On a functional equation of sum form

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Introduction

Functional equations of sum form appear in characterizations of entropies having the sum property (see e.g. ACZÉL—DARÓCZY [1] p. 110, LOSONCZI [3]). Examples are the equations

$$\sum_{i=1}^{k}\sum_{j=1}^{l} f(x_i y_j) = \sum_{i=1}^{k} f(x_i)+\sum_{j=1}^{l} f(y_j) \tag{1}$$

and

$$\sum_{i=1}^{k}\sum_{j=1}^{l} f(x_i y_j) = \sum_{i=1}^{k} f(x_i)\sum_{j=1}^{l} f(y_j). \tag{2}$$

Here $k, l \geqq 2$ are fixed integers, $(x_1, \ldots, x_k)\in\Gamma_k$, $(y_1, \ldots, y_l)\Gamma_l$ with

$$\Gamma_n = \Big\{(x_1, \ldots, x_n)\Big| x_i \geqq 0,\ \sum_{i=1}^{n} x_i = 1\Big\} \quad (n = 2, 3, \ldots)$$

and $f\colon [0, 1]\to\mathbf{C}$ is an unknown function.

Equations (1) and (2) have been solved under various assumptions on f. If $k, l\geqq 3$ and f is measurable then applying a general result of LOSONCZI ([4] Theorem 4) (1) and (2) can be reduced to the equations

$$f_1(xy) = yf_2(x)+xf_3(y) \quad x, y\in[0, 1]$$

and

$$f_1(xy) = f_2(x)f_3(y) \qquad x, y\in[0, 1]$$

respectively, where

$$f_1(x) = f(x)-f(0)+xklf(0),$$

$$f_2(x) = f_1(x)-xk(l-1)f(0), \quad f_3(x) = f_1(x)-x(k-1)\,lf(0).$$

Thus, solving (1), (2) and similar equations is simple if f is measurable and $k, l\geqq 3$. Difficulties arise however if $k=3$, $l=2$ ($k=2$, $l=3$), $k=l=2$ or if f is not measurable.

The measurable solutions and the solutions bounded on a set of positive measure of (1) are known if $k, l\geqq 2$ and $k\geqq 3$, $l\geqq 2$ respectively (DARÓCZY—JÁRAI [2], MAKSA [7]). The general solution is however not known even if $k, l \geqq 3$. The only result in this direction is a representation of the solution of (1) by help of certain additive functions (LOSONCZI—MAKSA [6]).

In case of equation (2) the situation is completely different. The general solution of (2) is known if $k \geqq 3$, $l \geqq 3$ (LOSONCZI—MAKSA [5]) or $k \geqq 3$, $l \geqq 2$ and $f(x)+f(1-x) \not\equiv 1$ (MAKSA [8]). Nothing has been known on the solutions if $k=l=2$.

The aim of this paper is to study equation (2) with $k=l=2$, that is the equation

$$f(xy)+f\big(x(1-y)\big)+f\big((1-x)y\big)+f\big((1-x)(1-y)\big) = \tag{3}$$

$$= [f(x)+f(1-x)][f(y)+f(1-y)] \quad x, y \in [0, 1].$$

We shall find all solutions of (3) in the class $C_3[0, 1]$, the space of all three times continuously differentiable complex valued functions on $[0, 1]$.

The behaviour of equation (3) is quite surprising.

First, the methods which have been used to find the general solution of (2) if $k \geqq 3$, $l \geqq 2$ completely fail.

Second, new solutions arise. Some polynomials of degree 5 are solutions of (3) but (2) with $k \geqq 3$, $l \geqq 2$ has no polynomial solution of degree 5 (different from x^5).

