On rank of apparition of primes in Lucas sequences

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Let $R=\{R_n\}_{n=0}^{\infty}$ be a sequence of Lucas numbers defined by the recursion

$$R_n = A\cdot R_{n-1}+B\cdot R_{n-2},$$

where A and B are fixed non-zero integers and the initial terms are $R_0=0$, $R_1=1$. Denote the roots of the characteristic polynomial x^2-Ax-B of the sequence by α and β. Throughout this paper we assume that R is a non-degenerated sequence, i.e. α/β is not a root of unity.

If p is a prime and $p\nmid B$ then, as it is known, there are terms in the sequence divisible by p. If $p|R_n$ but $p\nmid R_m$ for $0<m<n$, then we say that n is the rank of the apparition of p in the sequence and we denote it by $r(p)$. We also say that p is a primitive prime divisor of R_n if $r(p)=n$. We know that R_n has at least one primitive prime divisor for any $n>n_0$, where n_0 is an absolute constant (see e.g. C. L. STEWART [4]) and $r(p)$ is a divisor of $p-(D/p)$ for any prime with $p\nmid B$, where $D=A^2+$ $+4B$ and (D/p) is the Legendre's symbol with $(D/p)=0$ in case $p|D$ (see e.g. D. H. LEHMER [3]). Thus the primitive prime divisors of a term R_n are of the form $p=nk\pm1$ but we do not know how large k is in general or how small can be the ratios $r(p)/p$.

It is clear that $r(p)/p\leqq 1+1/p$, but from a result of [2] it follows that $r(p)/p$ can be arbitrarily small. We note that this also follows from a result of D. JARDEN ([1], p. 5) if R is the Fibonacci sequence. In this paper we give bounds for the average order of the ratios $r(p)/p$ which yield some results on the primitive prime divisors of Lucas numbers.

In the results and in their proofs $c_1, c_2, \ldots,$ $x_1, x_2, \ldots$ will denote positive constants which are absolute ones or they depend only on the sequence R. Furthermore we assume $r(p)=0$ if $p|B$.

Theorem 1. *There are positive constants c_1 and c_2 such that*

$$c_1\cdot\frac{\sqrt{x}}{\log x} < \sum_{p\leqq x}\frac{r(p)}{p} < c_2\cdot\frac{x}{\log x}$$

for any $x>x_1$.

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Theorem 2. *There are positive constants c_3 and c_4 such that*

$$c_3 \cdot \frac{\sqrt{x}}{\log x} < \sum_{\substack{p \\ r(p) \leqq x}} \frac{r(p)}{p} < c_4 \cdot x$$

for any $x > x_2$.

The magnitude of prime and primitive prime divisors of Lucas numbers was investigated in many papers (see C. L. Stewart [5] and its exhaustive references). Among others it was proved that for almost all natural numbers n the greatest primitive prime divisor of R_n is greater than $\varepsilon(n) \cdot n \cdot (\log n)^2/\log\log n$, where $\varepsilon(n)$ is any real valued function for which $\varepsilon(n) \to 0$ as $n \to \infty$. The following corollary shows that n^2 is an upper bound for infinitely many primitive prime divisors.

Corollary 1. *Let x and c be real numbers with the conditions $0 < c < 1/2$ and $x > x(c, R)$,* where *$x(c, R)$ depends only on c and the sequence R. Then there are at least x^c primes p for which $p \leqq x$ and $p < (r(p))^{1/c}$.*

It is known that $|R_n| < e^{c_5 n}$ for any $n > 0$ and, as we have seen, $p \geqq n-1$ if $r(p) = n$. From these it follows that R_n has at most $c_6 n/\log n$ primitive prime divisors and the trivial estimation

$$\sum_{\substack{p \\ r(p)=n}} \frac{1}{p} < \frac{c_7}{\log n}$$

holds for any $n > 1$. From Theorem 2 a better estimation follows for almost all n which shows a connection between the magnitude and the number of primitive prime divisors of the Lucas numbers.

