Invariant connections on a Whitney sum of bundles

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Introduction

Following M. MATSUMOTO's [2] ideas the theory of Finsler connections can be considered as the theory of certain connections on the pullback of a principal frame bundle. This theory can also be constructed using vector bundles instead of principal bundles (MIRON [4], OPRIS [5]). From this point of view a Finsler connection over a vector bundle ξ is given by a homogeneous connection in the vector bundle ξ and by a linear connection in its vertical subbundle $V\xi$. This approach is particularly fruitful if we represent the tangent bundle $\tau tl\xi$ of the total space $tl\xi$ as a special Whitney sum of its subbundles, one of them being the vertical subbundle $V\xi$, and the other determined by the homogeneous connection. In this way we can extend the linear connection to the whole tangent bundle $\tau tl\xi$ of the total space of ξ. If ξ is a tangent bundle of some manifold M this extension process turns out to be canonical, and gives a possibility to reinterpret the differential invariants of the Finsler connection under consideration in a compact form (OPROIU [6]).

As it can easily be seen the homogeneous connection is used in the extension process only for the construction of a special Whitney sum. In the present paper we follow a more general way: we will consider general Whitney sums instead of making use of homogeneous connections. Using Whitney sums several different possibilities emerge in order to generalize the original extension process. In this work we will study only the three most natural of them. To make the differences clearer, in our definitions general nonlinear connections are used (in place of the linear connections); however in the last part of the paper we will study the linear cases also.

2. Basic notions and notations

Whitney-sums. The notion of Whitney sum will be the central idea of the present paper.

1. Definition. Let ξ^1, η, ξ^2 be three vector bundles with common base spaces. We say that the vector bundle η is the Whitney-sum of the vector bundles ξ^1 and ξ^2 if there exists two short exact sequences

$$O \to \xi^1 \xrightarrow{\iota^1} \eta \xrightarrow{\varrho^2} \xi^2 \to O$$

and

$$O \to \xi^2 \xrightarrow{\iota^2} \eta \xrightarrow{\varrho^1} \xi^1 \to O$$

in which all the maps are of constant rank and satisfy the conditions

(a) $$\iota^1 \circ \varrho^1 + \iota^2 \circ \varrho^2 = \mathrm{id}_\eta$$

and

(b) $$\varrho^\nu \circ \iota^\nu = \mathrm{id}_{\xi^\nu}, \quad (\nu = 1, 2).$$

In the case of Whitney-sums we unite the above two diagrams into one

(*) $$O \leftrightarrows \xi^1 \underset{\iota^1}{\overset{\varrho^1}{\leftrightarrows}} \eta \underset{\varrho^2}{\overset{\iota^2}{\leftrightarrows}} \xi^2 \leftrightarrows O.$$

This diagram is the dual short exact sequence determined by the Whitney-sum.

Connection theory. *Let ξ be a vector bundle and denote its vertical subbundle by $V\xi$. A connection ∇ is simply a direct sum decomposition of the tangent bundle $\tau tl\xi$ of the total space $tl\xi$, where one of the summands is the vertical subbundle $V\xi$ and the other is the horizontal subbundle of ∇ denoted by $H_\nabla \xi$. We will refer to a connection by the special dual sequence*

(∇) $$O \leftrightarrows (\mathrm{pr}\,\xi)^! \,\xi \underset{J_\xi}{\overset{v_\nabla}{\leftrightarrows}} \tau tl\xi \underset{P_\xi}{\overset{h_\nabla}{\leftrightarrows}} (\mathrm{pr}\,\xi)^! \,\tau bs\xi \leftrightarrows O$$

The maps h_∇ and v_∇ are called the horizontal and the vertical maps of the connection ∇.

As it is well known the vector bundle map $\mathrm{ad}_\xi\,\mathrm{pr}\,\xi$ is an isomorphism on the fibers between $V\xi$ and ξ. In this notation the map

$$D_\nabla := \mathrm{ad}_\xi\,\mathrm{pr}\,\xi \circ v_\nabla \colon \tau tl\xi \to \xi$$

is called the Dombrovski map of the connection ∇.

The local form of maps arising in connection theory.

Denote the local coordinates of a point of the base space $bs\xi$ of ξ in a local coordinate system by x^1, and the coordinates of an element of ξ in the local coordinate system associated to the former by $(\mathrm{x}^1, \mathrm{y}^1)$. Then the coordinates of an element of $\tau tl\xi$ in the local coordinate system associated to $(\mathrm{x}^1, \mathrm{y}^1)$ are $(\mathrm{x}^1, \mathrm{y}^1, \mathrm{x}^2, \mathrm{y}^2)$, where $\mathrm{x}^1, \mathrm{x}^2 \in \mathbf{R}^n$, $n = \dim bs\xi$; $\mathrm{y}^1, \mathrm{y}^2 \in \mathbf{R}^m$, $m = \mathrm{rank}\,\xi$.