2. Solution of the equation (3)

Our main result is the following

Theorem. *If* $f\colon [0, 1] \to \mathbf{C}$, $f \in C_3[0, 1]$ *is a solution of* (3) *then*

$$f(x) = x^p \quad x \in [0, 1] \tag{4}$$

or

$$f(x) = qx + 1/2[1-q \pm (1-q)^{1/2}] \quad x \in [0, 1] \tag{5}$$

or

$$f(x) = rx^3 + 1/2[1-3r \pm (1+3r)^{1/2}]x^2 + 1/2[1+r \mp (1+3r)^{1/2}]x \quad x \in [0, 1] \tag{6}$$

or

$$f(x) = \tag{7}$$

$$= sx^5 + 1/2[1-5s \pm (1+15s)^{1/2}]x^4 + 1/54[1+3s \mp [1+15s)^{1/2}](40x^3-15x^2+2x)$$

$$x \in [0, 1]$$

where $\operatorname{Re} p \geqq 3$, p, q, r, s *are complex constants. Conversely, the functions* (4), (5), (6), (7) *are* $C_3[0, 1]$ *solutions of* (3) *if* $p, q, r, s \in \mathbf{C}$ *are arbitrary constants with* $\operatorname{Re} p \geqq 3$.

PROOF. Suppose that $f \in C_3[0, 1]$ satisfies (3). We shall deduce a differential equation for f, solve it and select the solutions of (3) out of the solutions of our differential equation. Let

$$h(x) = f(x) + f(1-x) \tag{8}$$

then

$$f(xy)+f(x(1-y))+f((1-x)y)+f((1-x)(1-y)) = h(x)h(y). \tag{9}$$

Differentiating with respect to y ones, twice, three times we obtain

$$xf'(xy)-xf'(x(1-y))+(1-x)f'((1-x)y)-(1-x)f'((1-x)(1-y)) = \\ = h(x)h'(y) \tag{10}$$

$$x^2f''(xy)+x^2f''(x(1-y))+(1-x)^2f''((1-x)y)+(1-x)^2f''((1-x)(1-y)) = \\ = h(x)h''(y) \tag{11}$$

$$x^3f'''(xy)-x^3f'''(x(1-y))+(1-x)^3f'''((1-x)y)-(1-x)^3f'''((1-x)(1-y)) = \\ = h(x)h'''(y). \tag{12}$$

Substituting $y=1$ in (9), (10), (11), (12) we get

$$x^nf^{(n)}(x)+(1-x)^nf^{(n)}(1-x)+(-1)^n[x^n+(1-x)^n]f^{(n)}(0) = h(x)h^{(n)}(1) \tag{13}$$

for $n=0, 1, 2, 3$; $x\in[0, 1]$.

Differentiation of (13) and multiplication by $1-x$ yields

$$n(1-x)^{n-1}f^{(n)}(x)+(1-x)x^nf^{(n+1)}(x)-n(1-x)^nf^{(n)}(1-x)- \\ -(1-x)^{n+1}f^{(n+1)}(1-x)+n(-1)^n[x^{n-1}-(1-x)^{n-1}]f^{(n)}(0) = (1-x)h'(x)h^{(n)}(1). \tag{14}$$

Adding equation (13) with n replaced by $n+1$ and equation (13) multiplied by n to equation (14) all derivatives at the point $1-x$ disappear and we have

$$nx^{n-1}f^{(n)}(x)+x^nf^{(n+1)}(x)+n(-1)^nx^{n-1}f^{(n)}(0)+ \\ +(-1)^{n+1}[x^{n+1}+(1-x)^{n+1}]f^{(n+1)}(0) = [nh^{(n)}(1)+h^{(n+1)}(1)]h(x)+h^{(n)}(1)(1-x)h'(x). \tag{15}$$

For $n=0, 1, 2$ (15) gives the following system of equations:

$$f'(x)-f'(0) = h'(1)h(x)+h(1)(1-x)h'(x) \tag{16}$$

$$f'(x)+xf''(x)-f'(0)+[x^2+(1-x)^2]f''(0) = [h'(1)+h''(1)]h(x)+h'(1)(1-x)h'(x) \tag{17}$$

$$2xf''(x)+x^2f'''(x)+2xf''(0)-[x^3+(1-x)^3]f'''(0) = \\ = [2h''(1)+h'''(1)]h(x)+h''(1)(1-x)h'(x). \tag{18}$$

