times n$ matrix A is said to be total positive (non-negative) if all subdeterminants of arbitrary order are positive (non-negative).

Conjecture 2.1. All positive definite Hermite-symmetric matrices A are convex with respect to the functional $FA=\text{Per } A$.

Conjecture 2.2. All total positive matrices A are convex with respect to the functional $FA=\text{Per } A$.

Examples for concave matrices with respect to functionals.

Theorem 2.4. *All positive definite Hermite-symmetric matrices* A *are concave with respect to the functional* $FA=\text{Det } A$.

PROOF. Condition (1.1) are satisfied automatically.

Let $B=\text{adj } A$. By the Hadamard's determinental inequality

$$\prod_{j=1}^{n} \text{Det } A_j \geqq \text{Det } B = (\text{Det } A)^{n-1}$$

with equality if and only if A is a diagonal matrix. This completes the proof.

Theorem 2.5. *All total positive matrices* A *are concave with respect to the functional* $FA=\text{Det } A$.

PROOF. Since A is a total positive matrix, conditions (1.1) are satisfied automatically.

Matrix $A^*=(a^*_{jk})^n_1$ is said to be the transsignation of $A=(a_{jk})^n_1$, if $a^*_{jk}=(-1)^{j+k}a_{jk}$ for $j, k=1, \dots, n$. It is obvious that

(2.11) $$\operatorname{Det} A^* = \operatorname{Det} A.$$

A matrix A is said to be sign-regular if its transsignation is a total positive matrix. It is known ([1], p. 87), if A is total positive matrix, then A^{-1} is sign-regular, and conversely. Thus we get

(2.12) $$\operatorname{Det} B = \operatorname{Det} B^* = (\operatorname{Det} A)^{n-1},$$

if here again $B=\operatorname{adj} A$. Moreover it is known also ([1] p. 108. Satz 8) if A is a total positive matrix, then inequality

(2.13) $$\operatorname{Det} A < a_{11} \dots a_{nn}$$

holds. Since B is sign-regular, then B^* is a total positive matrix, consequently

$$\prod_{j=1}^{n} \operatorname{Det} A_j > \operatorname{Det} B^* = (\operatorname{Det} A)^{n-1}$$

by (2.11), (2.12), and (2.13). *Thus the proof of the theorem is finished.*

Applying same procedure, which was used in the proof of Theorem 2.3, we get the following statement.

Theorem 2.6. *If the diagonal elements of $A=(a_{jk})^n_1$ are positive numbers, and the remaining entries of A are positive numbers too, but not larger that* 1, *then A is concave with respect to the functional $P(A)=\prod_{j,k=1}^{n} a_{jk}$.*

Examples for quasi-convex matrices with respect to functionals.

Theorem 2.7. *If the entries of $(A=(a_{jk})^n_1$ are non-negative numbers, moreover the elements of the diagonal of A are positive numbers, then A is quasi-convex with respect to the functional $S(A)=\sum_{j,k=1}^{n} a_{jk}$.*

Proof. Condition (1.1) is satisfied evidently.

For every matrix $A=(a_{jk})^n_1$ we have

(2.14) $$\sum_{1\le i_1<\dots<i_k\le n} S\big(A(i_1, \dots, i_k)\big) = \binom{n-2}{k-1} \sum_{j=1}^{n} a_{jj} + \binom{n-2}{k-2} S(A).$$

Using identity (2.14), and the inequality between the arithmetic and geometric means we get

$$\Big(\prod_{j=1}^{n} S(A_j)\Big)^{1/n} \le \frac{1}{n} \sum_{j=1}^{n} S(A_j) = \frac{1}{n} \Big[\sum_{j=1}^{n} a_{jj} + (n-2) S(A)\Big] < S(A),$$

which is the statement of the theorem.

The following statement is a special case of Theorem 2.7.