1. Proposition. *We can describe the above maps locally by the following formulas:*

$$h_\nabla(\mathrm{x}^1, \mathrm{y}^1, \mathrm{x}^2) = \big(\mathrm{x}^1, \mathrm{y}^1, \mathrm{x}^2, \omega(x^1, \mathrm{y}^1)(\mathrm{x}^2)\big)$$

$$v_\nabla(\mathrm{x}^1, \mathrm{y}^1, \mathrm{x}^2, \mathrm{y}^2) = \big(\mathrm{x}^1, \mathrm{y}^1, \mathrm{y}^2 - \omega(\mathrm{x}^1, \mathrm{y}^1)(\mathrm{x}^2)\big)$$

$$D_\nabla(\mathrm{x}^1, \mathrm{y}^1, \mathrm{x}^2, \mathrm{y}^2) = \big(\mathrm{x}^1, \mathrm{y}^2 - \omega(\mathrm{x}^1, \mathrm{y}^1)(\mathrm{x}^2)\big)$$

where $\omega(\mathrm{x}^1, \mathrm{y}^1)(\mathrm{x}^2)$ is smooth in the variables x^1, y^1 and linear in x^2.

PROOF. We can suppose that the local form of h_∇ is

$$h_\nabla(x^1, y^1, x^2) = \big(x^1, y^1, \tilde{\omega}(x^1, y^1)(x^2), \omega(x^1, y^1)(x^2)\big)$$

with some maps $\tilde{\omega}(x^1, y^1)(x^2)$ and $\omega(x^1, y^1)(x^2)$ which are smooth in the variables x^1, y^1, and linear in x^2. Because of the definition of a Whitney-sum the maps h_∇ and P_ξ satisfy the condition $P_\xi \circ h_\nabla = \mathrm{id}_{tl(\mathrm{pr}\,\xi)!\,\tau bs\xi}$. The local version of this equation is

$$P_\xi \circ h_\nabla(x^1, y^1, x^2) = P_\xi\big(x^1, y^1, \tilde{\omega}(x^1, y^1)(x^2), \omega(x^1, y^1)(x^2)\big) =$$

$$= \big(x^1, y^1, \tilde{\omega}(x^1, y^1)(x^2)\big) \equiv (x^1, y^1, x^2).$$

So $\tilde{\omega}(x^1, y^1)(x^2) \equiv x^2$.

The remaining part of the proposition can be proved by direct computations.

2. Definition. (1) We call the map ω the connection form of the connection ∇.
(2) The connection ∇ is linear if the map $\omega(x^1, y^1)(x^2)$ is linear in y^1.

Connections are completely determined by their connection forms. Using this fact and the local formulas the following statement is true

2. Proposition. *A vector bundle map $K\colon \tau tl\xi \to \xi$ is the Dombrovski map of some connection iff*

$$(D^*) \qquad K|_{V_\xi} = \mathrm{ad}_\xi\, \mathrm{pr}\, \xi \circ \tau_\xi$$

where τ_ξ denotes the canonical isomorphism between $V\xi$ and the pullback of ξ by its projection.

3. Definition. Let $\big(x(t), y(t)\big)$ be the local representant of a curve $\tilde{\gamma}$ of the total space $tl\xi$. We say that the curve $\tilde{\gamma}$ is parallel by ∇ if it satisfies the differential equation

$$(P) \qquad \dot{y}(t) + \omega\big(x(t), y(t)\big)\big(\dot{x}(t)\big) = 0.$$

In case of a Whitney-sum η as in $(*)$ the local coordinates of an element will be denoted by $(x^1, z^1, v^1, x^2, z^2, v^2)$, where $x^1, x^2 \in \mathbf{R}^n$; $z^1, z^2 \in \mathbf{R}^r$, $r = \operatorname{rank} \xi^1$; $v^1, v^2 \in \mathbf{R}^s$, $s = \operatorname{rank} \xi^2$; $r = s = \operatorname{rank} \eta$. Then the horizontal map h_∇ of a connection ∇ on η and its Dombrovski map are described by

$$h_{\nabla^1}(x^1, z^1, x^2, z^2) = \big(x^1, z^1, x^2, \Omega^1(x^1, z^1, 0)(x^2)\big)$$

and

$$D(x^1, z^1, v^1, x^2, z^2, v^2) = \big(x^1, z^2 - \Omega^1(x^1, z^1, v^1)(x^2), v^2 - \Omega^2(x^1, z^1, v^1)(x^2)\big),$$

where the connection forms Ω^1 and Ω^2 are smooth in their arguments and linear in x^2.

3. Whitney-sums and Whitney-decompositions of connections

Let ξ^1, ξ^2 and η be vector bundles as above $(*)$.