Denote by $L_1(x)$, $L_2(x)$, $L_3(x)$ the left hand side of (16), (17), (18) respectively. At any fixed $x\in[0, 1]$ (16), (17), (18) may be considered as a linear homogeneous system of algebraic equations for the unknowns -1, $h(x)$, $(1-x)h'(x)$. Since this system has nontrivial solution its determinant must be zero:

$$\begin{vmatrix} L_1(x) & h'(1) & h(1) \\ L_2(x) & h'(1)+h''(1) & h'(1) \\ L_3(x) & 2h''(1)+h'''(1) & h''(1) \end{vmatrix} = 0. \tag{19}$$

Expanding this determinant by the elements of the first column we get

$$A_1L_1(x)+A_2L_2(x)+A_3L_3(x)=0 \tag{20}$$

where A_i are the cofactors of L_i $(i=1,2,3)$ in (19).

To evaluate A_3 we substitute $x=0$ in (17) We obtain that

$$f''(0)=[h'(1)+h''(1)]h(0)+h'(1)h'(0).$$

Since $h(0)=h(1)$, $h'(0)=-h'(1)$ we have

$$A_3=h'(1)^2-[h'(1)+h''(1)]h(1)=-f''(0). \tag{21}$$

Similarly, from (18) with $x=0$

$$A_2=h(1)[2h''(1)+h'''(1)]-h'(1)h''(1)=-f'''(0). \tag{22}$$

To find A_1 we differentiate (17) with respect to x. The resulting equation is

$$2f''(x)+xf'''(x)+f''(0)(4x-2)=[h'(1)+h''(1)]h'(x)+h'(1)[(1-x)h''(x)-h'(x)].$$

Here the function $x\to xf'''(x)$ must be differentiable on $[0,1]$ since all other terms in the equation are differentiable. Thus we may differentiate again and obtain

$$2f'''(x)+(xf'''(x))'+4f''(0)=[h'(1)+h''(1)]h''(x)+h'(1)[(1-x)h'''(x)-2h''(x)].$$

Using the relation

$$(xf'''(x))'_{x=0}=\lim_{h\to 0}hf'''(h)/h=f'''(0)$$

we get with $x=0$

$$3f'''(0)+4f''(0)=[h'(1)+h''(1)]h''(0)+h'(1)[h'''(0)-2h''(0)].$$

By $h''(0)=h''(1)$, $h'''(0)=-h'''(1)$ the right hand side of this equation is exactly A_1 hence

$$A_1=3f'''(0)+4f''(0). \tag{23}$$

The form of our differential equation (20) thus depends on the values $f''(0)$, $f'''(0)$. We shall distinguish several cases.

Case 1. $h(1)=f(1)+f(0)\neq 1$. Then from (13) with $n=0$

$$h(x)[h(1)-1]=2f(0) \tag{24}$$

therefore $h(x)=\text{constant}$. By (16) $f'(x)-f'(0)=0$ i.e. f is a linear function. Substitution shows that this is a solution of (3) if and only if f is of the form (5).

Case 2. $h(1)=f(1)+f(0)=1$. Then by (24) $f(0)=0$ therefore $f(1)=1$.

Subcase 2.1. $h(1)=1$, $h'(1)=0$, $f''(0)=0$. By (21) $h''(1)=0$ and from (17)

$$f'(x)+xf''(x)=f'(0).$$

The general solution of this Euler's differential equation is

$$f(x)=c_1+c_2\ln x+f'(0)x \quad x\in[0,1]$$

with arbitrary constants c_1, c_2. Since $f(0)=0$, $f\in C_3[0, 1]$ the constants c_1, c_2 must be zero. $f(x)=f'(0)x$ is a solution of (3) if and only if $f'(0)=0$ or $f'(0)=1$. These solutions are included in (5).

