On certain properties of characters determined by centralizers*

By ERZSÉBET HORVÁTH (Budapest)

Dedicated to Professor Zoltán Daróczy on his 50th birthday

The aim of this paper is to prove monomiality and some more general properties of characters from certain assumptions on centralizers of elements in a finite group. As a consequence we get some classes of finite solvable groups where the derived length problem $\mathrm{d.l.}(G) \leqq |\mathrm{c.d.}(G)|$ can be solved positively. Throughout the paper we widely use methods, results and notations of [1] and [2].

Hypothesis 1.1. Let G be a finite group. If $M \lhd G$ and $M \leqq \mathrm{Ker}\,\varphi$ for every $\varphi \in \mathrm{Irr}(G)$ such that $\varphi(1) < \chi(1)$, where $\chi \in \mathrm{Irr}(G)$, then $M' \leqq \mathrm{Ker}\,\chi$.

Remark 1.2. It is easy to see that every M-group satisfies Hypothesis 1.1. In Theorem 2.2 of [2] BERGER proved that every finite solvable group of odd order also satisfies it. It can be easily proved by induction that if G satisfies Hypothesis 1.1 then $\mathrm{d.l.}(G) \leqq |\mathrm{c.d.}(G)|$.

In the following we shall need

Lemma 1.3. *Let $G = G_1 \times G_2$ be a direct procudt of finite groups. Let us suppose that G_1 and G_2 satisfy Hypothesis* 1.1. *Then G satisfies it as well.*

PROOF. Let $\chi \in \mathrm{Irr}(G)$, $M \lhd G$ such that for every $\varphi \in \mathrm{Irr}(G)$ satisfying $\varphi(1) < \chi(1)$, $M \leqq \mathrm{Ker}\,\varphi$. We have to prove that $M' \leqq \mathrm{Ker}\,\chi$. We may suppose that $\chi(1) > 1$. Let $\chi = \chi_1 \cdot \chi_2$, where $\chi_1 \in \mathrm{Irr}(G_1)$ and $\chi_2 \in \mathrm{Irr}(G_2)$. Let us suppose that one of the χ_i-s, e.g. χ_2 is linear. Let us denote by $\pi_i(M)$ for $i = 1, 2$ the image of M at the projection $\pi_i \colon G \to G_i$, e.g. $\pi_1(M) = \{g_1 \in G_1 \mid \exists g_2 \in G_2 \text{ such that } g_1 g_2 \in M\}$. Let $\psi_1 \in \mathrm{Irr}(G_1)$ such that $\psi_1(1) < \chi_1(1)$. Then $(\psi_1 \cdot 1_{G_2})(1) < \chi(1)$, so by our hypothesis $M \leqq \mathrm{Ker}(\psi_1 \cdot 1_{G_2})$, which gives that $\pi_1(M) \leqq \mathrm{Ker}\,\psi_1$. As Hypothesis 1.1 is valid in G_1 we have that $\pi_1(M)' \leqq \mathrm{Ker}\,\chi_1$. As χ_2 is linear, $\pi_2(M)' \leqq G_2' \leqq \mathrm{Ker}\,\chi_2$, so as $M \leqq \pi_1(M) \times \pi_2(M)$ we have that $M' \leqq \pi_1(M)' \times \pi_2(M)' \leqq \mathrm{Ker}\,\chi$. We would get a similar result if χ_1 would be linear. So we may assume that $\chi_i(1) < \chi(1)$ for $i = 1, 2$. According to our hypothesis then $M \leqq \mathrm{Ker}(\chi_1 \cdot 1_{G_2})$ so $\pi_i(M) \leqq \mathrm{Ker}\,\chi_i$ for $i = 1, 2$. So $M' \leqq M \leqq \pi_1(M) \times \pi_2(M) \leqq \mathrm{Ker}\,\chi_1 \times \mathrm{Ker}\,\chi_2 \leqq \mathrm{Ker}\,\chi$.

Lemma 1.4. *Let G be a finite group. Let us suppose that for every $x \in G^\# C_G(x)$ is the direct procudt of a group of odd order and of a 2-group. Then this property is inherited to every homomorphic image of G.*

* Research partially supported by Hungarian National Foundation for Scientific Research grant no. 1813.

Proof. Follows directly from Lemma 1 and Theorem B of [1].