4. Definition. (1) Let be given two vector bundle maps

$$K_\nu = \tau tl\xi^\nu \to \xi^\nu \quad (\nu = 1, 2)$$

determined on the vector bundles ξ^1 and ξ^2 respectively. Then the vector bundle map

$$K \equiv \iota^1 \circ K_1 \circ \mathrm{T}\varrho^1 + \iota^2 \circ K_2 \circ \mathrm{T}\varrho^2 : \quad \tau t l \eta \to \eta$$

is called the Whitney-sum of K_1 and K_2 by $(*)$. ($\mathrm{T}\varrho^1$ denotes the linearization of the mapping ϱ^1.)

(2) Let

$$\tilde{K}: \tau t l \eta \to \eta$$

be a vector bundle map. Then the maps

$$\tilde{K}_\nu \equiv \varrho^\nu \circ \tilde{K} \circ \mathrm{T}\iota^\nu : \tau t l \xi^\nu \to \xi^\nu \quad (\nu = 1, 2)$$

are called the Whitney-decompositions of the map $\tilde{K}$ by $(*)$.

(3) The map

$$S(K) = (\iota^1 \circ \varrho^1) \circ K \circ T(\iota^1 \circ \varrho^1) + (\iota^2 \circ \varrho^2) \circ K \circ T(\iota^2 \circ \varrho^2)$$

is the symmetric Whitney-projection of the map K by $(*)$.

(4) The maps

$$A^1(K) = \varrho^1 \circ K \circ T\iota^2 : \tau t l \xi^2 \to \xi^1$$

$$A^2(K) = \varrho^2 \circ K \circ T\iota^1 : \tau t l \xi^1 \to \xi^2$$

are called the asymmetric Whitney-projections of K by $(*)$.

3. Proposition. (1) *If in the above definition the vector bundle maps K_1 and K_2 are Dombrovski maps of some connections ∇^1 and ∇^2 defined on the vector bundles* ξ^1 and ξ^2, *and denoted by D_1, D_2 then there exists a connection ∇ on the vector bundle η such that its Dombrovski map is the Whitney-sum of the maps D_1 and D_2 by $(*)$.*

(2) *Let $\tilde{D}$ be the Dombrovski map of the connection $\tilde{\nabla}$ defined on η. Then there exist connections $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ defined on the vector bundles ξ^1 and ξ^2, whose Dombrovski maps are the Whitney-decompositions of $\tilde{D}$ by $(*)$.*

(3) *The symmetric Whitney-projection of the Dombrovski map of any connection defined on a vector bundle η is the Dombrovski map of some connection $\bar{\nabla}$ defined on η.*

PROOF. (1) Using trivializations of the vector bundles ξ^1 and ξ^2 with the help of $(*)$ one can construct a trivialization of η in which the local forms of the maps ι^1, ϱ^1, ι^2, ϱ^2 are:

$$\iota^1(\mathrm{x}, \mathrm{z}) = (\mathrm{x}, \mathrm{z}, 0)$$

$$\iota^2(\mathrm{x}, \mathrm{v}) = (\mathrm{x}, 0, \mathrm{v})$$

$$\varrho^2(\mathrm{x}, \mathrm{z}, \mathrm{v}) = (\mathrm{x}, \mathrm{z}) \qquad (\mathrm{x} \in \mathbf{R}^n, \mathrm{z} \in \mathbf{R}^{k_1}, \mathrm{v} \in \mathbf{R}^{k_2})$$

$$\varrho^2(\mathrm{x}, \mathrm{z}, \mathrm{v}) = (\mathrm{x}, \mathrm{v})$$

where $n = \dim bs\eta$, $k_1 = \operatorname{rank} \xi^1$, $k_2 = \operatorname{rank} \xi^2$.

Then the local form of $T\iota^1$ is:

$$(T\iota^1)(x^1, z^1, x^2, z^2) = \left(\iota^1(x^1, z^1), \frac{d}{dt}\left(\iota^1(x^1+tx^2, z^1+tz^2)\big|_{t=0}\right)\right) =$$

$$= \left(x^1, z^1, 0; \frac{d}{dt}(x^1+tx^2, z^1+tz^2, 0)\big|_{t=0}\right) = (x^1, z^1, 0, x^2, z^2, 0).$$

