Differences

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Notations and preliminary results

Let $\mathbf{R}$ be the set of real numbers and let $\mathbf{R}_+$ be the set of positive real numbers.

A function $\alpha\colon \mathbf{R}_+\to\mathbf{R}$ (or $\alpha\colon \mathbf{R}\to\mathbf{R}$) is a Jensen-function if

$$2\alpha\left(\frac{x+y}{2}\right) = \alpha(x)+\alpha(y) \tag{1}$$

is valid for all $x, y\in\mathbf{R}_+$ (or for all $x, y\in\mathbf{R}$).

Denote by $J(\mathbf{R}_+\to\mathbf{R})$ and $J(\mathbf{R}\to\mathbf{R})$ the class of Jensen-functions on $\mathbf{R}_+$ and $\mathbf{R}$, respectively.

The following theorem was proved by Z. DARÓCZY, L. SZÉKELYHIDI and myself in [1]:

Theorem 1. *Let $f\colon \mathbf{R}_+\to\mathbf{R}$ be a function, such that the function $\Delta_\lambda f\colon \mathbf{R}_+\to\mathbf{R}$ defined by the difference*

$$\Delta_\lambda f(x) = f(\lambda x)-f(x) \quad (x\in\mathbf{R}_+) \tag{2}$$

is a Jensen-function (i.e. $\Delta_\lambda f\in J(\mathbf{R}_+\to\mathbf{R})$) for all fixed $\lambda\in\mathbf{R}_+$. Then f has the form

$$f(x) = \alpha(x)+m(x) \quad (x\in\mathbf{R}_+),$$

where $\alpha\in J(\mathbf{R}_+\to\mathbf{R})$ and $m\colon \mathbf{R}_+\to\mathbf{R}$ satisfies the Cauchy functional equation

$$m(xy) = m(x)+m(y) \quad (x, y\in\mathbf{R}_+).$$

This theorem and its generalizations have applications to find the general solution of some functional equations (see [1], [2], [3]).

In this paper we consider the difference

$$\Delta_\lambda^+ f(x) = f(x+\lambda)-f(x) \quad (x, \lambda\in\mathbf{R})$$

instead of (2) and we prove a theorem analogous to Theorem 1.

2. The main result

Theorem 2. *Let $f\colon \mathbf{R}\to\mathbf{R}$ be an arbitrary function, such that the function*

$$\Delta_\lambda^+ f\colon \mathbf{R}\to\mathbf{R},\quad \Delta_\lambda^+ f(x)=f(x+\lambda)-f(x)\quad (x\in\mathbf{R}) \tag{3}$$

is a Jensen-function for all fixed $\lambda\in\mathbf{R}$. Then f has the form

$$f(x)=\alpha(x)+N(x)\quad (x\in\mathbf{R}), \tag{4}$$

where $\alpha\in J(\mathbf{R}\to\mathbf{R})$ and $N\colon \mathbf{R}\to\mathbf{R}$ satisfies the functional equation

$$N(x+y)+N(x-y)=2N(x)+2N(y)\quad (x, y\in\mathbf{R}). \tag{5}$$

Proof. If $\Delta_\lambda^+ f\in J(\mathbf{R}\to\mathbf{R})$ for all fixed $\lambda\in\mathbf{R}$, then the function $f\colon \mathbf{R}\to\mathbf{R}$ satisfies the functional equation

$$2f\left(\frac{x+y}{2}+\lambda\right)-2f\left(\frac{x+y}{2}\right)=f(x+\lambda)-f(x)+f(y+\lambda)-f(y)\quad (x, y, \lambda\in\mathbf{R}). \tag{6}$$

In (6) let us replace $-x$, $-y$ and $-\lambda$ by x, y and λ respectively, then we get

$$2f\left[-\left(\frac{x+y}{2}+\lambda\right)\right]-2f\left(-\frac{x+y}{2}\right)= \tag{7}$$

$$=f[-(x+\lambda)]-f(-x)+f[-(y+\lambda)]-f(-y)\quad (x, y, \lambda\in\mathbf{R}).$$

We easily obtain from (6) and (7) that the functions

$$\bar f\colon \mathbf{R}\to\mathbf{R},\quad \bar f(x)=f(x)-f(-x)\quad (x\in\mathbf{R})$$

and

$$f^*\colon \mathbf{R}\to\mathbf{R},\quad f^*(x)=f(x)+f(-x)\quad (x\in\mathbf{R})$$

satisfy the functional equation (6), moreover $\bar f$ is odd and f^* is even.

In addition, we have

$$f(x)=\frac{1}{2}[\bar f(x)+f^*(x)]\quad (x\in\mathbf{R}). \tag{8}$$

By substitution $\lambda=-y$ in (6), we get for $\bar f$ the functional equation

$$2\bar f\left(\frac{x-y}{2}\right)-2\bar f\left(\frac{x+y}{2}\right)=\bar f(x-y)-\bar f(x)-\bar f(y)\quad (x, y\in\mathbf{R}). \tag{9}$$

Interchanging x and y here, using that $\bar f$ is an odd function, we get

$$-2\bar f\left(\frac{x-y}{2}\right)-2\bar f\left(\frac{x+y}{2}\right)=-\bar f(x-y)-\bar f(x)-\bar f(y)\quad (x, y\in\mathbf{R}).$$

Adding this equation to (9), we obtain

$$2\bar f\left(\frac{x+y}{2}\right)=\bar f(x)+\bar f(y)\quad (x, y\in\mathbf{R}),$$

i.e. $\bar f\in J(\mathbf{R}\to\mathbf{R})$.

