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On the behaviour of Colombeau's generalized functions at a point Applications to semilinear systems

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Abstract. We define the quasiasymptotics at zero (or at any other point) of a Colombeau's generalized function, element of $\mathcal{G}$, and show that this definition restected on a Schwartz's distribution embedded into $\mathcal{G}$ gives the well-known notion of the quasiasymptotics at zero and, in a special case, the value at zero. We analyze the quasiasymptotics of a Cauchy problem for a strictly semilinear hyperbolic system and show that under suitable assumptions on the non-linear term, the behaviour of the solution is determined by the behaviour of initial data.

1. Introduction

The multiplication of generalized functions in Colombeau's generalized function space $\mathcal{G}$ is well-defined and because of that has a lot of advantages in solving nonlinear partial differential equations as well as linear ones with singular coefficients.

The aim of this paper is to present a new method of qualitative analysis of solutions to initial value non-linear problems in relation to initial data. Remark that the quasiasymptotics is the notion of linear analysis and we use it in non-linear problems in the frame of Colombeau's generalized functions.

We refer to [15] for the advantages of these notions in applications to some problems of theoretical physics in the frame of Schwartz distributions.

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For the relations of ordinary asymptotics and quasiasymptotics we refer to [15] and [9]. Clearly, under weaker assumptions ordinary asymptotics implies the quasiasymptotics but the converse does not hold in general.

In this paper we define the quasiasymptotics at zero of Colombeau's generalized functions and show that our definition restricted to Schwartz's distributions gives the well-known notion of the quasiasymptotics at zero.

We consider a semilinear hyperbolic system which, in general, can model two phenomena: advection (transport, propagation) and nonlinear interaction (or selfinteraction), see [6]. Hyperbolicity means that the time variable is distingvished and that a Cauchy problem is well posed in time, for arbitrary initial data. We apply our theory to a Cauchy problem and show that under suitable assumptions on the non-linear term the behaviour at zero of a solution which is in $\mathcal{G}$ is determined by the behaviour of the initial data. The results are interesting in the case when the initial condition does not have any ordinar behaviour but satisfy the assumptions concerning quasiasymptotic behaviour at zero. Variety of examples with different behaviour at zero is presented.

Notation. We will use the well known notions of Schwartz's theory. Basic spaces are $C_c^\infty(\Omega) = \mathcal{D}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$, and $\mathcal{S}(\mathbb{R}^n)$. Their strong duals are Schwartz distributions space $\mathcal{D}'(\Omega)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

We recall the definition of a sequence of seminorms on $C^\infty(\Omega)$. Let Ω_k be a sequence of open sets such that $\bigcup_{k=0}^\infty \Omega_k = \Omega$, $\Omega_k \subset\subset \Omega_{k+1}$, $k \in \mathbb{N}_0$. Then

$$\mu_k(f) = \sum_{|\alpha|\le k} \Big(\sup_{x\in\overline{\Omega}_k} |\partial^\alpha f(x)| \Big), \quad k \in \mathbb{N}_0. \tag{1}$$

The uniform structure on $C^\infty(\Omega)$, defined by this sequence of seminorms, does not depend on the choice of the sequence Ω_k.

The space of compactly supported distributions and of tempered distributions supported by $[0,\infty)$ (resp. $(-\infty,0]$) are $(C^\infty(\Omega))' = \mathcal{E}'(\Omega)$, and $\mathcal{S}'_+(\mathbb{R})$ (resp. $\mathcal{S}'_-(\mathbb{R})$).

We denote by L Karamata's slowly varying function at zero. Recall, it is measurable, positive and

$$\lim_{\varepsilon\to 0} \frac{L(\varepsilon t)}{L(\varepsilon)} = 1$$

uniformly for $t \in [a,b] \subset (0,\infty)$ (and $\varepsilon < \varepsilon_0/b$), ε_0 is fixed.

Throughout the paper C will denote the generic constant which is different in different appearences.

2. Colombeau's generalized functions

We will present the simplified version of Colombeau's theory (cf. [1], [3], [6], [11]).

Let V be a topological vector space whose topology is given by a countable set of seminorms μ_k, $k \in \mathbb{N}$, given by (1).

Then $\mathcal{E}_{M,V}$ is the set of locally bounded functions $R(\varepsilon)=R_\varepsilon : (0,1)\to V$ such that for every $k \in \mathbb{N}$ there exists $a \in \mathbb{R}$ with the property that

$$\mu_k(R_\varepsilon) = \mathcal{O}(\varepsilon^a),$$

where $\mathcal{O}(\varepsilon^a)$ means that the left side is smaller or equal than $C\varepsilon^a$ for some $C > 0$ and every $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$. The upper bound of such reals a is called the k-valuation of R_ε and it is denoted by $v_k(R_\varepsilon)$.

The space of all elements $H \in \mathcal{E}_{M,V}$ with the property that for any $k \in \mathbb{N}$ and for any $a \in \mathbb{R}$, $\mu_k(H_\varepsilon) = \mathcal{O}(\varepsilon^a)$ is denoted by $\mathcal{N}_V$. Note, $\mathcal{N}_V$ is the space of elements H_ε whose all valuations $v_k(H_\varepsilon)$, $k \in \mathbb{N}$, are equal to $+\infty$.

The quotient space $\mathcal{G}_V = \mathcal{E}_{M,V}/\mathcal{N}_V$ is called the polynomially generalized extension of V. If $R_\varepsilon - R'_\varepsilon \in \mathcal{N}_V$, then $v_k(R_\varepsilon) = v_k(R'_\varepsilon)$ for every $k \in \mathbb{N}_0$. The k-valuation of a class $[R_\varepsilon]$ is naturally defined (brackets [ ] are used to denote the equivalence class in the quotient space).