Subcase 2.2. $h(1)=1$, $h'(1)\neq 0$, $f''(0)=0$. From (21)

$$\big(h'(1)+h''(1)\big)/h'(1) = h'(1)/h(1) = p \neq 0.$$

Multiplying (16) by $-p$ and adding it to (17) we get

$$(1-p)f'(x)+xf''(x) = (1-p)f'(0). \tag{25}$$

The general solution of (25) is

$$f(x) = c_1+c_2x^p+f'(0)x$$

with arbitrary constants c_1, c_2. $f\in C_3[0, 1]$ implies that either $c_2=0$ or $p\in\{1, 2\}\cup$ $\cup\{s\in\mathbf{C}|\mathrm{Re}\, s\geqq 3\}$. By $f(0)=0$, $f(1)=1$ we have $c_1=0$, $c_2=1-f'(0)$, i.e.

$$f(x) = [1-f'(0)]x^p+f'(0)x. \tag{26}$$

If $p=1$ then $f(x)=x$ which is a solution of form (5).
If $p\neq 1$ then the substitution of (26) in (3) yields that

$$[1-f'(0)]f'(0)[x^p+(1-x)^p-1][y^p+(1-y)^p-1] = 0.$$

Hence (26) is a solution of (3) if and only if either $1-f'(0)=0$ or $f'(0)=0$. In the first case f is of the form (5), in the second f is of the form (4) if $p\neq 2$ while f is of the form (6) (with $r=0$) if $p=2$.

Subcase 2.3. $h(1)=1$, $f''(0)\neq 0$. By (21), (22), (23) we get from (20) that

$$L_3(x)+uL_2(x)-(3u+4)L_1(x) = 0$$

where $u=f'''(0)/f''(0)$. Using the definition of L_i this can be written as

$$x^2f'''(x)+(u+2)\,xf''(x)-2(u+2)f'(x)+2(u+2)f'(0)+2f''(0)x+f'''(0)(x-x^2) = 0.$$

Integrating from 0 to x and applying the formulas

$$\int_0^x tf''(t)\,dt = xf'(x)-f(x)+f(0),$$

$$\int_0^x t^2f'''(t)\,dt = x^2f''(x)-2xf'(x)+2f(x)-2f(0)$$

and the condition $f(0)=0$ we obtain

$$x^2f''(x)+uxf'(x)-(3u+4)f(x) = \frac{x^3}{3}f'''(0)-\frac{x^2}{2}(u+2)f''(0)-2x(u+2)f'(0). \tag{27}$$

Subcase 2.3.1. $h(1)=1$, $f''(0)\neq 0$, $u=f'''(0)/f''(0)=-2$. We show that this case cannot occur. The general solution of (27) is

$$f(x) = c_1x+c_2x^2+\frac{f'''(0)}{6}x^3 = c_1x+c_2x^2+c_3x^3$$

with arbitrary constants c_1, c_2. The conditions $f(1)=1, f''(0)\neq 0, u=-2$ give that $c_1+c_2+c_3=1$, $c_2\neq 0$, $6c_3/2c_2=-2$ that is

$$f(x) = c_3x^3-\frac{3}{2}c_3x^2+(1+c_3/2)x. \tag{28}$$

Substituting (28) in (3) we obtain, after a simple calculation, that

$$6c_3(x^2-x)(y^2-y) = 0.$$

Hence $c_3=0$ and $c_2=-\frac{3}{2}c_3=0$ which is a contradiction.

Subcase 2.3.2. $h(1)=1, f''(0)\neq 0, u=f'''(0)/f''(0)\neq -2$. In this case

$$g(x) = \frac{f'''(0)}{6}x^3+\frac{f''(0)}{2}x^2+f'(0)x$$

is a particular solution of (27). The homogeneous equation corresponding to (27) is an Euler's equation with characteristic equation

$$s^2+(u-1)s-(3u+4) = 0. \tag{29}$$

We have to distinguish 2 cases again.