Corollary 2. *For any* $\delta > 0$

$$\sum_{\substack{p \\ r(p)=n}} \frac{1}{p} < \frac{\delta \cdot \log n}{n}$$

for almost all n, i.e. for all natural numbers n except perhaps for those in a set of asymptotic density zero.

Now we prove our results.

Proof of Theorem 1. The right inequality trivially holds because

$$\frac{r(p)}{p} \leqq \frac{p+1}{p} = 1 + \frac{1}{p}$$

for any p and

$$\sum_{p \leqq x} \frac{1}{p} = \mathcal{O}(\log\log x).$$

Let $a(n)$ be an arithmetical function such that $a(n) = r(n)$ if n is a prime number and $a(n) = 0$ otherwise. Then for any sufficiently large x

$$(1) \qquad S_x = \sum_{p \leqq x} \frac{r(p)}{p} = \sum_{n \leqq x} a(n) \cdot \frac{1}{n}.$$

Before we estimate the sum S_x we give a lower estimation for the sum

$$A(x)=\sum_{n\leqq x} a(n).$$

There are $\pi(x)-c_8$ summands in $A(x)$ different from zero and, as we have seen above, there are at most $c_6 n/\log n$ primes for which $r(p)=n$. But, using Euler's summation formula,

$$\sum_{n\leqq c_9\sqrt{x}} \frac{c_6 n}{\log n}=c_6\cdot\int_2^{c_9\sqrt{x}} \frac{t}{\log t}\,dt+\mathcal{O}\left(\frac{\sqrt{x}}{\log x}\right)<c_{10}\cdot\frac{x}{\log x},$$

where c_9 can be chosen such that

$$c_{10}\cdot\frac{x}{\log x}<\pi(x)-c_8,$$

and so

$$A(x)>\sum_{n\leqq c_9\sqrt{x}} n\cdot\frac{c_6 n}{\log n}=c_6\cdot\int_2^{c_9\sqrt{x}}\frac{t^2}{\log t}\,dt+\mathcal{O}\left(\frac{x}{\log x}\right)>c_{11}\cdot\frac{x\cdot\sqrt{x}}{\log x}. \tag{2}$$

Using Abel's identity, by (1) and (2)

$$S_x=\frac{A(x)}{x}+\int_2^x\frac{A(t)}{t^2}\,dt+\mathcal{O}(1)>$$

$$>c_{11}\cdot\frac{\sqrt{x}}{\log x}+c_{11}\cdot\int_2^x\frac{dt}{\sqrt{t}\cdot\log t}+\mathcal{O}(1)>c_{12}\cdot\frac{\sqrt{x}}{\log x}$$

follows which proves the left inequality of the theorem.

PROOF OF THEOREM 2. The left inequality follows from Theorem 1 since $r(p)\leqq x$ for any prime for which $p\leqq x-1$.

Let x be a sufficiently large number and let p be a prime such that $p\leqq x$ and $r(p)=n\leqq x$. Then there is an integer k such that $p=kn+1$ or $p=kn-1$ and $k\leqq\dfrac{x+1}{n}$ and so, using that $\sum_{k\leqq t} 1/k=\log t+\mathcal{O}(1)$,

$$\sum_{\substack{p\\ r(p)=n}}\frac{1}{p}\leqq\frac{1}{n-1}+\frac{1}{n+1}+\frac{1}{2n-1}+\frac{1}{2n+1}+\ldots= \tag{3}$$

$$=2\cdot\sum_{k\leqq(x+1)/2n}\frac{1}{kn}+\mathcal{O}\left(\frac{1}{n}\right)=\frac{2}{n}(\log x-\log n)+\mathcal{O}\left(\frac{1}{n}\right)$$

follows. Since

$$\sum_{\substack{p\\ r(p)\leqq x}}\frac{r(p)}{p}=\sum_{n\leqq x}\left(n\cdot\sum_{\substack{p\\ r(p)=n}}\frac{1}{p}\right),$$

by (3) we have

$$\sum_{\substack{p\\ r(p)\leqq x}}\frac{r(p)}{p}\leqq 2\cdot\sum_{n\leqq x}(\log x-\log n)+\sum_{\substack{p>x\\ r(p)\leqq x}}\frac{r(p)}{p}+\mathcal{O}(x). \tag{4}$$