Scarpalézos has used valuations for the definition of the metric in $\mathcal{G}_V$ and the so called sharp topology. If the space V is an algebra whose products are continuous for all the seminorms, then $\mathcal{N}_V$ is an ideal of the algebra $\mathcal{E}_{M,V}$ and $\mathcal{G}_V$ becomes a Hausdorff topological ring.

If $V = \mathbb{C}$, then $\mathcal{G}_V$ is called the algebra of generalized constants and it is denoted by $\bar{\mathbb{C}}$; $\mathcal{E}_{M,V}$ is denoted by $\mathcal{E}^0$ and $\mathcal{N}_V$ is denoted by $\mathcal{N}^0$.

If $V = \mathbb{C}^\infty(\Omega)$, where Ω is an open set in $\mathbb{R}^n$ and μ_k are given by (1) then $\mathcal{G}_V$ is called the algebra of generalized functions on Ω and it is denoted by $\mathcal{G}(\Omega)$; $\mathcal{E}_{M,V}$ is denoted by $\mathcal{E}_M(\Omega)$ and $\mathcal{N}_V$ is denoted by $\mathcal{N}(\Omega)$.

Then, $\mathcal{G}(\Omega)$ is a differential topological ring where derivations ∂_x are continuous for its sharp topology. In n-dimensional case, $\bar{\mathbb{C}}$ can be considered as a subalgebra of $\mathcal{G}(\Omega)$.

In order to embed $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$ we recall the following assertion of Colombeau, slightly changed in [11] for the sake of simplified version of Colombeau's theory.

Let $\psi \in C_c^\infty(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *such that it is even,* $\mathcal{F}(\phi) = \hat{\phi} \in \mathcal{D}(\mathbb{R})$ *and* $\hat{\phi} \equiv 1$ *on a neighbourhood of zero. Put* $\phi_{\varepsilon^2}(x) = \frac{1}{\varepsilon^2}\phi\left(\frac{x}{\varepsilon^2}\right)$, $x \in \mathbb{R}^n$, $\varepsilon \in (0,1)$. *Then,*

$$N_\varepsilon(x) = (\psi * \phi_{\varepsilon^2}(x) - \psi(x)) \quad \textit{belongs to} \quad \mathcal{N}(\Omega),$$

where $*$ *is a convolution.*

We fix once for all such a function ϕ and call it the "vision" function. Put $I_\phi(\psi) = [\psi * \phi_{\varepsilon^2}]$. Usually, ε is used instead of ε^2 but later it will be clear why we use ε^2. It can be easily verified that if φ, ψ belong to $\mathcal{D}(\mathbb{R})$, then

$$I_\phi(\varphi \cdot \psi) = I_\phi(\varphi) \cdot I_\phi(\psi).$$

If $T \in \mathcal{E}'(\Omega)$ then $I_\phi(T) = [T * \phi_{\varepsilon^2}]$. We will use $[T]$ for $I_\phi(T)$ in order to simplify the notation.

Since the presheaf $U \to \mathcal{G}(U)$ (U is open in $\mathbb{R}^n$) is a sheaf, it follows that the above embeddings can be extended to embeddings of $C^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$. The support of a generalized function H is defined as the complement of the largest open subset $\Omega' \subset \Omega$ such that $H_{|\Omega'} = 0$. If $T \in \mathcal{D}'(\Omega)$, then $\operatorname{supp} T = \operatorname{supp}(I_\phi(T))$.

If G is a generalized function with compact support $K \subset\subset \Omega$ ($G \in \mathcal{G}_c(\Omega)$) and $G_\varepsilon(x)$ is a representative of G, then its integral is defined by

$$\int G dx = \left[\int \psi(x) G_\varepsilon(x) dx\right],$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi = 1$ on K. This definition does not depend on ψ.

If $G, F \in \mathcal{G}(\Omega)$ then they are equal in the sense of distributions, $G \overset{\mathcal{D}'}{=} F$ if

$$\int (G_\varepsilon(x) - F_\varepsilon(x))\psi(x) dx \in \mathcal{N}^0 \quad \text{for any} \quad \psi \in \mathcal{D}(\Omega),$$

G_ε and F_ε being the representatives of G and F, respectively.

3. Quasiasymptotics at zero

This notion is defined by Droshinov and Zavialov for elements of S'_+ (cf. [14]). We will use slightly modified definition [9]. In the sequel ω will denote an open set in $\mathbb{R}^n$ which contains 0.

Definition 1. Let $f \in \mathcal{D}'(\omega)$, (resp. $f \in \mathcal{S}'(\mathbb{R}^n)$) and c be a positive measurable function in an interval $(0, \varepsilon_0)$. If

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon x)}{c(\varepsilon)} = g(x) \neq 0, \quad \text{in } \mathcal{D}'(\omega) \quad (\text{resp. in } \mathcal{S}'(\mathbb{R}^n)), \tag{2}$$

then it is said that f has the quasiasymptotics at zero with respect to $c(\varepsilon)$ in $\mathcal{D}'(\omega)$, (resp. in $\mathcal{S}'(\mathbb{R}^n)$). We write, $f \overset{q}{\sim} g$ at zero with respect to $c(\varepsilon)$.

For f to have value at zero in the sense of Lojasiewich [5] g needs to be a constant and $c = 1$.