Subcase 2.3.2.1. $h(1)=1$, $f''(0)\neq 0$, $u\neq -2$, $u^2+10u+17=0$. Now $u=-5\pm\sqrt{8}$ and the corresponding root of (29) $s_1=\frac{u-1}{2}=-3\pm\sqrt{2}$ has multiplicity 2. The general solution of (27) is

$$f(x) = c_1x^{s_1}+c_2x^{s_1}\ln x+g(x) \quad x\in(0, 1].$$

Since $f(0)=0$ and $f\in C_3[0, 1]$ c_1, c_2 must be zero. Thus

$$f(x) = g(x) = \frac{f'''(0)}{6}x^3+\frac{f''(0)}{2}x^2+f'(0)x = d_1x^3+d_2x^2+d_3x \tag{30}$$

and by $f(1)=1$ $d_1+d_2+d_3=1$. Substituting (30) in (3) and using the condition $d_1+d_2+d_3=1$ we obtain

$$[(3d_1+2d_2)^2-(9d_1+4d_2)](x^2-x)(y^2-y) = 0$$

hence (30) is a solution of (3) if and only if

$$(3d_1+2d_2)^2 = 9d_1+4d_2, \quad d_3 = 1-d_1-d_2$$

hold. Therefore, with $d_1=r$

$$d_2 = \frac{1}{2}[1-3r\pm(1+3r)^{1/2}], \quad d_3 = \frac{1}{2}[1+r\mp(1+3r)^{1/2}].$$

We obtained solution (6) (with $d_2\neq 0$, $3d_1/d_2=-5\pm\sqrt{8}$).

Subcase 2.3.2.2. $h(1)=1, f''(0)\neq 0, u\neq -2, u^2+10u+17\neq 0$. In this case (29) has two distinct roots s_1, s_2. We may suppose that $\operatorname{Re} s_1 \geqq \operatorname{Re} s_2$. Due to the condition $u\neq -2$ we have $\{s_1, s_2\}\cap\{1, 2, 3\}=\emptyset$ therefore the general solution of (27) is

(31) $$f(x) = c_1 x^{s_1}+c_2 x^{s_2}+g(x) \quad x\in(0, 1]$$

with arbitrary constants c_1, c_2.

It would be a very long calculation to substitute (31) in (3). Therefore first we try to satisfy (12). Substituting (31) in (12) we obtain after some rearrangements that

(32) $$e_1 k_1(x) l_1(y)+e_2 k_2(x) l_2(y) = h(x) h'''(y) \quad x, y\in(0, 1)$$

where

(33) $$l_i(y) = y^{s_i-3}-(1-y)^{s_i-3} \quad y\in(0, 1)$$

(34) $$k_i(y) = x^{s_i}+(1-x)^{s_i} \qquad x\in(0, 1)$$

(35) $$e_i = c_i s_i(s_i-1)(s_i-2) \quad (i = 1, 2).$$

Subcase 2.3.2.2.1. $h(1)=1,\ f''(0)\neq 0,\ u\neq -2,\ u^2+10u+17\neq 0,\ h'''(y)=0$ *if* $y\in(0, 1)$. Now h is a polynomial of degree $\leqq 2$ and from (16) f is a polynomial of degree $\leqq 3$ with $f(0)=0, f(1)=1$. Thus, as in the Subcase 2.3.2.1, f must be of the form (6) (with $d_2\neq 0,\ 3d_1/d_2\neq -5\pm\sqrt{8}$).

Subcase 2.3.2.2.2. $h(1)=1,\ f''(0)\neq 0,\ u\neq -2,\ u^2+10u+17\neq 0,\ h'''(y_0)\neq 0$ *for some* $y_0\in(0, 1)$, $\left(y_0\neq\frac{1}{2}\right)$. From (32) we obtain that

(36) $$h(x) = B_1 k_1(x)+B_2 k_2(x)$$

with suitable constants B_1, B_2. With (36) we get from (32) that

(37) $$k_1(x)[e_1 l_1(y)-B_1 h'''(y)]+k_2(x)[e_2 l_2(y)-B_2 h'''(y)] = 0, \quad x, y\in(0, 1).$$

At this point we have to know if k_1, k_2 (and l_1, l_2) are linearly independent or not. First we deal with this question.