Theorem 1.5. *Let G be a finite solvable group where for every $x \in G^\# C_G(x)$ is the direct product of a group of odd order and of a 2-group. Then G satisfies Hypothesis* 1.1, *in particular* d.l. $(G) \leqq |$c.d. $(G)|$.

Proof. Let G be a couterexample of minimal order. Let $\chi \in \operatorname{Irr}(G)$, $M \lhd G$ such that $M \leqq \operatorname{Ker} \varphi$ for every $\varphi \in \operatorname{Irr}(G)$ satisfying $\varphi(1) < \chi(1)$ and $M' \nleqq \operatorname{Ker} \chi$. As the hypotheses of the theorem are valid in every subgroup of G, we can prove, similarly as in Theorem 2.2 of [2], that we may assume that χ is primitive. As we have seen in Lemma 1.4, the hypotheses of the theorem are inherited to every homomorphic image of G. So we may assume that $\operatorname{Ker} \chi = 1$. As G is solvable, $1 \neq Z(F(G)) \leqq Z(G)$, so G is the direct product of a 2-group and of a group of odd order. By Remark 1.2 and Lemma 1.3, Hypothesis 1.1 is satisfied in G, which is a contradiction. The second statement follows also from Remark 1.2.

Corollary 1.6. *Let G be a finite solvable group and let us suppose that for every $x \in G^\# C_G(x)$ has an abelian normal* 2-*complement, then G satisfies Hypothesis* 1.1 *and so* d.l. $(G) \leqq |$c.d. $(G)|$.

Proof. Let G be a counterexample of minimal order, $\chi \in \operatorname{Irr}(G)$, $M \lhd G$ such that $M \leqq \operatorname{Ker} \varphi$ for every $\varphi \in \operatorname{Irr}(G)$ satisfying $\varphi(1) < \chi(1)$ and $M' \nleqq \operatorname{Ker} \chi$. As every subgroup satisfies the hypothesis of the corollary, we may assume, as in the above proof, that χ is primitive. By Lemma 1 and Theorem B of [1] we have that $C_{\bar{G}}(\bar{x})$ is 2-nilpotent for every $\bar{x} \in \bar{G}^\#$ in an arbitrary homomorphic image $\bar{G}$ of G. Let $p > 2$ prime, $P \in \operatorname{Syl}_p(G)$. If $1 \neq x \in Z(P)$ then $C_G(x) \geqq P$ so $P' = 1$. If $p, q > 2$ primes then $(p, q) \nmid C_G(x)$ for every $x \in G^\#$ so by Theorem B of [1] $(p, q) \nmid C_{\bar{G}}(\bar{x})$ for every $\bar{x} \in \bar{G}^\#$. So the normal 2-complement of $C_{\bar{G}}(\bar{x})$ is nilpotent and so it is abelian. So the hypothesis of the corollary is inherited to every homomorphic image of G. So we may assume that $\operatorname{Ker} \chi = 1$. As above we have that $Z(G) \neq 1$, so G has an abelian normal 2-complement which is in the centre. So G is the direct product of a 2-group and of a group of odd order and by Theorem 1.5 we have a contradiction.

Corollary 1.7. *Let G be a finite solvable group. Let us suppose that for every $x \in G^\#$ the* 2-*Sylow subgroup of $C_G(x)$ is normal and abelian. Then G satisfies Hypothesis* 1.1 *and so* d.l. $(G) \leqq |$c.d. $(G)|$.

Proof. The proof is similar to that of Corollary 1.6.

In the following we shall prove a generalization of Corollary 2 and 3 of [1].

Definition 2.1. Let G be a finite group, π a set of primes. We say that G has property T_π with respect to the partition $\pi = \bigcup_1^k \pi_i$ if the following conditions are satisfied:

a) $\pi_i \cap \pi_j = \emptyset$ for $i \neq j$

b) $|\pi_i| \leqq 2$ for $i = 1, \ldots, k$

c) G has a normal π-complement K and $G/K = \bigtimes_1^k A_i$, where $A_i \in \operatorname{Hall}_{\pi_i}(G/K)$ and all A_i are supersolvable.