It follows from (2) that

$$c(\varepsilon) = \varepsilon^\nu L(\varepsilon), \quad \varepsilon \in (0, \varepsilon_0). \tag{3}$$

Remark. Let the space dimension be $n = 1$. Then, one can show [9] that (2) implies that the limit distribution is of the form

$$g(x) = C_+ f_{\nu+1}(x) + C_- f_{\nu+1}(-x),\ x \in \mathbb{R}, \tag{4}$$

where $(C_+, C_-) \neq (0,0)$

$$f_{\nu+1}(x) = \begin{cases} H(x)x^\nu/\Gamma(\nu), & \nu > -1 \\ f^{(m)}_{\nu+m+1}(x), & \nu \leq -1,\ \nu + m > -1,\ m \in \mathbb{N}, \end{cases}$$

where H is Heaviside's function and ${}^{(m)}$ is the distributional derivative.

The following two theorems relate the notion of the quasiasymptotics at zero and ∞ in $\mathcal{D}'(\omega)$, and $\mathcal{S}'(\mathbb{R}^n)$. Their proofs are combinations of Theorem 2 in [9] and Lemma 6 in [15], as well as of Theorem 1 in [10] and again Lemma 6 in [15].

Theorem A. *Let $f \in \mathcal{D}'(\omega)$, $f \overset{q}{\sim} g$, at zero in $\mathcal{D}'(\omega)$, with respect to $\varepsilon^\nu L(\varepsilon)$, where $0 \in \omega \subset \mathbb{R}$. Let $\theta \in C_0^\infty$, $\theta = 1$ in $[-s, s]^n \subset \omega$. Then, $f_1 = \theta f \overset{q}{\sim} g$, at zero with respect to $\varepsilon^\nu L(\varepsilon)$, in $\mathcal{S}'(\mathbb{R}^n)$.*

Moreover, if we assume $\nu \notin -\mathbb{N}$, then there exist a continuous function F and $m \in \mathbb{N}_0$, $m + \nu > 0$ such that

$$f_1 = F^{(m)}, \quad \lim_{\varepsilon \to \pm 0} \frac{F(\varepsilon)}{|\varepsilon|^{m+\nu} L(|\varepsilon|)} = (C_+, C_-) \neq (0,0).$$

Theorem B. *Let $f \in \mathcal{D}'(\mathbb{R})$, $c(k)$, $k > 0$ be positive and measurable. Assume that for every $\psi \in \mathcal{D}(\mathbb{R})$ there exists the limit*

$$\lim_{k\to\infty} \langle f(kx)/c(k), \psi(x)\rangle$$

and it is different from zero for some ψ. Then, $f \in \mathcal{S}'(\mathbb{R})$ and this limit exists for every $\psi \in \mathcal{S}(\mathbb{R})$.

Further characterizations of the quasiasymptotics at zero are given in the next proposition.

Proposition 1. *Let $f \in \mathcal{E}'(\mathbb{R})$, $\operatorname{supp} f = K \ni 0$. The following conditions are equivalent:*

(a) $f \overset{q}{\sim} g \neq 0$, *at zero with respect to $c(\varepsilon)$, in $\mathcal{S}'(\mathbb{R})$.*

(b) $f \overset{q}{\sim} g \neq 0$, *at zero with respect to $c(\varepsilon)$, in $\mathcal{D}'(\mathbb{R})$.*

(c)
$$\lim_{\varepsilon\to 0} \frac{(f * \phi_{\varepsilon^2})(\varepsilon x)}{c(\varepsilon)} = g \quad \text{in } \mathcal{S}',\ g \neq 0,$$
where ϕ is the "vision" function.

(d) *For every $\theta \in \mathcal{D}(\mathbb{R})$, $\theta(0) \neq 0$, $f\theta \overset{q}{\sim} \theta(0)g \neq 0$ at zero with respect to $c(\varepsilon)$, in $\mathcal{D}'(\mathbb{R})$.*

Remark. Clearly, if (c) holds, then the limit in (c) exists in $\mathcal{D}'$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from Theorem A.

(a) $\Rightarrow$ (c). Let $\alpha \in \mathcal{S}$, $\eta > 0$ and $\varepsilon > 0$. We have

$$\left\langle \frac{(f * \phi_\eta)(\varepsilon x)}{c(\varepsilon)}, \alpha(x) \right\rangle = \left\langle \frac{f(x)}{\varepsilon c(\varepsilon)}, (\check{\phi}_\eta(t) * \alpha(t/\varepsilon))(x) \right\rangle,$$

which implies, for $\eta = \varepsilon^2$,

$$\left\langle \frac{(f * \phi_{\varepsilon^2})(\varepsilon x)}{c(\varepsilon)}, \alpha(x) \right\rangle = \left\langle \frac{f(\varepsilon x)}{c(\varepsilon)}, \int_{-\infty}^{\infty} \check{\phi}_{\varepsilon^2}(t)\alpha(x - t/\varepsilon)dt \right\rangle$$
$$= \left\langle \frac{f(\varepsilon x)}{c(\varepsilon)}, \psi_\varepsilon(x) \right\rangle,$$

where

$$\psi_\varepsilon(x) = \int \check{\phi}_{\varepsilon^2}(t)\alpha(x - t/\varepsilon)dt = \int \check{\phi}(t)\alpha(x - \varepsilon t)dt, \quad x \in \mathbb{R}.$$

One can easily prove that ψ_ε converges to α in $\mathcal{S}(\mathbb{R})$. This implies

$$\lim_{\varepsilon\to 0}\left\langle \frac{(f*\phi_{\varepsilon^2})(\varepsilon x)}{c(\varepsilon)}, \alpha(x)\right\rangle = \langle g,\alpha\rangle,\ \alpha\in\mathcal{S}(\mathbb{R}).$$

