On the continuity of causal operators

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. There are two possible way to define the stability of an input-output system: ([1] p. 109).

a) A bounded input u produces a bounded output Tu.

b) If the bound of u is $\|u\|$ and the bound of Tu is $\|Tu\|$, then there is $c>0$ such that $\|Tu\|<c\cdot\|u\|$. Obviously, b)$\Rightarrow$a) but not the contrary. In a "real" system a) rather than b) can be verified however, for the operator-theoretic approach b) is needed.

So, for the application of functional analysis in the theory of linear systems, a natural problem is to find a class of linear operators when a) and b) are equivalent. Hence the following question arises:

When will an everywhere defined linear operator T be bounded in a Banach space?

FEINTUCH and SAEKS in their book [3] define a generalization of causal operator. Our main result in this paper is that for every generalized causal operator T there is a non-trivial invariant subspace $p_t\mathbf{B}$ such that T is continuous on $p_t\mathbf{B}$. Next we shall show by the method of LOY [7] that a causal and time-invariant operator in the sense of [3] p. 119, is always continuous if the shift operators are isometries. Finally, using a slight modification of HACKENBROCK's proofs [5], we prove that an operator which is passive in the sense of [3] p. 196, is also causal and continuous.

2. Let T be a linear operator on a Banach space $\mathbf{B}(\mathbf{D}(T)=\mathbf{B}$ i.e. T is everywhere defined on $\mathbf{B})$; Λ be a partially ordered set such that for every $t\in\Lambda$ there is $s\in\Lambda$ so that $t<s$; $\{p^s;\ s\in\Lambda\}$ be bounded linear operators on $\mathbf{B}$ with the properties:

I. $$\|p^s\|\leqq 1 \quad\text{and}\quad p^s p^t = p^{\min(s,t)}$$

II. $$p^s z=\theta \quad\text{for every}\quad s\in\Lambda \quad\text{implies}\quad z=\theta.$$

Moreover, we define $p_s:=I-p^s$ (I is the identity operator).

Definition 1. T is *causal with respect to* $\{p^s;\ s\in\Lambda\}$ if

(∗) $$p^sT=p^sTp^s.$$

There are several forms to express causality:

Lemma 1. *The following assertions are equivalent:*

a) T *is causal;*

b) $p_s\mathbf{B}:=\{p_s x;\ x\in\mathbf{B}\}$ *is an invariant subspace for every* $s\in\Lambda$;

c) *if* $p^s x=p^s y$ *then* $p^s Tx=p^s Ty$.

PROOF. a)⇒b): If $p_s\mathbf{B}$ is *not* an invariant subspace, then there is $y\in\mathbf{B}$ such that $Tp_s y\notin P_s\mathbf{B}$. On the other hand, $p_s Tp_s y\in p_s\mathbf{B}$ by definition, and hence

$$Tp_s\neq p_sTp_s$$

in this case.

b)⇒a): If $p_s\mathbf{B}$ is an invariant subspace, i.e. $T(p_s\mathbf{B})\subseteq p_s\mathbf{B}$, then for every $x\in\mathbf{B}$ there is $z\in\mathbf{B}$ such that

$$Tp_sx=p_sz$$

and so $p_sTp_sx=p_s^2z=p_sz$; hence

$$p_sTp_s=Tp_s.$$

Substituting $p_s:=I-p^s$ in this equality, $(*)$ is obtained.

a)⇒c): If $p^sx=p^sy$ and $(*)$ is satisfied, then

$$p^sTx=p^sTp^sx=p^sTp^sy=p^sTy$$

and hence c) is obtained.

c)⇒a): If $p^sx=\theta$, then it follows from c) that $p^sTx=\theta$, since T is linear. By definition

$$p^s(I-p^s)=0$$

for every $s\in\Lambda$ and hence

$$p^sT(I-p^s)x=0$$

for every $s\in\Lambda$ and $x\in\mathbf{B}$. Hence $(*)$ is obtained.

Lemma 2. *For every* $s\in\Lambda$, $p_s\mathbf{B}$ *is a closed subset of* $\mathbf{B}$.

PROOF. If $x_n\in p_s\mathbf{B}$, then x_n has the form p_sz_n and hence $p_sx_n=p_s^2z_n=p_sz_n=x_n$. Moreover, if $x_n\to x$ then $p_sx_n\to p_sx$. We conclude

$$p_sx=x$$

and hence $x\in p_s\mathbf{B}$.

Our main result is the following

Theorem 1. *If* T *is causal with respect to* $\{p^s;\ s\in\Lambda\}$, *then there exist* $t\in\Lambda$ *and* $K_t>0$ *(common for every* $s\in\Lambda$*!) such that*

$$\|p^sTp_tx\|\leqq K_t\|p_tx\|\quad s\in\Lambda,\quad x\in\mathbf{B}.$$

This will say that each operator p^sT is continuous on the invariant subspace $p_t\mathbf{B}$.