Theorem 2.8. *Suppose that the diagonal elements of matrix A are positive numbers. Then A is quasi-convex with respect to the functional $FA = \operatorname{tr} A$.*

Theorem 2.9. *Let the elements of $A=(a_{jk})_1^n$ be positive numbers. Let*

$$A^- = \left(\frac{1}{a_{jk}}\right)_1^n.$$

Then matrix A (A^-) is quasi-convex with respect to the functional $FA = FA^- = \operatorname{Per} A \cdot \operatorname{Per} A^-$.

Proof. Inequality (1.1) is satisfied trivially.

It is well-known that

$$a_{11} \dots a_{nn} \prod_{j=1}^n \operatorname{Per} A_j < (\operatorname{Per} A)^n,$$

thus similarly

$$\frac{1}{a_{11}} \dots \frac{1}{a_{nn}} \prod_{j=1}^n \operatorname{Per} A_j^- < (\operatorname{Per} A^-)^n.$$

Multiplying these we get the statement of the theorem.

Examples for quasi-concave matrices with respect to functionals.

Theorem 2.10. *If all entries of $A=(a_{jk})_1^n$ are positive numbers, then A is quasi-concave with respect to the functional*

$$FA = \frac{n}{\sum_{j,k=1}^n \frac{1}{a_{jk}}} = H(A).$$

Proof. Condition (1.1) is satisfied trivially.

Applying the inequality between the geometric and harmonic mean, we get

$$\Big(\prod_{j=1}^n H(A_j)\Big)^{1/n} \geqq \frac{n}{\sum_{j=1}^n \frac{1}{H(A_j)}} = \frac{n(n-1)}{\sum_{1 \leqq i_1 < \dots < i_{n-1} \leqq n} \sum_{k,l=1}^n \frac{1}{a_{i_k i_l}}} > \frac{n}{\sum_{j,k=1}^n \frac{1}{a_{jk}}} = H(A)$$

using equality (2.14) for $k=n-1$. And this is the statement of the theorem.

The following theorem can be proved in a similar way.

Theorem 2.11. *If the diagonal elements a_j $(j=1, \dots, n)$ of the matrix A are positive numbers, then A is quasi-concave with respect to the functional*

$$FA = \frac{n}{\sum_{j=1}^n \frac{1}{a_j}}.$$

The following Theorem gives a comparison result between the functionals $\operatorname{Per} A$ and $P(A)$.

Theorem 2.12. *Let the entries of matrix* $A=(a_{jk})_1^n$ *be positive numbers. Then*

$$\operatorname{Per} A \geqq n! \sqrt[n]{\prod_{j,k=1}^{n} a_{jk}}$$

with equality if and only if $A=a^*b$, *where a and b are n-dimensional column vectors with positive components, and* a^* *denotes the transpose of a.*

Proof. By the well-known relation between the aritmetic and geometric means, we obtain

$$\frac{1}{n!} \operatorname{Per} A \geqq \sqrt[n!]{\prod_{(i_1, \dots, i_n) \in R} a_{1i_1} \dots a_{ni_n}} = \sqrt[n]{P(A)},$$

where R is the set of permutations of the elements $1, \dots, n$ without repetition. Equality holds if and only if

$$(2.15) \qquad a_{1i_1} \dots a_{ni_n} = a_{11} \dots a_{nn}, \quad (i_1, \dots, i_n) \in R.$$

By (2.15) it can be obtained easily that

$$\operatorname{Det} \begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{pmatrix} = 0, \quad 1 \leqq i_1 < i_2 \leqq n, \; 1 \leqq j_1 < j_2 \leqq n$$

indicating the rank of A is one. Then we have

$$(a_{i1} \dots a_{in}) = b_i(a_1, \dots, a_n), \quad b_i > 0 \quad (i = 1, \dots, n),$$

which is the statement of the theorem.

The following theorem of Van der Waerden type result is a special case of Theorem 2.12.

Theorem 2.13. *If* $A=(a_{jk})_1^n$ *is a doubly stochastic matrix with positive elements, then*

$$\operatorname{Per} A \geqq n! \sqrt[n]{\prod_{j,k=1}^{n} a_{jk}}$$

with equality if and only if all entries of A equal to $\frac{1}{n}$.