(c) $\Rightarrow$ (a) We will use the following well-known equalities:

$$\mathcal{F}\left(\frac{1}{\varepsilon}f\left(\frac{\cdot}{\varepsilon}\right)\right)(\xi) = \hat{f}(\varepsilon\xi)$$

$$\langle f,\psi\rangle = \langle \hat{f}(\xi),\hat{\psi}(-\xi)\rangle,\quad \psi\in\mathcal{S}(\mathbb{R}),\ f\in\mathcal{S}'(\mathbb{R}).$$

Let θ be an arbitrary element of $\mathcal{D}(\mathbb{R})$ and $\psi\in\mathcal{S}(\mathbb{R})$ such that $\hat{\psi}=\theta\in\mathcal{D}(\mathbb{R})$. Then,

$$\left\langle \frac{(f*\phi_{\varepsilon^2})(\varepsilon x)}{c(\varepsilon)}, \psi(x)\right\rangle = \left\langle \frac{\hat{f}(k\xi)}{c_1(k)}\hat{\phi}(\xi/k), \theta(\xi)\right\rangle,$$

where $c_1(k)=\varepsilon c(\varepsilon)$, $\varepsilon = 1/k$. Since $\hat{\phi}(\xi/k)=1$ on $\operatorname{supp}\theta$ for enough large k, for every $\theta\in\mathcal{D}$

$$\lim_{k\to\infty}\left\langle \frac{\hat{f}(k\xi)}{c_1(k)}, \theta(\xi)\right\rangle = \langle g_1(\xi),\theta(\xi)\rangle. \tag{5}$$

Theorem B implies that (5) holds for every $\theta\in\mathcal{S}(\mathbb{R})$.

Let $\psi=\mathcal{F}(\theta),\theta\in\mathcal{S}(\mathbb{R})$. It follows

$$\lim_{\varepsilon\to 0}\left\langle \frac{f(\varepsilon x)}{c(\varepsilon)}, \phi(x)\right\rangle = \langle \mathcal{F}^{-1}(g_1),\theta\rangle$$

what is assertion (a).

(b) $\Rightarrow$ (d) Since for every $\alpha\in\mathcal{D}(\mathbb{R})$, $\theta(\varepsilon\cdot)\alpha\to\theta(0)\alpha$ in $\mathcal{D}(\mathbb{R})$ the proof is simple.

(d) $\Rightarrow$ (b) Take $\theta\in\mathcal{D}(\mathbb{R})$ such that $\theta(x)=1$ for $|x|<\varepsilon_0$. Then for $\varepsilon<\varepsilon_0$ we have

$$\left\langle \frac{f(\varepsilon x)\theta(\varepsilon x)}{c(\varepsilon)}, \alpha(x)\right\rangle = \left\langle \frac{f(\varepsilon x)\theta(0)}{c(\varepsilon)}, \alpha(x)\right\rangle$$

which implies (b).

4. Quasiasymptotics at zero of Colombeau's generalized functions

Let $\mathcal{K}$ be the set of positive measurable functions defined on $(0,1)$ with the property

$$A^{-1}\varepsilon^p \le c(\varepsilon) \le A\varepsilon^{-p}, \quad \varepsilon \in (0,1)$$

for some $A > 0$ and $p > 0$.

Definition 2. Let $F \in \mathcal{G}(\omega)$. It is said that F has the quasiasymptotics at zero with respect to $c(\varepsilon) \in \mathcal{K}$ if there is F_ε, a representative of F, such that for every $\psi \in \mathcal{D}(\omega)$ and some $s > 0$ there is $C_{\psi,s} \in \mathbb{C}$ such that

$$\lim_{\varepsilon \to 0} \left\langle \frac{F_\varepsilon(\varepsilon s x)}{c(\varepsilon)}, \psi(x) \right\rangle = C_{\psi,s} \tag{6}$$

and $C_{\psi,s} \neq 0$ for some ψ and s.

Remark. If $F \in \mathcal{G}_t(\mathbb{R})$ — the space of Colombeau's tempered generalized functions, then we say that F has the quasiasymptotics in $\mathcal{G}_t$ if (6) holds for every $\psi \in \mathcal{S}(\mathbb{R})$. We will not use this notion and because of that we do not recall the properties of $\mathcal{G}_t(\mathbb{R})$.

It follows from (6) that this limit exists for every $s > 0$ and that for every $s > 0$ there exists ψ such that $C_{\psi,s} \neq 0$.

Note, in general, $F_{s\varepsilon}$, $s \neq 1$, is not a representative of $F = [F_\varepsilon]$.

The consequences of this definition are given in the next proposition.

Proposition 2.

(a) *Let $R_\varepsilon \in \mathcal{N}(\omega)$. Then, for every $c \in \mathcal{K}$, $s > 0$ and every $\psi \in \mathcal{D}(\omega)$,*

$$\lim_{\varepsilon \to 0} \left\langle \frac{R_\varepsilon(\varepsilon s x_1, \dots, \varepsilon s x_n)}{c(\varepsilon)}, \psi(x_1, \dots, x_n) \right\rangle = 0.$$

(b) *Let $F \in \mathcal{G}(\omega)$, $c \in \mathcal{K}$ and let* (6) *hold. Then,* (6) *holds for every representative $\tilde{F}_\varepsilon$ of F.*

(c) *If* (6) *holds, then there exists $g \in \mathcal{D}'(\omega)$ such that for $s = 1$*

$$C_{\psi,1} = C_\psi = \langle g, \psi \rangle, \quad \psi \in \mathcal{D}(\omega).$$

Proof. (a) It follows from the assumption on $c \in \mathcal{K}$, (b) follows from (a) and (c) follows from the Banach–Steinhaus theorem.

