On Hosszú's functional inequality

By GY. MAKSA and ZS. PÁLES (Debrecen)

Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. At the International Conference on Functional Equations and Inequalities held at Szczawnica in Poland (June 21—27, 1987) K. LAJKÓ gave a survey talk about Hosszú's functional equation

$$f(x+y-xy)+f(xy) = f(x)+f(y).$$

The speaker mentioned numerous conditions under which this equation is equivalent to Jensen's functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}.$$

A great part of these results can be found in [1], [2] and [4]. Following this talk, Z. DARÓCZY raised the question: What can be said about the connection of the inequalities corresponding to these equations supposing the solutions to be continuous? In this note we deal with this problem.

It is known (see [3]) that every continuous solution $f\colon]0, 1[\to\mathbf{R}$ of Jensen's inequality

$$f\left(\frac{x+y}{2}\right) \geqq \frac{f(x)+f(y)}{2} \quad x, y\in]0, 1[\tag{1}$$

is concave, i.e.

$$f(\lambda x+(1-\lambda)y) \geqq \lambda f(x)+(1-\lambda)f(y) \tag{2}$$

holds for all $x, y\in]0, 1[$ and $\lambda\in[0, 1]$. The inequality which corresponds to Hosszú's equation is

$$f(x+y-xy)+f(xy) \leqq f(x)+f(y) \tag{3}$$

where $f\colon]0, 1[\to\mathbf{R}$ and (3) holds for all $x, y\in]0, 1[$. Concerning Z. DARÓCZY's question we will prove that concave functions satisfy Hosszú's inequality (3) (Theorem 1) but there exist nonconcave continuous solutions of (3).

2. For the sake of brevity, we introduce the notation

$$x\circ y = x+y-xy \quad x, y\in]0, 1[.$$

Since $x\circ y$ is the weighted arithmetic mean of 1 and y with the weights x and $1-x$, furthermore $x\circ y=y\circ x$, we obtain that x and y lie between xy and $x\circ y$ for all $x, y\in]0, 1[$. This observation is the key for the proof of our first theorem.

Theorem 1. *Suppose that* $f\colon]0, 1[\to \mathbf{R}$ *is concave. Then f satisfies* (3).

PROOF. If $x, y\in]0, 1[$ then for some $\lambda\in[0, 1]$ we have

$$x = \lambda xy+(1-\lambda)(x\circ y)$$

and

$$y = xy+(x\circ y)-x = (1-\lambda)xy+\lambda(x\circ y).$$

Since f is concave we get

$$f(x) \geqq \lambda f(xy)+(1-\lambda)f(x\circ y)$$

and

$$f(y) \geqq (1-\lambda)f(xy)+\lambda f(x\circ y).$$

Adding these inequalities, we obtain (3). □

In our second theorem we will use the identities

$$\text{(4)}\qquad \begin{cases}(x\circ y)^2-x^2\circ y^2 = 2xy(1-x)(1-y) > 0\\ (x\circ y)^3-x^3\circ y^3 = \dfrac{3}{2}(x+y)[(x\circ y)^2-x^2\circ y^2] \quad \text{and}\\ (x\circ y)^4-x^4\circ y^4 = (x^2y^2-x^2y-xy^2+2x^2+2y^2+3xy)[(x\circ y)^2-x^2\circ y^2]\end{cases}$$

which hold for all $x, y\in]0, 1[$ and can be checked by a simple computation.

Theorem 2. *If* $a\in\left[\dfrac{23}{16}, \dfrac{3}{2}\right[$ *is fixed then the function f defined on* $]0, 1[$ *by*

$$f(x) = -x^4+2x^3-ax^2$$

is a continuous solution of (3) *and f is not concave.*

PROOF. The inequality (3) is satisfied by f if and only if

$$\text{(5)}\qquad -[(x\circ y)^4-x^4\circ y^4]+2[(x\circ y)^3-x^3\circ y^3] \leqq a[(x\circ y)^2-x^2\circ y^2]$$

for all $x, y\in]0, 1[$. According to (4), (5) is equivalent to the inequality

$$\text{(6)}\qquad -x^2y^2+x^2y+xy^2-2x^2-2y^2-3xy+3x+3y \leqq a \quad x, y\in]0, 1[.$$

Applying the Taylor formula to the left hand side of (6) at the point $(1/2, 1/2)$, (6) can be written as

$$-\frac{7}{4}\left(x-\frac{1}{2}\right)^2-2\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)-\frac{7}{4}\left(y-\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\left(y-\frac{1}{2}\right)^2+\frac{23}{16} \leqq a,$$

that is

$$-\frac{3}{4}\left(x-\frac{1}{2}\right)^2-\frac{3}{4}\left(y-\frac{1}{2}\right)^2-(x+y-1)^2-\left(x-\frac{1}{2}\right)^2\left(y-\frac{1}{2}\right)^2+\frac{23}{16} \leqq a.$$

Since $a \geqq \frac{23}{16}$ this inequality holds for all $x, y \in]0, 1[$ thus f is a continuous solution of (3). On the other hand $a < \frac{3}{2}$ thus $f''\left(\frac{1}{2}\right) = 3 - 2a > 0$ which shows that f is not concave. □

Finally, we formulate an open problem: Suppose that $f\colon]0, 1[\to \mathbf{R}$ is Jensen-concave (i.e. f is a solution of (1)). Does this imply that f is a solution of Hosszú's inequality (3)?

References

[1] Z. DARÓCZY, Über die Funktionalgleichung $f(x+y-xy)+f(xy)=f(x)+f(y)$, *Publ. Math, Debrecen*, **16** (1969), 129—132.

[2] T. M. K. DAVISON, On the functional equation $f(m+n-mn)+f(mn)=f(m)+f(n)$, *Aequationes Math.*, **10** (1975), 206—211.

[3] M. KUCZMA, An Introduction to the Theory of Functional Equations and Inequalities, PWN, *Warszawa—Krakow—Katowice*, 1985.

[4] K. LAJKÓ, Applications of extensions of additive functions, *Aequationes Math.*, **2** (1974), 68—76.

(Received August 1, 1987)