Lemma 1. *Let* $s_1, s_2\in\mathbf{C}, s_1\neq s_2$. *The functions* k_1, k_2 *defined by* (34) *are linearly dependent on* $(0, 1)$ *if and only if* $s_1=0, s_2=1$ *or* $s_1=1, s_2=0$.

PROOF OF LEMMA 1. $x^0+(1-x)^0=2=2[x+(1-x)]$ thus k_1, k_2 are linearly dependent for $s_1=0, s_2=1$ and $s_1=1, s_2=0$.

We show that apart from these cases k_1, k_2 are linearly independent. Suppose that

(38) $$\lambda_1 k_1(x)+\lambda_2 k_2(x) = 0 \quad x\in(0, 1)$$

holds with constants λ_1, λ_2. Then

$$\lambda_1 k_1\left(\frac{1}{2}\right)+\lambda_2 k_2\left(\frac{1}{2}\right)=0 \tag{39}$$

$$\lambda_1 k_1''\left(\frac{1}{2}\right)+\lambda_2 k_2''\left(\frac{1}{2}\right)=0 \tag{40}$$

$$\lambda_1 k_1^{\mathrm{IV}}\left(\frac{1}{2}\right)+\lambda_2 k_2^{\mathrm{IV}}\left(\frac{1}{2}\right)=0 \tag{41}$$

are valid too. If $s_1+s_2\neq 1$ then

$$\begin{vmatrix} k_1\left(\frac{1}{2}\right) & k_2\left(\frac{1}{2}\right) \\ k_1''\left(\frac{1}{2}\right) & k_2''\left(\frac{1}{2}\right) \end{vmatrix} = 2^{4-s_1-s_2}(s_2-s_1)(s_1+s_2-1)\neq 0$$

thus by (39), (40) $\lambda_1=\lambda_2=0$, k_1, k_2 are linearly independent.

If $s_1+s_2=1$ but $(|s_1|+|s_2-1|)(|s_1-1|+|s_2|)\neq 0$ then

$$\begin{vmatrix} k_1''\left(\frac{1}{2}\right) & k_2''\left(\frac{1}{2}\right) \\ k_1^{\mathrm{IV}}\left(\frac{1}{2}\right) & k_2^{\mathrm{IV}}\left(\frac{1}{2}\right) \end{vmatrix} = 2^{8-s_1-s_2}s_1 s_2(s_1-1)(s_2-1)(s_1-s_2)(s_1+s_2-5)\neq 0$$

therefore by (40), (41) $\lambda_1=\lambda_2=0$ and k_1, k_2 are linearly independent.

Lemma 2. *Let* $s_1, s_2\in \mathbf{C}$, $s_1\neq s_2$. *The functions* l_1, l_2 *defined by* (33) *are linearly dependent on* $(0, 1)$ *if and only if* $s_1=3$ $(s_2\neq 3)$ *or* $s_2=3$ $(s_1\neq 3)$ *or* $s_1=5, s_2=4$ *or* $s_1=4$, $s_2=5$.

Proof of Lemma 2. It is easy to see that l_1, l_2 are linearly dependent if s_1, s_2 have the values listed in Lemma 2. We show that apart from these cases l_1, l_2 are linearly independent. This follows from the fact that one of the determinants

$$\begin{vmatrix} l_1'\left(\frac{1}{2}\right) & l_2'\left(\frac{1}{2}\right) \\ l_1'''\left(\frac{1}{2}\right) & l_2'''\left(\frac{1}{2}\right) \end{vmatrix} = 2^{12-s_1-s_2}(s_1-3)(s_2-3)(s_2-s_1)(s_1+s_2-9)$$

and

$$\begin{vmatrix} l_1'''\left(\frac{1}{2}\right) & l_2'''\left(\frac{1}{2}\right) \\ l_1^{\mathrm{V}}\left(\frac{1}{2}\right) & l_2^{\mathrm{V}}\left(\frac{1}{2}\right) \end{vmatrix} =$$

$$=2^{14-s_1-s_2}(s_1-3)(s_2-3)(s_1-4)(s_2-4)(s_1-5)(s_2-5)(s_2-s_1)(s_1+s_2-13)$$

is nonzero if s_1, s_2 are not among the pairs listed in Lemma 2.