According to Proposition 2 (c), for Colombeau's generalized functions we will write

$$F \overset{q.c.}{\sim} g \text{ at zero with respect to } c(\varepsilon).$$

Proposition 3.

(a) *Let $f \in \mathcal{E}'(\mathbb{R}^n)$, and $f \overset{q}{\sim} g$ at zero with respect to $c(\varepsilon)$. Then*

$$I_\phi(f) \overset{q.c.}{\sim} g \text{ at zero with respect to } c(\varepsilon). \tag{7}$$

Conversely, if $f \in \mathcal{E}'(\mathbb{R}^n)$ and (7) holds, then $f \overset{q}{\sim} g$ at zero with respect to $c(\varepsilon)$.

(b) *Let $F \overset{q.c.}{\sim} g$ at zero with respect to $c(\varepsilon)$. Then, for every $\psi \in C^\infty$*

$$\psi F \overset{q.c.}{\sim} \psi(0) g$$

at zero with respect to $c(\varepsilon)$.

Proof. (a) Recall, the representative of $I_\phi(f)$ is $f * \phi_{\varepsilon^2}$. Then Proposition 1 (c) implies the assertion in both directions.

(b) Proposition 1 (d) implies the assertion.

Examples. Recall that (2) implies (3) and (4), but it is not true in general in $\mathcal{G}$. The existence of the quasiasymptotics at zero for an element of $\mathcal{G}$ does not imply that $c(\varepsilon)$ must be of the given form as well as g.

Example 1. We consider the δ^2-potential (cf. [7]) determined by

$$\Theta_{\phi,\varepsilon}(x) = \frac{1}{\varepsilon^2}\phi^2\left(\frac{x}{\varepsilon}\right), \ \phi \in C_0^\infty, \text{ with } \operatorname{supp}\phi \subset [-1,1], \ \int \phi(x)dx = 1,$$

as the example of Colombeau's generalized function (element of $\mathcal{G}$) which is not obtained by embedding of a distribution in $\mathcal{G}$ (i.e. which is not generated by some distribution). We shall prove that $\left[\frac{1}{\varepsilon^2}\phi^2\left(\frac{x}{\varepsilon}\right)\right]$ has the quasiasymptotics at zero with respect to $c(\varepsilon) = \varepsilon^{-2}$. It holds

$$\begin{aligned} \lim_{\varepsilon\to 0}\left\langle \frac{\Theta_{\phi,\varepsilon}(\varepsilon x)}{\varepsilon^{-2}}, \psi(x)\right\rangle &= \lim_{\varepsilon\to 0}\int \phi^2(x)\psi(x)dx \\ &= \langle \phi^2, \psi\rangle = C_{\psi,1}, \quad C_{\psi,1} \in \mathbb{C}. \end{aligned} \tag{8}$$

Thus, $[\Theta_{\phi,\varepsilon}]$ has the quasiasymptotics at zero in the sense of Definition 2 with respect to ε^{-2}. The limit distribution in (8) is not of the form Cx^α.

Example 2. The generalized function $\left[\left(2+\sin\frac{1}{\varepsilon}\right)\phi^2\left(\frac{x}{\varepsilon}\right)\right]$ has the quasiasymptotics at zero with respect to $c(\varepsilon)=2+\sin\frac{1}{\varepsilon}$, but it is not of the form $c(\varepsilon)=\varepsilon^{\alpha}L(\varepsilon)$. Also $\left\langle\frac{1}{c(\varepsilon)}\left(2+\sin\frac{1}{\varepsilon}\right)\phi^2(x),\psi(x)\right\rangle\to\langle g(x),\psi(x)\rangle$, $\psi\in\mathcal{D}$ and g is not of the form Cx^{α}, $C\neq 0$.

Example 3. The generalized function $\frac{1}{\varepsilon^2}\phi^2\left(\frac{x}{\varepsilon^2}\right)$ has the quasiasymptotics with the limit $\delta\int\phi^2(t)dt$ with respect to $c(\varepsilon)=\varepsilon^{-1}$ because

$$\left\langle\frac{1}{\varepsilon^{-1}}\frac{1}{\varepsilon^2}\phi^2\left(\frac{\varepsilon x}{\varepsilon^2}\right),\psi(x)\right\rangle=\left\langle\frac{1}{\varepsilon}\phi^2\left(\frac{x}{\varepsilon}\right),\psi(x)\right\rangle$$
$$=\langle\phi^2(t),\psi(\varepsilon t)\rangle=\psi(0)\int\phi^2(t)dt.$$

Example 4. Let $F,G\in\mathcal{G}(\omega)$. If $F\overset{q.c.}{\sim}g$ at zero with respect to $c(\varepsilon)$ and $G\overset{q.c.}{\sim}g_1$, at zero with respect to $c_1(\varepsilon)$, then it is not true, in general, that $GF\overset{q.c.}{\sim}gg_1$ at zero with respect to $c_2(\varepsilon)=c(\varepsilon)c_1(\varepsilon)$. For example, $\frac{1}{\varepsilon^2}\phi\left(\frac{-x}{\varepsilon^2}\right)\overset{q.c.}{\sim}\delta$ with respect to $c(\varepsilon)=\varepsilon^{-1}$ but $\frac{1}{\varepsilon^4}\phi^2\left(\frac{-x}{\varepsilon^2}\right)\overset{q.c.}{\sim}(\int\phi^2(t)dt)\delta$ with respect to $c(\varepsilon)=\varepsilon^{-3}$.