Theorem 2.2. *Let G be a finite π-solvable group,* $\pi=\bigcup_{1}^{k}\pi_i$, $\pi_i\cap\pi_j=\emptyset$ for $i\neq j$ *and* $|\pi_i|\leqq 2$ *for* $i=1, \dots, k$. *Let us suppose that for every π-element* $x\in G^{\#}C_G(x)$ *has property T_π with respect to this partition. Then this property is inherited to every homomorphic image of G.*

PROOF. Let G be a couterexample of minimal order, $H\lhd G$ such that $C_{\bar{G}}(\bar{x})$ does not have property T_π, where $\bar{G}=G/H$ and $\bar{x}\in\bar{G}^{\#}$ is a π-element.

1. $C_{\bar{G}}(\bar{x})$ has a normal π-complement K_2 by Theorem B and Lemma 1 of [1].

2. As in the proof of Theorem B of [1] we may assume that H is a minimal normal subgroup, $|x|$ is prime power, $|\bar{x}|$ is prime, $\bar{x}\in Z(G/H)$, $H\leqq\Phi(G)$, $(|x|, |H|)\neq 1$ and $B=\langle x, H\rangle$ is elementary abelian p-group for some prime p.

3. We may suppose that $C_G(x)\geqq T\in \mathrm{Hall}_{p'}(G)$:

By Theorem D7* in [3] if G is π-solvable then for every $\pi_1\subseteq\pi$ G has property D_{π_1}, so there exists a $T\in\mathrm{Hall}_{p'}(G)$. By the theorem of Maschke $B=H\oplus Y$, where Y is a T-invariant complement to H in B. We may suppose that $\langle x\rangle=Y$ so $[x, T]\leqq$ $\leqq\langle x\rangle\cap H=1$.

4. Let $B\leqq P\in\mathrm{Syl}_p(G)$. Then $H\cap Z(P)\neq 1$ and we may assume that $x\notin Z(P)$:

The first statement is obvious. If $x\in Z(P)$ then by Step 3. $C_G(x)\geqq PT=G$ so G would have property T_π and so would $\bar{G}=C_{\bar{G}}(\bar{x})$.

5. If $p\in\pi_i$ then $|\pi_i|=2$ and p is the bigger prime in π_i. Let $K\in\mathrm{Hall}_{\pi'}(G)$, then $PK\lhd G$:

Let K_1 be the normal π-complement in $C_G(x)$. If $|\pi_i|=1$ or p is the smaller prime then $C_G(x)/K_1$ and also $C_G(x)$ has a normal p-complement for every p-element $x\in G^{\#}$. By Theorem B of [1] $C_{\bar{G}}(\bar{x})$ also has this property. By the Frattini argument we have that $T\lhd G$ and so $C_G(H\cap Z(P))\geqq PT=G$ has property T_π, contradiction. So $|\pi_i|=2$ and p is the bigger prime. Then the Sylow p-subgroup of $C_G(x)/K_1$ is normal for every π-element $x\in G^{\#}$. So the $\{\pi', p\}$ Hall-subgroup of $C_G(x)$ is normal for every π-element $x\in G^{\#}$ and by Theorem B of [1] the $\{\pi', p\}$ Hall-subgroup of G/H is also normal, so we have $PK\lhd G$.

6. Let $K_2\in\mathrm{Hall}_{\pi'}(\bar{G})$, then $\bar{G}/K_2\cong\bigtimes_{1}^{k}B_i$, where $\pi(B_i)\subseteq\pi_i$ and $\pi=\bigcup_{1}^{k}\pi_i$ is the above partition:

Let us suppose that it is not true, then $C_{\bar{G}}(\bar{x})/K_2\geqq(r, s)$ for some $r\in\pi_j$, $s\in\pi_m j\neq m$. By Theorem A in [1] then $G/H\geqq(r, s)$ and $G\geqq(r, s)$. We can distinguish three cases: a) $\{r, s\}\not\ni p$, b) $r=p$, c) $s=p$.

In the case a) we apply property D_p, and we have $(r, s)\leqq T\leqq C_G(x)$, contradiction.

In the case b) if there exists a $t\in\pi$, $t\notin\{p, s\}$ then because of property D_t, there exists a $U\in\mathrm{Hall}_{t'}(G)$ such that U/H contains a (p, s)-subgroup of G/H. As $P\leqq U<G$, by the inductive hypothesis $U/H=C_{U/H}(\bar{x})$ has property T_π in contradiction with the existence of a (p, s)-subgroup.

If $\pi=\{p, s\}$ then if $\pi'=1$ then as $p\in\pi_j$, $s\in\pi_m j\neq m$, $C_G(x)$ is nilpotent for every π-element $x\in G^{\#}$ and so by Theorem B in [1] G/H is nilpotent, contradiction.

