On a fixed point theorem

By I. JOÓ (Budapest)

Dedicated to Professor Zoltán Daróczy on his 50th birthday

In [1] is proved the

Theorem A. *Let E be a topological vector space, F be a vector space, $K \subset F$ be a convex set, and let $\{H_y: y \in K\}$ be a family of non-empty convex compact subsets of E with the following properties:*

(1) $$H_{\lambda y_1 + (1-\lambda) y_2} \subset H_{y_1} \cup H_{y_2} \quad (y_1, y_2 \in K;\ 0 \leqq \lambda \leqq 1),$$

(2) $$H_{\lambda y_1 + (1-\lambda) y_2} \cap H_{y_1} \subset H_{\mu y_1 + (1-\mu) y_2} \cap H_{y_1} \quad (y_1, y_2 \in K;\ 0 \leqq \lambda \leqq \mu \leqq 1),$$

(3) $$H_{\lambda_0 y_1 + (1-\lambda_0) y_2} \cap H_{y_1} = \bigcap_{\lambda > \lambda_0} (H_{\lambda y_1 + (1-\lambda) y_2} \cup H_{y_1})$$

for arbitrary $0 \leqq \lambda_0 < 1$.

Then $\bigcap_{y \in K} H_y \neq \emptyset$, *where* $\emptyset$ *denotes the empty set.*

According to a well known theorem of F. RIESZ (any family of compact sets with finite intersection property has a common point) the essential part of Theorem A is: *the system* $\{H_y: y \in K\}$ *has the finite intersection property.*

The aim of this note is to prove a theorem of similar type. Our theorem is more general than Theorem 1. We consider sets H_y in an arbitrary topological space, on the other hand it is more special in the sense the sets H_y are defined as the level sets of a function. We shall give applications of Theorem B (below) in a subsequent paper of the same journal.

For the formulation of Theorem B, let A and B be arbitrary topological spaces, $f: B \times B \to P(B)$ be any mapping of $B \times B$ into the power set $P(B)$ of B such that $\{y_1, y_2\} \subset f(y_1, y_2)$ holds for all $y_1, y_2 \in B$. Suppose that the values of f are connected, non-empty, closed sets and $x, y \in f(x, y)$ for all x, y. Let $g: A \times B \to \overline{R}$ be any function and denote by c a real (fixed) number such that

$$H_y^c = H_y = \{x: g(x, y) > c\} \neq \emptyset$$

for every $y \in B$.

Let $H_x^c = H_x = \{y: g(x, y) > c\}$. We prove the following

Theorem B. *Suppose, for any* $x \in A$ *and* $y_1, y_2 \in B$ *that*

(4) $$(B \setminus H_x) \cap f(y_1, y_2)$$

is closed in $f(y_1, y_2)$

(5) $$y_1, y_2 \in B \setminus H_x \quad \textit{implies} \quad f(y_1, y_2) \subset B \setminus H_x,$$

(6) $$\textit{for any finite set } \{y_1, y_2, \ldots, y_n\} \subset B, \ \bigcap_{i=1}^{n} H_{y_i}$$

is open and connected (may be empty). Then the system $\{H_y: y \in B\}$ *has the finite intersection property.*

Proof. Use induction. Suppose we know that for any subset $\{y_1, \ldots, y_k\} \subset B$ having at most n elements we have

$$\bigcap_{i=1}^{k} H_{y_i} \neq \emptyset$$

and then we prove the same for $n+1$ elements. (One can prove the starting case $n=1$ similarly). Suppose there exist $y_1, \ldots, y_{n+1}$ such that

(7) $$\bigcap_{i=1}^{n+1} H_{y_i} = \emptyset.$$

Using the notation $H_y^* := H_y \cap \bigcap_{i=3}^{n+1} H_{y_i}$ we can write (7) in the form

$$H_{y_1}^* \cap H_{y_2}^* = \emptyset.$$

According to our induction assumption and taking into account also (6) we know that the sets H_y^* $(y \in B)$ are open, connected, non-empty. The assumption (5) means: $g(x, y_1) \leqq c$ and $g(x, y_2) \leqq c$ implies $g(x, z) \leqq c$ for every $z \in f(y_1, y_2)$, i.e.

$$H_z \subset H_{y_1} \cup H_{y_2} \quad \text{and} \quad H_z^* \subset H_{y_1}^* \cup H_{y_2}^* \quad \text{for} \quad z \in f(y_1, y_2).$$

Let $S_i := \{z \in f(y_1, y_2): H_z^* \subset H_{y_i}^*\}$ $(i=1, 2)$. We see that $S_i \neq \emptyset$ $(i=1, 2)$, $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = f(y_1, y_2)$ is connected. Hence we arrive to a contradiction, if we prove that S_1 and S_2 are closed. But this follows from the equality

$$S_1 = \{z \in f(y_1, y_2): \text{ for every } x \in H_{y_2}^* \ g(x, z) \leqq c\} =$$
$$= \bigcap_{x \in H_{y_2}^*} \{z \in f(y_1, y_2): g(x, z) \leqq c\}.$$

A similar equality holds for S_2. □

References

[1] I. Joó and A. P. Sövegjártó, A fixed point theorem, *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* **24** (1981), 9—11.

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

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