Example 5. The $\sqrt{\delta}$-potential (cf. [7]) determined by

$$\Theta_{\psi,\varepsilon}(x)=\frac{1}{\sqrt{\varepsilon}}\psi\left(\frac{x}{\varepsilon}\right),\ \psi\in C_0^{\infty}\text{ with }\operatorname{supp}\psi\subset[-1,1],\ \int\psi^2(x)dx=1$$

is associated with the zero distribution. It has the quasiasymptotics at zero in the sense of Definition 2 with respect to $c(\varepsilon)=\varepsilon^{-1/2}$.

$$\lim_{\varepsilon\to 0}\left\langle\frac{\Theta_{\psi,\varepsilon}(\varepsilon x)}{\varepsilon^{-1/2}},\eta(x)\right\rangle=\lim_{\varepsilon\to 0}\int\psi(x)\eta(x)dx=C_{\eta,1},\quad\eta\in\mathcal{D},\ C_{\eta,1}\neq 0.$$

Example 6. We can find the quasiasymptotic behaviour in the case of a more general potential determined by

$$\Theta_{\psi,\varepsilon}(x)=\frac{1}{\mu(\varepsilon)}\psi\left(\frac{x}{\nu(\varepsilon)}\right),\ \text{(cf. [7])},$$

where $\psi\in\mathcal{D}$, $\operatorname{supp}\psi\in[-1,1]$ and $\sup_{x\in\mathbb{R}}\psi(x)>0$, $\mu(\varepsilon)\to 0$, $\nu(\varepsilon)\to 0$ as $\varepsilon\to 0$. In general, such potentials are not the images of distributions in $\mathcal{G}(\mathbb{R})$.

Let $c(\varepsilon)=\nu(\varepsilon)(\varepsilon\mu(\varepsilon))^{-1}$. Then,

$$\lim_{\varepsilon\to 0}\left\langle \frac{\Theta(\varepsilon x)}{c(\varepsilon)},\eta(x)\right\rangle = \lim_{\varepsilon\to 0}\int \frac{\varepsilon}{\nu(\varepsilon)}\psi\left(\frac{\varepsilon x}{\nu(\varepsilon)}\right)\eta(x)dx$$

$$\lim_{t\to 0}\int \psi(t)\eta\left(\frac{\nu(\varepsilon)}{\varepsilon}t\right)dt = \int \psi(x)\eta_1(x)dx = C_{\eta_1,1}, \quad C_{\eta_1,1}\in\mathbb{C},\ \eta\in\mathcal{D},$$

if $\frac{\mu(\varepsilon)}{\varepsilon}$ has a limit.

5. Application

Let $(x,t,u)\mapsto F_i(x,t,u)$, $i=1,\dots,n$ be smooth functions on $\mathbb{R}^{2n+2}$ such that the following conditions hold:

(9) $\mathbb{C}^n\ni u\mapsto F_i(x,t,u)$, $i=1,\dots,n$, is polynomially bounded together with all derivatives, uniformly for $(x,t)\in K$, for any compact set $K\subset\mathbb{R}^2$;

(10) $\mathbb{C}^n\ni u\mapsto \nabla_u F_i(x,t,u)$, $i=1,\dots,n$, is globally bounded uniformly with respect to $(x,t)\in K$, for any compact set $K\subset\mathbb{R}^2$.

Let K_0 be a compact set such that the interior of K_0, $\overset{\circ}{K}_0$, contains 0. We denote by K_T a domain of determinancy bounded by extremal characteristics emanating from the end points of K_0 and the lines $t=\pm T$.

The Cauchy problem for a semilinear strictly hyperbolic $(n\times n)$-system in two independent variables, $(x,t)\in\mathbb{R}^2$

$$(\partial_t+\Lambda(x,t)\partial_x)u(x,t)=F(x,t,u(x,t)) \tag{11}$$
$$u(x,0)=(u_1(x,0),\dots,u_n(x,0))=(a_1(x),\dots,a_n(x))\in(\mathcal{G}(\mathbb{R}))^n,$$

where $\Lambda(x,t)$ is a diagonal matrix with the real distinct smooth functions on the diagonal and F_i satisfy conditions (9) and (10), is uniquely solvable in $(\mathcal{G}(K_T))^n$, for some $T>0$ (cf. [6]).

The integral curves for (11), which pass through (x_0,t_0) at time $\tau=t_0$, are the solutions to

$$\frac{\partial}{\partial\tau}\gamma_i(x_0,t_0,\tau)=\lambda_i(\gamma_i(x_0,t_0,\tau),\tau),\ \gamma_i(x_0,t_0,t_0)=x_0.$$

They are denoted by $x = \gamma_i(x_0, t_0, \tau)$, $i \in \{1, \dots, n\}$, and called characteristic curves of the system. Then,

$$u_i(x,t) = a_i(\gamma_i(x,t,0)) + \int_0^t F_i(\gamma_i(x,t,\tau), \tau, u(\gamma_i(x,t,\tau), \tau))d\tau,$$
$$(x,t) \in K_T.$$

The next proposition shows that the quasiasymptotic behaviour of initial data to (11) implies the quasiasymptotic behaviour of the solution.