If $\pi'\neq 1$ then let $U\in\mathrm{Hall}_\pi(G)$ such that U/H contains a (p, s)-subgroup of G/H, so by induction we have contradiction as above. In the case c) let $RS\leqq G$ be a subgroup of type (r, s). Then $[R, S]\leqq R\cap KP=1$, contradiction.

7. $\bar{G}/K_2$ is supersolvable:

Let us consider $\bar{G}/K_2/\langle\bar{x}\rangle=\bigtimes_1^k \bar{B}_i$. If there are at least two direct components, then by induction $\bar{G}/K_2/\langle\bar{x}\rangle$ and so $\bar{G}/K_2$ are supersolvable. If there is only one direct component, then we can distinguish two cases: a) $p\notin\pi(\bar{B}_1)$, b) $p\in\pi(\bar{B}_1)$.

Case a) means that $\langle x, H\rangle\in \mathrm{Syl}_p(G)$ abelian and we are done by Step 4.

In the case b) we are done if $|\pi(\bar{B}_1)|=1$. Otherwise we have that $|\pi(\bar{B}_1)|=2$ and so $|\pi|=2$. Let $\bar{G}=K_2\bar{N}$, where $\bar{x}\in\bar{N}\in\mathrm{Hall}_\pi(\bar{G})$. Then $G=KN$, where $K\in\mathrm{Hall}_{\pi'}(G)$ and N is the inverse image of $\bar{N}$ in G. Then $KH\lhd G$ and by the Frattini argument $K\lhd G$. If $K\neq 1$ then we apply induction on N to have that $\bar{N}=\bar{G}/K_2$ is supersolvable. Otherwise $N=G$ and we are done by Theorem C of [1].

Corollary 2.3. *If G is a π-solvable group and for every π-element $x\in G^\# C_G(x)$ has property T_π for a fixed partition of π then every π-character of* Irr (G) *is monomial.*

Proof. Let G be a counterexample of minimal order, $\chi\in\mathrm{Irr}(G)$ is a non-monomial π-character. By Theorem 2.2 we may suppose that $\mathrm{Ker}\,\chi=1$. As our conditions are hereditary to subgroups we can assume that χ is primitive. Let $K=O_{\pi'}(G)$ then $\chi|K=e\vartheta$ and $\vartheta(1)=1$. As $\mathrm{Ker}\,\vartheta\geqq K'$ K has to be abelian. If $F(G)_\pi\neq 1$ then $1\neq Z(O_p(G))\leqq Z(G)$ for some $p\in\pi$. So G has property T_π. Then G/K is supersolvable, so by Theorem 6.22 and 6.23 of [4] G is an M-group. If $F(G)_\pi=1$ then $F(G)\leqq K\leqq Z(G)\leqq F(G)$. By the π-solvability of G there exists a $p\in\pi$ such that $O_p(G/Z(G))\neq 1$. Its inverse image is a nilpotent normal subgroup in G in contradiction with $F(G)_\pi=1$.

Corollary 2.4. *Let G be solvable, $\pi=\pi(G)$ and $\pi=\bigcup_1^k\pi_i$ $\pi_i\cap\pi_j=\emptyset$ for $i\neq j$, $|\pi_i|\leqq 2$ for $i=1, \ldots, k$. Let us suppose that for every $x\in G^\# C_G(x)=\bigtimes_1^k A_i$, where A_i are supersolvable, $\pi(A_i)\subseteqq\pi_i$, then G is an M-group.*

Acknowledgement. The author would like to express her gratitude to Professor K. Corrádi for suggesting the theme and for his valuable comments.

References

[1] K. Corrádi, On certain properties of centralizers hereditary to the factor group. *To appear.*
[2] T. R. Berger, Characters and derived length in groups of odd order, *Journal of Algebra* **39**, 199—207, 1976.
[3] P. Hall, Theorems like Sylow's, *Proc. London Math. Soc.* **6**, 286—304, 1956.
[4] I. M. Isaacs, Character theory of finite groups, *Acad. Press, N. Y.* 1976.

DEPARTMENT OF MATHEMATICS
FACULTY OF MECHANICAL ENGINEERING
UNIVERSITY OF TECHNOLOGY
BUDAPEST
MŰEGYETEM RKP. 3/9
1111

(Received March 13, 1987)