The local form of the maps $T\iota^2$, $T\varrho^1$, $T\varrho^2$ can be computed similarly: Let $x^1, x^2 \in \mathbf{R}^n$, $z^1, z^2 \in \mathbf{R}^{k_1}$, $v^1, v^2 \in \mathbf{R}^{k_2}$. Then

$$T\iota^2(x^1, v^1, x^2, v^2) = (x^1, 0, v^1, x^2, 0, v^2)$$

$$T\varrho^1(x^1, z^1, v^1, x^2, z^2, v^2) = (x^1, z^1, x^2, z^2)$$

$$T\varrho^2(x^1, z^1, v^1, x^2, z^2, v^2) = (x^1, v^1, x^2, v^2).$$

If the local representatives of D_1 and D_2 are

$$D_1(x^1, z^1, x^2, z^2) = (x^1, z^2-\omega^1(x^1, z^1)(x^2))$$

and

$$D_2(x^1, v^1, x^2, v^2) = (x^1, v^2-\omega^2(x^1, v^1)(x^2)),$$

then the local form of the map

$$D = \iota^1\circ D_1\circ T\varrho^1+\iota^2\circ D_2\circ T\varrho^2$$

is the following:

$$D(x^1, z^1, v^1, x^2, z^2, v^2) = (\iota^1\circ D_1)(x^1, z^1, x^2, z^2)+(\iota^2\circ D_2)(x^1, v^1, x^2, v^2) =$$

$$= (x^1, z^2-\omega^1(x^1, z^1)(x^2), v^2-\omega^2(x^1, v^1)(x^2)).$$

By condition (D*) D is a Dombrovski map iff its restriction to the vertical subbundle is equal to the map $\mathrm{ad}_\eta\, \mathrm{pr}\, \eta\circ r_\eta$.

A typical element of $V\eta$ locally has the form

$$(x^1, z^1, v^1, 0, z^2, v^2)$$

and the map $\mathrm{ad}_\eta\, \mathrm{pr}\, \eta\circ r_\eta$ sends this into (x^1, z^2, v^2); but this is equal to $D(x^1, z^1, v^1, 0, z^2, v^2)$.

(2) The map $\tilde{D}$ has the following local form (see page 3):

$$\tilde{D}(x^1, z^1, v^1, x^2, z^2, v^2) = (x^1, z^2-\tilde{\Omega}^1(x^1, z^1, v^1)(x^2), v^2-\tilde{\Omega}^2(x^1, z^1, v^1)(x^2)).$$

Then the map $\tilde{D}_1 \equiv \varrho^1\circ\tilde{D}\circ T\varrho^1$ can be written locally as

$$\tilde{D}_1(x^1, z^1, x^2, z^2) = (\varrho^1\circ\tilde{D}\circ T\varrho^1)(x^1, z^1, x^2, z^2) =$$

$$(\varrho^1\circ\tilde{D})(x^1, z^1, 0, x^2, z^2, 0) = (x^1, z^2-\tilde{\Omega}^1(x^1, z^1, 0)(x^2))$$

and this map satisfies condition (D*), so it is a Dombrovski-map.

(3). This statement is a direct consequence of statements (1) and (2). Q.E.D.

With the aid of the connections ∇, ∇^1, ∇^2, $\tilde{\nabla}$, $\tilde{\nabla}^1$, $\tilde{\nabla}^2$ and $\overline{\nabla}$ appearing in Proposition 3, we give the following

5. Definition. (1) The connection ∇ is called the Whitney-sum of the connections ∇^1 and ∇^2 by $(*)$.

(2) The connections $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ are called the Whitney-decompositions of $\tilde{\nabla}$ by $(*)$.

(3) We call the connection ∇ the symmetric Whitney-projection of the connection $\overline{\nabla}$, and denote it by $F_*(\overline{\nabla})$ if its Dombrovski map is equal to the symmetric Whitney-projection of the Dombrovski map of $\overline{\nabla}$ by $(*)$.

4. Proposition. (1) *The Whitney-decomposition operation is the conjugate of the Whitney-sum operation: the Whitney-decompositions of the Whitney-sum of two connections are the original connections.*

(2) *Geometric characterization of a connection's Whitney-sum. The Whitney-sum of the connections ∇^1 and ∇^2 is that connection $\tilde{\nabla}$ which can be characterized by the following condition: any curve $\tilde{\gamma}$ of $tl\eta$ is parallel by $\tilde{\nabla}$ iff the curves $\varrho^1\circ\tilde{\gamma}$ and $\varrho^2\circ\tilde{\gamma}$ are parallel by the connections ∇^1 and ∇^2 respectively.*

PROOF. (1) The statement can easily be proved by the local formulas given in the proof of the previous statement.