Proposition 4. *Let* $c(\varepsilon) \in \mathcal{K}$, $\lim_{\varepsilon\to 0} \frac{\varepsilon}{c(\varepsilon)} = 0$ *and* $\lim_{\varepsilon\to 0} \frac{a_{i\varepsilon}(\gamma_i(\varepsilon s x, \varepsilon s t, 0))}{c(\varepsilon)}$, $i = 1, \dots, n$, *exist in* $\mathcal{D}'(\mathbb{R}^2)$. *for some* $s > 0$. *Then, the solution* $u(x,t) = [(u_{1\varepsilon}(x,t), \dots, u_{n\varepsilon}(x,t))]$ *has the quasiasymptotics at zero with respect to* $c(\varepsilon)$, *i.e.*

$$\lim_{\varepsilon\to 0} \left\langle \frac{u_{i\varepsilon}(\varepsilon s x, \varepsilon s t)}{c(\varepsilon)}, \psi(x,t) \right\rangle = C_{i,\psi,s}, \tag{12}$$
$$C_{i,\psi,s} \in \mathbb{C},\ \psi \in \mathcal{D}(\mathbb{R}^2),\ i = 1, \dots, n,$$

for some $s > 0$ *(which implies, for every* s*) provided one of the following conditions hold:*

(a) $\mathbb{C}^n \ni u \mapsto F_i(x,t,u)$, *uniformly bounded for* $(x,t) \in K$ *for any compact set* $K \subset \mathbb{R}^2$, $i = 1, \dots, n$*;*

(b)

$$\frac{\varepsilon}{c(\varepsilon)} \sup_{(x,t)\in K} |a_\varepsilon(\gamma_i(\varepsilon s x, \varepsilon s t, 0))| \to 0, \quad \varepsilon \to 0,\ i = 1, \dots, n,$$

for any compact set $K \subset\subset \mathbb{R}^2$.

Proof. Let $\psi \in \mathcal{D}(\mathbb{R}^2)$, $\operatorname{supp}\psi \subset K$, K_0 and T be choosen so that $K \subset\subset \overset{\circ}{K}_T$. The representative of the solution to (11) $(u_{1\varepsilon}, \dots, u_{n\varepsilon})$ belongs $(\mathcal{E}_M(\overset{\circ}{K}_T))^n$ and satisfies

$$u_{i\varepsilon}(\varepsilon s x, \varepsilon s t) = a_{i\varepsilon}(\gamma_i(\varepsilon s x, \varepsilon s t, 0)) + \int_0^{\varepsilon s t} F_i(\gamma_i(\varepsilon s x, \varepsilon s t, \tau), \tau, u_\varepsilon(\gamma_i(\varepsilon s x, \varepsilon s t, \tau), \tau))d\tau, \tag{13}$$

when $(x,t) \in K_T$, and $i = 1, \dots, n$.

First, we shall give the estimate for $u_{i\varepsilon}(\varepsilon sx, \varepsilon st)$, then the estimate for the integral part in (13), and finally prove the assertion (12).

Putting $F(x,t,u) = F(x,t,0) + \nabla_u F(x,t,\theta u)u$, with $0 \le \theta \le 1$, in (13) we obtain

$$u_{i\varepsilon}(\varepsilon sx, \varepsilon st) = a_{i\varepsilon}(\gamma_i(\varepsilon sx, \varepsilon st, 0)) + \int_0^{\varepsilon st} F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau, 0)d\tau$$
$$\int_0^{\varepsilon st} (u_{1\varepsilon}(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau), \dots, u_{n\varepsilon}(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau))$$
$$\nabla_u F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau, \theta(\tau) u_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau))d\tau, \;\; 0 \le \theta(\tau) \le 1.$$

Gronwall's inequality and assumptions (10) imply that there exist $C > 0$ and $\varepsilon_0 > 0$ such that for $t \in (-T, T)$, $\varepsilon \in (0, \varepsilon_0)$

$$\text{(14)} \qquad \sup_{(x,y)\in K_T} |u_\varepsilon(\varepsilon sx, \varepsilon st)| \le \Big\{ \sup_{(x,t)\in K_T} |a_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, 0))|$$
$$+ |\varepsilon sT| \sup_{(x,t)\in K_T} |F(x,t,0)| \Big\}$$
$$\exp\Big(n\varepsilon sT \sup_{\substack{(x,t)\in K_T \\ u\in\mathbb{C}^n}} |\nabla_u F_i(x,t,u_\varepsilon)| \Big)$$
$$\le C\Big(\sup_{(x,t)\in K_T} |a_{i\varepsilon}(\gamma_i(\varepsilon sx, \varepsilon st, 0))| + \varepsilon \Big).$$

We shall estimate the integral part of (13). Let $(x,t) \in K_T$. We have

$$\int_0^{\varepsilon st} |F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau, u_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau))| d\tau$$
$$\le \int_0^{\varepsilon st} |F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau, 0)| d\tau$$
$$+ \int_0^{\varepsilon st} |u_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau)| d\tau \sup_{\substack{(x,t)\in K_T \\ u\in\mathbb{C}^n}} |\nabla_u F_i(x,t,u)|$$
$$\le |\varepsilon sT| \Big\{ \sup_{(x,t)\in K_T} |F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau, 0)|$$
$$+ C \sup_{(x,t)\in K_T} |u_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau))| \Big\}.$$

Now, (14) implies that there exists $C > 0$ such that

$$\tag{15} \int_0^{\varepsilon st} |F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau, u_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau))| d\tau$$
$$\leq C\varepsilon\Big(\sup_{(x,t)\in K_T} |a_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, 0))| + 1 + \varepsilon\Big),$$

when $t \in (-T, T),\ \varepsilon \in (0, \varepsilon_0)$.