Now we *return to the proof of our theorem.* We remark that the assumptions on s_1, s_2 in the Subcase 2.3.2.2 and in Lemmas 1, 2 are slightly different. In Subcase 2.3.2.2 $\operatorname{Re} s_1 \geqq \operatorname{Re} s_2$, $s_1 \neq s_2$ and $\{s_1, s_2\} \cap \{1, 2, 3\} = \emptyset$ are supposed while in Lemmas 1, 2 only $s_1 \neq s_2$ is a priori assumed.

By $\{s_1, s_2\} \cap \{1, 2, 3\} = \emptyset$ and by Lemma 1 k_1, k_2 are linearly independent thus by (37)

$$e_1 l_1(y) = B_1 h'''(y) \quad y \in (0, 1) \tag{42}$$

$$e_2 l_2(y) = B_2 h'''(y) \quad y \in (0, 1). \tag{43}$$

If $e_1 = e_2 = 0$ then $c_1 = c_2 = 0, f(x) = g(x)$ which is a solution of (3) if and only if f is of the form (6).

If $e_1 e_2 = 0$ *but one of* e_1, e_2, *say* e_1, *is nonzero* then $e_2 = 0$, $c_2 = 0$ and

$$f(x) = c_1 x^{s_1} + g(x). \tag{44}$$

Substituting this function in (3) we get

$$\begin{aligned} c_1 k_1(x) k_1(y) + g(xy) + g\big(x(1-y)\big) + g\big((1-x)y\big) + g\big((1-x)(1-y)\big) = \\ = [c_1 k_1(x) + g(x) + g(1-x)][c_1 k_1(y) + g(y) + g(1-y)]. \end{aligned} \tag{45}$$

Differentiating (45) three times with respect to y we obtain

$$c_1 k_1(x) k_1'''(y) = [c_1 k_1(x) + g(x) + g(1-x)] c_1 k_1'''(y)$$

therefore, by

$$c_1 k_1'''(y) = c_1 s_1 (s_1 - 1)(s_1 - 2)[y^{s_1 - 3} - (1-y)^{s_1 - 3}] \neq 0 \quad \text{if} \quad y \neq \frac{1}{2}$$

we get

$$k_1(x) = c_1 k_1(x) + g(x) + g(1-x).$$

Differentiation three times shows that $c_1 = 1$ and $g(x) + g(1-x) = 0$. Hence from (45)

$$g(xy) + g\big(x(1-y)\big) + g\big((1-x)y\big) + g\big((1-x)(1-y)\big) = 0$$

which implies that

$$g(x) = 0 = \frac{f'''(0)}{6} x^3 + \frac{f''(0)}{2} x^2 + f'(0) x, \quad \text{i.e.} \quad f''(0) = 0$$

which is a contradiction. Thus this case cannot occur.

If $e_1 e_2 \neq 0$ then $B_1 B_2 \neq 0$ since $e_1 l_1$, $e_2 l_2$ are nonzero functions. From (42), (43)

$$B_2 e_1 l_1(y) - B_1 e_2 l_2(y) = 0 \quad y \in (0, 1),$$

i.e. l_1, l_2 are linearly dependent on $(0, 1)$. By Lemma 2 and our condition $\operatorname{Re} s_1 \geqq \operatorname{Re} s_2$ this holds if and only if $s_1 = 5$, $s_2 = 4$. From (31) with

$$c_3 = \frac{f'''(0)}{6}, \quad c_4 = \frac{f''(0)}{2}, \quad c_5 = f'(0)$$

$$f(x) = c_1 x^5 + c_2 x^4 + c_3 x^3 + c_4 x^2 + c_5 x \tag{46}$$

and by $f(1)=1$

$$c_5 = 1-(c_1+c_2+c_3+c_4). \tag{47}$$

Substituting (46) in (3) we obtain after some simple calculations an equation of the form