(2) Let $(\mathrm{x}(t), \mathrm{z}(t), \mathrm{v}(t))$ be the local form of $\tilde{\gamma}(t)$. Then the local forms of $\varrho^1\circ\tilde{\gamma}$ and $\varrho^2\circ\tilde{\gamma}$ are $(\mathrm{x}(t), \mathrm{z}(t))$ and $(\mathrm{x}(t), v(t))$ respectively. The local condition for the parallelity by the connections ∇^1 resp. ∇^2 are the equations

$$\dot{\mathrm{z}}(t) = \omega^1(\mathrm{x}(t), \mathrm{z}(t))(\dot{\mathrm{x}}(t))$$
$$\dot{\mathrm{v}}(t) = \omega^2(\mathrm{x}(t), \mathrm{v}(t))(\dot{\mathrm{x}}(t)),$$

where ω^1 and ω^2 are the connection forms of ∇^1 and ∇^2. If $\tilde{\Omega}$ denotes the connection form of $\tilde{\nabla}$, then by (P), the local condition for parallelity of $\tilde{\gamma}$ is

$$(\dot{\mathrm{z}}(t), \dot{\mathrm{v}}(t)) = \tilde{\Omega}(\mathrm{x}(t), \mathrm{z}(t), \mathrm{v}(t))(\dot{\mathrm{x}}(t)).$$

Comparing these equations and using the linearity in $\dot{\mathrm{x}}(t)$ we get

$$\tilde{\Omega}(\mathrm{x}(t), \mathrm{z}(t), \mathrm{v}(t))(\dot{\mathrm{x}}(t)) = \omega^1(\mathrm{x}(t), \mathrm{z}(t))(\dot{\mathrm{x}}(t)) + \omega^2(\mathrm{x}(t), \mathrm{v}(t))(\dot{\mathrm{x}}(t)),$$

so the connection $\tilde{\nabla}$ is the Whitney-sum of ∇^1 and ∇^2 by $(*)$. The converse way can be proved with similar direct calculations. Q.E.D.

4. The invariance conditions and their local form

We can define three different types of invariance for connections on a Whitney-sum of boundles:

6. Definition. (1) A connection ∇ defined on a Whitney-sum bundle η is called W-invariant by $(*)$ if $\nabla = F_*(\nabla)$.

(2) A connection ∇ is called K-invariant by $(*)$ if its horizontal map h_∇ and vertical map v_∇ satisfy the following equations:

$$v_\nabla \circ T(\iota^v \circ \varrho^v) \circ h_\nabla \equiv 0 \quad (v = 1, 2).$$

(3) We call the connection ∇P-invariant by $(*)$ if for any curve γ of $bs\eta$ the parallel translation of any element from $\operatorname{Im}\iota^{\nu}$ by ∇ along γ is also an element of $\operatorname{Im}\iota^{\nu}$ $(\nu=1,2)$.

The local description of W-invariance. Let η be the Whitney-sum of ξ^1 and ξ^2 as in $(*)$. The Dombrovski map D of a connection on η is described at the end of Section 2.

1. Theorem. (1) *The local form of the Dombrovski map $D_{F^*(\nabla)}$ of ∇'s symmetric Whitney-projection $F^*(\nabla)$ is*

$$D_{F^*(\nabla)}(x^1, z^1, v^1, x^2, z^2, v^2) = \big(x^1, z^2 - \Omega^1(x^1, z^1, 0)(x^2), v^2 - \Omega^2(x^1, 0, v^1)(x^2)\big).$$

(2) *The local forms of the asymmetric Whitney-projections of ∇'s Dombrovski maps are*

$$A^1(D)(x^1, v^1, x^2, v^2) = \big(x^1, -\Omega^1(x^1, 0, v^1)(x^2)\big)$$

and

$$A^2(D)(x^1, v^1, x^2, v^2) = \big(x^1, -\Omega^2(x^1, z^1, 0)(x^2)\big).$$

PROOF. We make use of the local forms introduced in the previous part.

(1) The proof of the first statement is given by the following computations

$$D_{F^*(\nabla)}(x^1, z^1, v^1, x^2, z^2, v^2) =$$

$$\big((\iota^1\circ\varrho^1)\circ D\circ T(\iota^1\circ\varrho^1)\big)(x^1, z^1, v^1, x^2, z^2, v^2) + \big((\iota^2\circ\varrho^2)\circ D\circ T(\iota^2\circ\varrho^2)\big)(x^1, z^1\, v^1\, x^2\, z^2\, v^2) =$$

$$= (\iota^1\circ\varrho^1)\circ D(x^1, z^1, 0, x^2, z^2, 0) + (\iota^2\circ\varrho^2)\circ D(x^1, 0, z^1, x^2, 0, z^2) =$$

$$= (\iota^1\circ\varrho^1)\big(x^1, z^2 - \Omega^1(x^1, z^1, 0)(x^2), -\Omega^2(x^1, z^1, 0)(x^2)\big) +$$

$$+ (\iota^2\circ\varrho^2)\big(x^1, -\Omega^1((x^1, 0, v)(x^2), v^2 - \Omega^2(x^1, 0, v^1(x^2))\big) =$$

$$= \big(x^1, z^2 - \Omega^1(x^1, z^1, 0)(x^2), v^2 - \Omega^2(x^1, 0, v^1)(x^2)\big).$$