Proof. It is enough to prove the case of equality.

Since A is a stochastic matrix, i.e. $b_i \sum_{j=1}^{n} a_j = 1$, thus $b_i = \frac{1}{\sum_{j=1}^{n} a_j}$ $(i=1, \dots, n)$.
Moreover A is a double stochastic matrix. These two conditions are satisfied in the case only if

$$a_{jk} = \frac{1}{n} \quad (j, k = 1, \dots, n).$$

3. Inequalities of Szász type

In the following some consequences of theorems in the first, and the second section are discussed.

Inequality (2.2) gives the following theorem.

Theorem 3.1. *For fixed* $x>0$ *sequence*

$$\{(n!\,S_n)^{1/n}\}_0^\infty$$

is strictly increasing, where

$$S_n=\sum_{k=0}^{n}\frac{x^k}{k!}.$$

By conjectures 2.1 and 2.2 are suggested the following conjectures.

Conjecture 3.1. If A is a $n\times n$ positive definite Hermite-symmetric matrix, P_k denotes the product of all principal k rowed permanental minors of A, then

$$P_1\leqq P_2^{\frac{1}{\binom{n-1}{1}}}\leqq\ldots\leqq P_{n-1}^{\frac{1}{\binom{n-1}{n+2}}}\leqq P_n$$

with equality if and only if A is a diagonal matrix.

Conjecture 3.2. If A is a total positive $n\times n$ matrix, and P_k denotes the product of all principal k rowed permanental minors of A, then sequence

$$\{P_k^{\frac{1}{\binom{n-1}{k-1}}}\}_1^n$$

is increasing.

The next theorem follows immediately from Theorems 1.1 and 2.4.

Theorem 3.2. *If A is a positive definite Hermite-symmetric $n\times n$ matrix, and P_k denotes the product of all principal k rowed determinental minors of A, then*

$$P_1\geqq P_2^{\frac{1}{\binom{n-1}{1}}}\geqq P_3^{\frac{1}{\binom{n-1}{2}}}\geqq\ldots\geqq P_{n-1}^{\frac{1}{\binom{n-1}{n-2}}}\geqq P_n$$

with equality if and only if A is a diagonal matrix.

This is the theorem of O. Szász ([3]), we mentioned in the introduction.

Theorems 1.1 and 2.5 imply the following theorem.

Theorem 3.3. *If A is a total positive $n\times n$ matrix, and P_k denotes the product of all principal k rowed determinental minor of A, then sequence*

$$\{P_k^{\frac{1}{\binom{n-1}{k-1}}}\}_1^n$$

is striclty decreasing.

The next theorem follow from Theorems 1.2 and 2.7.

Theorem 3.4. *If all entries of the $n \times n$ matrix A are positive numbers, and P_k denotes the product of the sums of entries of all principal k rowed matrices of A, then sequence*

$$\{P_k^{\frac{1}{\binom{n}{k}}}\}_1^n$$

is strictly increasing.

Theorems 1.2 and 2.8 imply the following result.

Theorem 3.5. *If the diagonal elements of the $n \times n$ matrix A are positive, and P_k denotes the product of sums of the diagonal elements of all principal k rowed matrices of A, then sequence*

$$\{P_k^{\frac{1}{\binom{n}{k}}}\}_1^n$$

is strictly increasing.

The following theorem is a consequence of the Theorems 1.2 and 2.9.

Theorem 3.6. *Let the entries of the $n \times n$ matrix A be positive numbers. Let A^- denote the matrix constructed by the inverts of the entries of A. If P_k denotes the product of all principal k rowed permanental minors of A and of A^-, respectively, then sequence*

$$\{P_k^{\frac{1}{\binom{n}{k}}}\}_1^n$$

is strictly increasing.

By Theorems 1.2 and 2.10 we have the following result.