$$\sum_{k=0}^{3}\sum_{l=0}^{3} a_{kl}\, p_k(x)\, p_l(y) = 0 \tag{48}$$

where $p_k(x)=x^k$ $(k=0, 1, 2)$ and $p_3(x)=x^4-2x^3$. Since p_0, p_1, p_2, p_3 are linearly independent (48) holds if and only if $a_{kl}=0$ $(k, l=0, 1, 2, 3)$. A simple but long calculation gives that

$$a_{33} = 25c_1+4c_2-(5c_1+2c_2)^2$$

$$a_{32} = a_{23} = 50c_1+12c_2-(5c_1+2c_2)(10c_1+6c_2+3c_3+2c_4)$$

$$a_{31} = a_{13} = -25c_1-8c_2-(5c_1+2c_2)(-5c_1-4c_2-3c_3-2c_4)$$

$$a_{30} = a_{03} = 0$$

$$a_{22} = 100c_1+36c_2+9c_3+4c_4-(10c_1+6c_2+3c_3+2c_4)^2$$

$$a_{21} = a_{12} = -50c_1-24c_2-9c_3-4c_4-(10c_1+6c_2+3c_3+2c_4)(-5c_1-4c_2-3c_3-2c_4)$$

$$a_{20} = a_{02} = 0$$

$$a_{11} = 25c_1+16c_2+9c_3+4c_4-(-5c_1-4c_2-3c_3-2c_4)^2$$

$$a_{10} = a_{01} = a_{00} = 0.$$

It can be seen that

$$a_{31} = a_{33}-a_{32}$$

$$a_{21} = a_{32}-a_{22}$$

$$a_{11} = a_{33}+a_{22}-2a_{32}$$

therefore $a_{kl}=0$ $(k, l=0, 1, 2, 3)$ holds if and only if

$$a_{33} = 0, \quad a_{32} = 0, \quad a_{22} = 0$$

is valid. This gives the following system of equations:

$$25c_1+4c_2 = (5c_1+2c_2)^2 \tag{49}$$

$$50c_1+12c_2 = (5c_1+2c_2)(10c_1+6c_2+3c_3+2c_4) \tag{50}$$

$$100c_1+36c_2+9c_3+4c_4 = (10c_1+6c_2+3c_3+2c_4)^2. \tag{51}$$

Choosing $c_1=s$ arbitrarily we get from (49) that

$$c_2 = \frac{1}{2}[1-5s\pm(1+15s)^{1/2}]. \tag{52}$$

In solving (50), (51) we may suppose that $5c_1+2c_2\neq 0$ (otherwise $5c_1+2c_2=0$, $50c_1+12c_2=0$ hence $c_1=c_2=0$ and then f would be a polynomial of degree $\leqq 3$, i.e. f would be of the form (6)). From (50), (51) we get

$$3c_3+2c_4 = \frac{50c_1+12c_2}{5c_1+2c_2}-(10c_1+6c_2) = D_1$$

$$9c_3+4c_4 = \left(\frac{50c_1+12c_2}{5c_1+2c_2}\right)^2-(100c_1+36c_2) = D_2$$

thus

$$c_3 = \frac{1}{3}(D_2-2D_1), \quad c_4 = \frac{1}{2}(3D_1-D_2).$$

Expressing D_1, D_2 by help of the parameter s we obtain

$$D_1 = 5/3[1+3s\mp(1+15s)^{1/2}], \quad D_2 = 50/9[1+3s\mp(1+15s)^{1/2}]$$

further

$$c_3 = 20/27[1+3s\mp(1+15s)^{1/2}] \tag{53}$$

$$c_4 = -5/18[1+3s\mp(1+15s)^{1/2}] \tag{54}$$

and by (47)

$$c_5 = 1/27[1+3s\mp(1+15s)^{1/2}]. \tag{55}$$

With $c_1=s$, (52), (53), (54), (55) we get exactly solution (7) from (46). This concludes the proof of our theorem.

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(Received August 18, 1988)