(2) Because of symmetry we prove the first part only:

$$A^1(D)(x^1, v^1, x^2, v^2) = (\varrho^1\circ D)(x^1, 0, v^1, x^2, 0, v^2) =$$

$$= \varrho^1\big(x^1, -\Omega^1(x^1, 0, v^1)(x^2), v^2 - \Omega^2(x^1, 0, v^1)(x^2)\big) = \big(x^1, -\Omega^1(x^1, 0, v^1)(x^2)\big).$$

5. Proposition. *A connection ∇ is W-invariant iff the equations*

$$\Omega^1(x^1, z^1, v^1) = \Omega^1(x^1, z^1, 0)$$

and

$$\Omega^2(x^1, z^1, v^1) = \Omega^2(x^1, 0, v^1)$$

are satisfied.

PROOF. By direct computations using the previous proposition.

Now we will describe the conditions for the structure of the horizontal subbundle of ∇ implied by the invariance conditions. First we prove a statement about the horizontal subbundles of the Whitney-projection of a connection.

6. Proposition. *Let the connections ∇^1 and ∇^2 be the Whitney-projections of the connection ∇ by $(*)$. Then the horizontal subbundles of ∇^1 and ∇^2 have the fol-*

lowing forms:

$$H_{\nabla^\nu} = T\varrho^\nu(H_\nabla|_{\operatorname{Im}\iota^\nu}) \quad (\nu = 1, 2)$$

where H_∇ is the horizontal subbundle of ∇.

PROOF. We prove the case $\nu=1$ only. The Dombrovski map of the connection ∇^1 can be represented locally in the form

$$D_{\nabla^1}(\mathrm{x}^1, \mathrm{z}^1, \mathrm{x}^2, \mathrm{z}^2) = \big(\mathrm{x}^1, \mathrm{z}^2 - \Omega(\mathrm{x}^1, \mathrm{z}^1, 0)(\mathrm{x}^2)\big),$$

so its horizontal map is

$$h_{\nabla^1}(\mathrm{x}^1, \mathrm{z}^1, \mathrm{x}^2, \mathrm{z}^2) = \big(\mathrm{x}^1, \mathrm{z}^1, \mathrm{x}^2, \Omega^1(\mathrm{x}^1, \mathrm{z}^1, 0)(\mathrm{x}^2)\big)$$

which means that a typical element of $H_{\nabla^1}(=\operatorname{Im} h_{\nabla^1})$ has the following form

$$(\diamond) \qquad \big(\mathrm{x}^1, \mathrm{z}^1, \mathrm{x}^2, \Omega^1(\mathrm{x}^1, \mathrm{z}^1, 0)(\mathrm{x}^2)\big).$$

The typical element of the horizontal subbundle of ∇ is

$$\big(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1, \mathrm{x}^2, \Omega^1(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1)(\mathrm{x}^2), \Omega^2(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1)(\mathrm{x}^2)\big).$$

If we restrict H_∇ to the image of ι^1 we get

$$\big(\mathrm{x}^1, \mathrm{z}^1, 0, \mathrm{x}^2, \Omega^1(\mathrm{x}^1, \mathrm{z}^1, 0)(\mathrm{x}^2), \Omega^2(\mathrm{x}^1, \mathrm{z}^1, 0)(\mathrm{x}^2)\big),$$

and the map $T\varrho^1$ sends this exactly into $(\diamond)$. Q.E.D.

Suppose that ∇^1 and ∇^2 are connections on vector bundles ξ^1 and ξ^2.

2. Theorem. *If the horizontal subbundles of the connections ∇^1 and ∇^2 are denoted by H_{∇^1} and H_{∇^2} then the horizontal subbundle of their Whitney-sum by* $(*)$ *is*

$$H_\nabla = (T\varrho^1)^{-1}(H_{\nabla^1}) \cap (T\varrho^2)^{-1}(H_{\nabla^2}).$$

PROOF. The horizontal subbundle of a connection is exactly the kernel of its Dombrovski map, and the Dombrovski map of the Whitney-sum is

$$D = \iota^1 \circ D_{\nabla^1} \circ T\varrho^1 + \iota^2 \circ D_{\nabla^2} \circ T\varrho^2.$$

Then

$$D(v) = (\iota^1 \circ D_{\nabla^1} \circ T\varrho^1)(v) + (\iota^2 \circ D_{\nabla^2} \circ T\varrho^2)(v) = 0$$

is true just in the case when $(\iota^1 \circ D_{\nabla^1} \circ T\varrho^1)(v)=0$ and $(\iota^2 \circ D_{\nabla^2} \circ T\varrho^2)(v)=0$ is satisfied, since $\operatorname{Im}\iota^1 \cap \operatorname{Im}\iota^2 = \emptyset$. But maps ι^1, ι^2 are injections, so this is equivalent with $(D_{\nabla^1} \circ T\varrho^1)(v)=0$ and $(D_{\nabla^2} \circ T\varrho^2)(v)=0$. This means that $D(\mathrm{v})=0$ iff $(T\varrho^1)(v) \in \operatorname{Ker} D_{\nabla^1}$ and $(T\varrho^2)(v) \in \operatorname{Ker} D_{\nabla^2}$,

$$v \in (T\varrho^1)^{-1}(H_{\nabla^1}) \cap (T\varrho^2)^{-1}(H_{\nabla^2}),$$

and this proves our statement.