We have

$$\left\langle \frac{u_{i\varepsilon}(\varepsilon sx, \varepsilon st)}{c(\varepsilon)}, \psi(x,t) \right\rangle = \left\langle \frac{a_{i\varepsilon}(\gamma_i(\varepsilon sx, \varepsilon st, 0))}{c(\varepsilon)}, \psi(x,t) \right\rangle$$
$$+ \iint\Big(\frac{1}{c(\varepsilon)} \int_0^{\varepsilon st} F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau, u_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau)) d\tau\Big)\, \psi(x,t) dx dt.$$

In case (a) we immediately obtain the assertion since the double integral on the right-hand-side tends to zero as $\varepsilon \to 0$.

Let us prove the assertion with the assumption in case (b). By (15) we have

$$\frac{1}{c(\varepsilon)} \int_0^{\varepsilon st} |F_i(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau, u_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, \tau), \tau))| d\tau$$
$$\leq C\left(\varepsilon \frac{\sup_{(x,t)\in K_T} |a_\varepsilon(\gamma_i(\varepsilon sx, \varepsilon st, 0))|}{c(\varepsilon)} + \frac{\varepsilon}{c(\varepsilon)} + \frac{\varepsilon^2}{c(\varepsilon)}\right) \to 0, \text{ as } \varepsilon \to 0.$$

Thus,

$$\lim_{\varepsilon\to 0} \left\langle \frac{u_{i\varepsilon}(\varepsilon sx, \varepsilon st)}{c(\varepsilon)}, \psi(x,t) \right\rangle = \lim_{\varepsilon\to 0} \left\langle \frac{a_{i\varepsilon}(\gamma_i(\varepsilon sx, \varepsilon st, 0))}{c(\varepsilon)}, \psi(x,t) \right\rangle.$$

□

Consider a Cauchy problem

$$\tag{16} u'(t) = F(t,u),\ u(0) = a = [a_\varepsilon] \in \overline{\mathbb{C}},$$

where $(t,u) \mapsto F(t,u)$ is a smooth function on $\mathbb{R}^2$ such that (9) and (10) hold for F (with x canceled). It is uniquely solvable in $\mathcal{G}(-T,T)$ for some $T > 0$ (cf. [6]).

For the behaviour of the solution to (16) we can use somewhat stronger concept of asymptotic behaviour at zero since the initial data does not depend on x.

Definition 3. Let $F \in \mathcal{G}(\omega)$. It is said that F has the strong quasi-asymptotics at zero with the limit $g \in C^{\infty}(\omega)$ with respect to $c(\varepsilon) \in \mathcal{K}$ if there exists F_ε, a representative of F, such that for every $K \subset\subset \mathbb{R}$ and some $s > 0$

$$\lim_{\varepsilon \to 0} \frac{F_\varepsilon(\varepsilon s x)}{c(\varepsilon)} = g(x) \quad \text{uniformly for } x \in K.$$

We have the following Proposition.

Proposition 5. *Let $c(\varepsilon) \in \mathcal{K}$, and $a = [a_\varepsilon] \in \overline{\mathbb{C}}$ such that*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{c(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{a_\varepsilon}{c(\varepsilon)} = 1.$$

Then the solution $u(t) \in \mathcal{G}(-T, T)$ to the Cauchy problem (16) satisfies $\lim_{\varepsilon \to 0} \frac{u_\varepsilon(\varepsilon s t)}{c(\varepsilon)} = 1$ uniformly in $t \in K$, for some $s > 0$, where K is an arbitrary compact set of $(-T, T)$.

Proof. There holds

$$\text{(17)} \qquad \frac{u_\varepsilon(\varepsilon s t)}{c(\varepsilon)} = \frac{a_\varepsilon}{c(\varepsilon)} + \frac{1}{c(\varepsilon)} \int_0^{\varepsilon s t} F(\tau, u_\varepsilon(\tau)) d\tau, \quad t \in (-T, T).$$

Condition (10) and Gronwall's inequality [4] imply that there exist $C > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in (-T,T)} |u_\varepsilon(\varepsilon s t)| \leq C(|a_\varepsilon| + \varepsilon), \quad \varepsilon \in (0, \varepsilon_0).$$

This implies that there exists $C > 0$ such that

$$\left| \int_0^{\varepsilon s t} F(\tau, u_\varepsilon(\tau)) d\tau \right| \leq C\varepsilon(1 + |a_\varepsilon| + \varepsilon), \quad t \in (-T, T),\ \varepsilon \in (0, \varepsilon_0).$$

The last summand in (17) is then

$$\frac{1}{c(\varepsilon)} \left| \int_0^{\varepsilon t} F(\tau, u_\varepsilon(\tau)) d\tau \right| \leq C \left(\varepsilon \frac{|a_\varepsilon|}{c(\varepsilon)} + \frac{\varepsilon}{c(\varepsilon)} + \frac{\varepsilon^2}{c(\varepsilon)} \right),$$

$$t \in (-T, T),\ \varepsilon \in (0, \varepsilon_0).$$

Because $\frac{a_\varepsilon}{c(\varepsilon)} \to 1$ and $\frac{\varepsilon}{c(\varepsilon)} \to 0$, as $\varepsilon \to 0$, the second summand in (17) tends to zero and

$$\lim_{\varepsilon\to 0} \frac{u_\varepsilon(\varepsilon st)}{c(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{c(\varepsilon)} = 1.$$

□

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