We have seen that there are at most $c_6 n/\log n$ primes p for which $r(p)=n$, furthermore

$$\sum_{n\leqq x} \frac{c_6 n}{\log n} < c_{13}\cdot\frac{x^2}{\log x}$$

and there are at most $c_{13}x^2/\log x$ primes not exceeding $c_{14}x^2$, therefore

$$(5)\qquad \sum_{\substack{p>x\\ r(p)\leqq x}} \frac{r(p)}{p} \leqq x\cdot \sum_{\substack{p>x\\ r(p)\leqq x}} \frac{1}{p} \leqq x\cdot\left(\sum_{p\leqq c_{14}x^2}\frac{1}{p}-\sum_{p\leqq x}\frac{1}{p}\right) = \mathcal{O}(x).$$

On the other hand

$$(6)\qquad \sum_{n\leqq x}\log x = x\cdot\log x+\mathcal{O}(\log x)$$

and

$$(7)\qquad \sum_{n\leqq x}\log n = x\cdot\log x - x+\mathcal{O}(\log x)$$

and so (4), by (5), (6) and (7), implies the right hand inequality of the theorem which completes the proof.

Proof of Corollary 1. Suppose that there are at most x^c primes p such that $p\leqq x$ and

$$\frac{r(p)}{p} > \frac{1}{p^{1-c}}.$$

Then, using that $r(p)\leqq p+1$ and so $r(p)/p\leqq 1+\frac{1}{p}$ for any prime, with the notation $y=\pi(x)-x^c$ we have

$$(8)\qquad \sum_{p\leqq x}\frac{r(p)}{p} < \sum_{n\leqq y}\frac{1}{p_n^{1-c}}+x^c+\sum_{n\leqq x^c}\frac{1}{p_n} < \sum_{p\leqq x}\frac{1}{p^{1-c}}+x^c+\mathcal{O}(\log\log x),$$

where p_n is the n^{th} prime. But

$$\sum_{p\leqq x}\frac{1}{p^{1-c}} = \mathcal{O}\left(\frac{x^c}{\log x}\right)$$

and so (8) contradicts Theorem 1 if x is sufficiently large since $c<1/2$.

Thus there are at least x^c prime numbers p not exceeding x for which

$$(9)\qquad \frac{r(p)}{p} > \frac{1}{p^{1-c}}.$$

For these primes by (9)

$$p < \left(r(p)\right)^{1/c}$$

which proves the assertion.

Proof of Corollary 2. Let x and δ be positive numbers and let N_x be the set of natural numbers n for which $n\leqq x$ and

$$\sum_{\substack{p\\ r(p)=n}}\frac{1}{p} \geqq \frac{\delta\cdot\log n}{n}.$$

If $|N_x| = \varepsilon x$, where $|N_x|$ denotes the cardinality of the set N_x, then

$$\sum_{\substack{p \\ r(p) \leqq x}} \frac{r(p)}{p} = \sum_{n \leqq x} \left(n \cdot \sum_{\substack{p \\ r(p)=n}} \frac{1}{p} \right) \geqq \sum_{n \leqq \varepsilon x} \delta \cdot \log n =$$
$$= \delta \cdot \left(\varepsilon x \cdot \log \varepsilon x - \varepsilon x + \mathcal{O}(\log x) \right)$$

which does not contradict Theorem 2 only if $\varepsilon \to 0$ as $x \to \infty$. From this the corollary follows.

References

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