The above consideration yields also the

7. Proposition. *The horizontal subbundle of the symmetric Whitney-projection of the connection ∇ is*

$$(T\varrho^1)^{-1}[(T\varrho^1)(H_\nabla|_{\operatorname{Im}\iota^1})] \cap (T\varrho^2)^{-1}[(T\varrho^2)(H_\nabla|_{\operatorname{Im}\iota^2})].$$

8. Corollary. *The necessary and sufficient condition for the W-invariance of the connection* ∇ *is that its horizontal subbundle satisfies the relation*

$$H_\nabla = (T\varrho^1)^{-1}[(T\varrho^1)(H_\nabla|_{\mathrm{Im}\,\iota^1})]\cap(T\varrho^2)^{-1}[(T\varrho^2)(H_\nabla|_{\mathrm{Im}\,\iota^2})].$$

The local description of P-invariance.

3. Theorem. *Denote by* ∇ *the same connection as in the previous part. Then the necessary and sufficient conditions for the P-invariance of* ∇ *are the equations*

I) $$\Omega^1(\mathrm{x}^1, 0, \mathrm{z}^1) \equiv 0$$

and

II) $$\Omega^2(\mathrm{x}^1, \mathrm{v}^1, 0) \equiv 0.$$

Proof. We will prove only II) because of the symmetry of the above equations.

Sufficiency. The local form of the curve $\gamma(t)$ of $tl\eta$ is $(\mathrm{x}(t), \mathrm{z}(t), \mathrm{v}(t))$. Suppose that $\gamma(t_0)$ is an element of $\mathrm{Im}\,\iota^1$. Then $\gamma(t_0)$ has the form $(\mathrm{x}(t_0), \mathrm{z}(t_0), 0)$.

Now consider the following initial value problems for $\bar{\mathrm{z}}(t)$:

$$\dot{\bar{\mathrm{z}}}(t) = \Omega^1(\mathrm{x}(t), \bar{\mathrm{z}}(t), 0)(\dot{\mathrm{x}}(t)) \quad \bar{\mathrm{z}}(t_0) = \mathrm{z}(t_0).$$

If $\tilde{\mathrm{z}}(t)$ is the solution of this system, then in view of our assumption II) $\tilde{\mathrm{z}}(t)$ and $\tilde{\mathrm{v}}(t) \equiv 0$ is a solution of the following first order system of differential equations:

$$\dot{\bar{\mathrm{z}}}(t) = \Omega^1(\mathrm{x}(t), \bar{\mathrm{z}}(t), \bar{\mathrm{v}}(t))(\dot{\mathrm{x}}(t))$$

$$\dot{\bar{\mathrm{v}}}(t) = \Omega^1(\mathrm{x}(t), \bar{\mathrm{z}}(t), \bar{\mathrm{v}}(t))(\dot{\mathrm{x}}(t)).$$

This means the parallelity of $(\tilde{\mathrm{z}}(t), 0)\in \mathrm{Im}\,\iota^1$ in η with respect to ∇. Now the sufficiency is a direct consequence of the unicity of the solutions of this system of differential equations and of the fact that $(\tilde{\mathrm{z}}(t), 0)$ is the parallel translated of $(\mathrm{z}(t_0), 0)$.

Necessity. Let us suppose that the above equation of parallelity with initial condition $(\mathrm{x}_0, \mathrm{z}_0, 0)$ has a solution of the form $(\mathrm{x}(t), \mathrm{z}(t), \mathrm{v}(t))$ $(\mathrm{v}(t)\equiv 0)$. This implies that $\Omega^2(\mathrm{x}(t), \mathrm{z}(t), 0)=0$, and because $\mathrm{x}(t)$ and $\mathrm{z}(t)$ can take any value this equation gives exactly the necessity part of the proposition. Q.E.D.

The local form of K-invariance. At the end of this part we give an equivalent condition for the *K*-invariance in terms of connection forms and/or horizontal subbundles.