Now consider the functional equation (6) for function f^* and substitute here $\lambda=-y$, then replace y with $-y$. Thus we get

$$(10)\quad 2f^*\left(\frac{x+y}{2}\right)-2f^*\left(\frac{x-y}{2}\right)=f^*(x+y)-f^*(x)-f^*(y)+f^*(0)\quad (x, y\in\mathbf{R}).$$

With the substitution $x=y=\frac{t}{2}$, we get from (10) that

$$f^*\left(\frac{t}{2}\right)=\frac{1}{4}f^*(t)+\frac{3}{4}f^*(0)\quad (t\in\mathbf{R}).$$

Using this equation, we obtain from (10) the equation

$$f^*(x+y)+f^*(x-y)=2f^*(x)+2f^*(y)-2f^*(0)\quad (x, y\in\mathbf{R}),$$

which implies that the function

$$N\colon \mathbf{R}\to\mathbf{R},\quad N(x)=f^*(x)-f^*(0)\quad (x\in\mathbf{R})$$

satisfies the functional equation (5). Thus f^* has the form

$$(11)\qquad f^*(x)=N(x)+f^*(0)\quad (x\in\mathbf{R}),$$

where $f^*(0)\in\mathbf{R}$ is an arbitrary constant.

From (8) and (11) it follows that

$$f(x)=\frac{1}{2}\bar{f}(x)+\frac{1}{2}f^*(0)+\frac{1}{2}N(x)\quad (x\in\mathbf{R}).$$

Since $\frac{1}{2}\bar{f}+\frac{1}{2}f^*(0)\doteq\alpha\in J(\mathbf{R}\to\mathbf{R})$ and $\frac{1}{2}N$ also satisfies the functional equation (5), we obtain the form (4) for f.

Thus the Theorem 2 is proved.

3. Remarks and applications

Remark 1. The converses of Theorems 1 and 2 are also true.

E.g. if f has the form (4), then using the property $4N\left(\frac{x}{2}\right)=N(x)$ $(x\in\mathbf{R})$ of the function N, we have

$$2\Delta_\lambda^+ f\left(\frac{x+y}{2}\right)=2\alpha\left(\frac{x+y}{2}+\lambda\right)-2\alpha\left(\frac{x+y}{2}\right)+2N\left(\frac{x+y}{2}+\lambda\right)-2N\left(\frac{x+y}{2}\right)=$$

$$=2\alpha\left(\frac{(x+\lambda)+(y+\lambda)}{2}\right)-2\alpha\left(\frac{x+y}{2}\right)+2N\left(\frac{x+\lambda}{2}+\frac{y+\lambda}{2}\right)-2N\left(\frac{x}{2}+\frac{y}{2}\right)=$$

$$= \alpha(x+\lambda)+\alpha(y+\lambda)-\alpha(x)-\alpha(y)-2N\left(\frac{x+\lambda}{2}-\frac{y+\lambda}{2}\right)+4N\left(\frac{x+\lambda}{2}\right)+4\left(\frac{y+\lambda}{2}\right)+$$

$$+2N\left(\frac{x-y}{2}\right)-4N\left(\frac{x}{2}\right)-4N\left(\frac{y}{2}\right) = \alpha(x+\lambda)-\alpha(x)+\alpha(y+\lambda)-\alpha(y)+$$

$$+N(x+\lambda)-N(x)+N(y+\lambda)-N(y) = \Delta_\lambda^+ f(x)+\Delta_\lambda^+ f(y),$$

i.e. $\Delta_\lambda^+ f \in J(\mathbf{R}\to\mathbf{R})$.

Remark 2. A similar theorem was used by GY. MAKSA (see [4]) in connection with the functional equation

$$f(x+y)+g(xy) = h(x)+h(y) \quad (x, y\in \mathbf{R}_+),$$

which was considered for the first time by Z. DARÓCZY in 1969.

Remark 3. Theorem 2 can be generalized for functions $f\colon K\to A$, where K and A are additive Abelian groups.

Remark 4. Theorem 2 can be used to solve e.g. the functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x)+f(y)+g(x-y) \quad (x, y\in \mathbf{R})$$

for the unknown functions $f, g\colon \mathbf{R}\to\mathbf{R}$.

References

[1] Z. DARÓCZY, K. LAJKÓ and L. SZÉKELYHIDI, Functional equations on ordered fields, *Publ. Math. (Debrecen)*, **24** (1977), 173—179.
[2] K. LAJKÓ, Remark to a paper by J. A. BAKER, *Aequationes Math.*, **19** (1979), 227—231.
[3] K. LAJKÓ, On the general solution of rectangle-type functional equations, *Publ. Math. (Debrecen)*, **28** (1981), 137—143.
[4] GY. MAKSA, On the functional equation $f(x+y)+g(xy)=h(x)+h(y)$, *Publ. Math. (Debrecen)*, **24** (1977), 25—29.

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