Theorem 3.7. *Let all entries of the $n \times n$ matrix A be positive numbers. Let P_k denote the product of the harmonic means of the entries of all principal k rowed matrices of A devided by k. Then*

$$\{P_k^{\frac{1}{\binom{n}{k}}}\}_1^n$$

is strictly decreasing.

The next theorem follows from Theorem 1.2 and 2.11.

Theorem 3.8. *Let the diagonal elements of the $n \times n$ matrix A be positive numbers. Let P_k denote the product of harmonic means of the diagonal elements of all principal k rowed matrices of A, then sequence*

$$\{P_k^{\frac{1}{\binom{n}{k}}}\}_1^n$$

is strictly decreasing.

Remark. The reason of not deriving more Szász type theorem by making use of Theorems 2.3 and 2.6 is that they would not provide new results but only trivialities.

Finally we show an example for quasi-concave functional defined on a probability space.

Theorem 3.9. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Let* $n \geqq 2$ *be an integer. If*

$$A_j \in \mathcal{F} \quad (j = 1, \ldots, n), \quad P(A_1 \ldots A_n) > 0,$$

moreover if

$$P_k = \prod_{1 \leqq i_1 < \ldots < i_k \leqq n} P(A_i \ldots A_i) \quad (k = 1, \ldots, n),$$

then

(3.1)
$$P_1^{\frac{1}{\binom{n}{1}}} \geqq P_2^{\frac{1}{\binom{n}{2}}} \geqq \ldots \geqq P_{n-1}^{\frac{1}{\binom{n}{n-1}}} \geqq P_n$$

with equality everywhere if and only if the representations

(3.2)
$$A_k = A + B_k \quad (k = 1, \ldots, n)$$

hold with

(3.3) $\quad A \in \mathcal{F}, \quad P(A) > 0; \quad B_k \in \mathcal{F}, \quad AB_k = \emptyset, \quad P(B_k) = 0 \quad (k = 1, \ldots, n).$

PROOF. Since

$$C_k = A_1 \ldots A_{k-1} A_{k+1} \ldots A_n \supset A_1 \ldots A_n \quad (k = 1, \ldots, n),$$

we have $P_{n-1} \geqq P_n^n$, i.e. the condition of Theorem 1.2 for quasi-concave system is satisfied, consequently (3.1) holds.

Equality is in (3.1) everywhere if and only if

(3.4)
$$P(A_1) \ldots P(A_n) = P^n(A_1 \ldots A_n).$$

Since

$$A_k \supset A_1 \ldots A_n \quad (k = 1, \ldots, n),$$

we get

$$P(A_k) \geqq P(A_1 \ldots A_n) \quad (k = 1, \ldots, n).$$

Thus (3.4) is satisfied if and only if

(3.5)
$$P(A_k) = P(A_1 \ldots A_n) \quad (k = 1, \ldots, n).$$

It is obvious that

$$A_k = A_k C_k + A_k \bar{C}_k = A + B_k \quad (k = 1, \ldots, n)$$

using the notations

$$A = A_1 \ldots A_n, \quad B_k = A_k \bar{C}_k \quad (k = 1, \ldots, n),$$

where

$$AB_k = \emptyset \quad (k = 1, \ldots, n),$$

and we get that

$$P(B_k) = 0 \quad (k = 1, \ldots, n)$$

by (3.5). I.e. we obtained a representation (3.2) satisfying condition (3.3).

Conversely, when representation (3.2) with conditions (3.3) hold, we have

$$P(A_k) = P(A) + P(B_k) = P(A) \quad (k = 1, \ldots, n).$$

Using condition

$$AB_k = \emptyset \quad (k = 1, \ldots, n)$$

again, we have

$$A_1 \dots A_n = \prod_{j=1}^{n} (A+B_k) = A+B,$$

where

$$B = B_1 \dots B_n,$$

consequently

$$AB = \emptyset, \quad P(B) = 0.$$

Thus

$$P(A_1 \dots A_n) = P(A)+P(B) = P(A),$$

i.e. condition (3.4) is satisfied. This completes the proof.

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