4. Theorem. *The necessary and sufficient conditions for the K-invariance of the connection* ∇ *are the local equations*

$$\Omega^1(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1) = \Omega^1(\mathrm{x}^1, \mathrm{z}^1, 0) \quad \Omega^1(\mathrm{x}^1, 0, \mathrm{v}^1) = 0$$

$$\Omega^2(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1) = \Omega^1(\mathrm{x}^1, 0, \mathrm{v}^1) \quad \Omega^2(\mathrm{x}^1, \mathrm{z}^1, 0) = 0.$$

PROOF. Suppose that ∇ is K-invariant by $(*)$. Then according to 6. Definition (2) $v_\nabla \circ T(\iota^1 \circ \varrho^1) \circ h_\nabla = 0$. Calculating this in a local form we obtain

$$\big(v_\nabla \circ T(\iota^1 \circ \varrho^1) \circ h_\nabla\big)(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1, \mathrm{x}^2) =$$

$$\big(\mathrm{x}^1, \mathrm{z}^1, 0, \mathrm{x}^2, +\Omega^1(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1)(\mathrm{x}^2) - \Omega^1(\mathrm{x}^1, \mathrm{z}^1, 0)(\mathrm{x}^2), -\Omega^2(\mathrm{x}^1, \mathrm{z}^1, 0)(\mathrm{x}^2) \equiv (\mathrm{x}^1, \mathrm{z}^1, 0, \mathrm{x}^2, 0, 0).$$

This gives the first and fourth equations of our Theorem. The second and third equations follow from $v_\nabla \circ T(\iota^2 \circ \varrho^2) \circ h_\nabla = 0$. Thus the conditions of the Theorem are necessary. — The sufficiency part of the statement can be proved simply by retracing the above computations. Q.E.D.

9. Proposition. *The connection ∇ is K-invariant by $(*)$ iff its maps $T(\iota^1 \circ \varrho^1)$ and $T(\iota^2 \circ \varrho^2)$ leave the horizontal subbundle H_∇ invariant.*

PROOF. If ∇ K-invariant, then for any $a \in tl(\mathrm{pr}\ \xi)!\tau bs\xi$

$$\big(v_\nabla \circ T(\iota^\nu \circ \varrho^\nu) \circ h_\nabla\big)(a) = 0 \quad (\nu = 1, 2)$$

is satisfied, and so

$$v_\nabla\big((T(\iota^\nu \circ \varrho^\nu) \circ h_\nabla)(a)\big) = 0.$$

This means that in this case $\big(T(\iota^\nu \circ \varrho^\nu)\big)\big(h_\nabla(a)\big) \in \mathrm{Ker}\, v_\nabla = H_\nabla$ i.e. $T(\iota^\nu \circ \varrho^\nu)$ leave H_∇ invariant. But we have proved with this also the sufficiency part of the statement because the above the equations are equivalent. Q.E.D.

5. Relationships between the different invariances

5. Theorem. *A connection ∇ is K-invariant by $(*)$ iff it is both W-invariant and P-invariant.*

PROOF. The local condition of W-invariance is $\Omega^1(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1) \equiv \Omega^1(\mathrm{x}^1, \mathrm{z}^1, 0)$ and $\Omega^2(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1) \equiv \Omega^2(\mathrm{x}^1, 0, \mathrm{v}^1)$. The local condition of P-invariance is $\Omega^1(\mathrm{x}^1, 0, \mathrm{v}^1) \equiv 0$ and $\Omega^2(\mathrm{x}^1, \mathrm{z}^1, 0) \equiv 0$. If we compose these two local conditions we get the local condition of K-invariance and the decomposition of the condition of K-invariance yields the conditions of W- and of P-invariance. Q.E.D.

As it can be seen from the local conditions, in the nonlinear case the condition of [P-invariance and that of W-invariance] are independent in general.

6. Theorem. *If the connection ∇ is linear then the W-, K-, and P-invariances by $(*)$ are equivalent.*

PROOF. Because of the previous Theorem it is sufficient to prove that the notions of W- and P-invariance are the same when the connection ∇ is linear.

If ∇ is linear, then Ω^1, Ω^2 appearing in its Dombrovski map locally has the form

$$D(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1, \mathrm{x}^2, \mathrm{z}^2, \mathrm{v}^2) = \big(\mathrm{x}^1, \mathrm{z}^2 - \Omega^1(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1)(\mathrm{x}^2), \mathrm{v}^2 - \Omega^2(\mathrm{x}^1, \mathrm{z}^1, \mathrm{v}^1)(\mathrm{x}^2)\big)$$

where the maps Ω^1 and Ω^2 are linear, and so

$$\Omega^1(x^1, z^1, v^1) = \Omega^1(x^1, z^1, 0) + \Omega^1(x^1, 0, v^1)$$

$$\Omega^2(x^1, z^1, v^1) = \Omega^2(x^1, z^1, 0) + \Omega^2(x^1, 0, v^1).$$

Substracting from these the corresponding local conditions of W-invariance we obtain the local conditions of P-invariance , and conversely. Q.E.D.

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(Received April 25, 1988)