PROOF. Indirect. We suppose that for every $t\in\Lambda$ there is $s\in\Lambda$ such that p^sT is unbounded on $p_t\mathbf{B}$ and we shall construct $x_0\in B$ so that $x_0\notin\mathbf{D}(T)$.

The construction is the following. For $t_1 \in \Lambda$ there is $s_1 \in \Lambda$ such that $p^{s_1}T$ is unbounded on $p_{t_1}\mathbf{B}$ and hence there is $p_{t_1}x_1$ so that

$$\|p_{t_1}x_1\| = 1 \quad \text{and} \quad \|p^{s_1}Tp_{t_1}x_1\| > 1.$$

For $t_2 = s_1$ there is $s_2 \in \Lambda$ such that $p^{s_2}T$ is unbounded on $p_{t_2}B$ and hence there is $p_{t_2}x_2$ so that

$$\|p_{t_2}x_2\| = 1 \quad \text{and} \quad \|p^{s_2}Tp_{t_2}x_2\| > 2^2\left(2+\frac{1}{2}\|Tp_{t_1}x_1\|\right)$$

moreover, $t_2 < s_2$ can be supposed since

$$\|p^s z\| = \|p^s p^{s_2} z\| \leqq \|p^{s_2} z\| \quad \text{for} \quad s < s_2.$$

In the sequel the following abreviations will be used:

$$p^{s_k} := p^k \quad \text{resp.} \quad p_{s_k} := p_k.$$

For any integer $k > 0$ and $t_{k+1} = s_k$ there is s_{k+1} so that $p^{k+1}T$ is unbounded on p_kB and hence there is p_kx_k so that

$$\|p_kx_k\| = 1,\ s_{k+1} > s_k \quad \text{and} \quad \|p^{k+1}Tp_kx_k\| > 2^k\cdot\left(k+\sum_{i=1}^{k-1}\frac{1}{2^k}\|Tp_ix_i\|\right).$$

Now, it is obvious that

$$x_0 = \sum_{k=1}^{\infty}\frac{1}{2^k}p_kx_k \in \mathbf{B}$$

i.e. the infinite series on the right hand side is convergent.

We shall show, that for *any* integer $N > 0$

$$\|Tx_0\| > N$$

and hence $x_0 \notin \mathbf{D}(T)$. In fact

$$\|Tx_0\| \geqq \|p^{N+1}Tx_0\| = \sum_{k=1}^{N-1}\frac{1}{2^k}p^{N+1}Tp_kx_k + \frac{1}{2^N}p^{N+1}Tp_Nx_N + p^{N+1}T\sum_{k=N+1}^{\infty}\frac{1}{2^k}p_kx_k.$$

Since T is causal, $p^{N+1}p_k = p^{N+1}(I-p^k) = 0$ for $k \geqq N+1$ therefore we have

$$p^{N+1}T\sum_{k=N+1}^{\infty}\frac{1}{2^k}p_kx_k = p^{N+1}T\sum_{k=N+1}^{\infty}\frac{1}{2^k}p^{N+1}p_kx_k = \theta$$

moreover

$$\left\|\sum_{k=1}^{N-1}\frac{1}{2^k}p^{N+1}p_kx_k\right\| \leqq \sum_{k=1}^{N-1}\frac{1}{2^k}\|Tp_kx_k\|.$$

We conclude

$$\|Tx_0\| \geqq \left\|\frac{1}{2^N}p^{N+1}Tp_Nx_N + \sum_{k=1}^{N-1}\frac{1}{2^k}p^{N+1}Tp_kx_k\right\| \geqq$$

$$\geqq \frac{1}{2^N}\|p^{N+1}Tp_Nx_N\| - \sum_{k=1}^{N-1}\frac{1}{2^k}\|Tp_kx_k\| \geqq$$

$$\geqq \left(N+\sum_{k=1}^{N-1}\frac{1}{2^k}\|Tp_kx_k\|\right) - \sum_{k=1}^{N-1}\frac{1}{2^k}\|Tp_kx_k\| = N.$$

From [2] II.2.7 and from the properties I and II for $\{p^s;\ s\in\Lambda\}$ we get the following

Corollary. *If T is causal with respect to $\{p^s;\ s\in\Lambda\}$ $(\mathbf{D}(T)=\mathbf{B}!)$, then there exist $t\in\Lambda$ such that T is a continuous operator on the invariant subspace $p_t\mathbf{B}$.*

Remarks. I. According to Lemmas 1 and 2, if T is causal, then there are infinitely many closed invariant subspaces $p_s\mathbf{B}$ of T such that

$$p_s\mathbf{B}\subset p_t\mathbf{B}$$

for every $t\in\Lambda$. So, this is the condition which implies that there is an invariant subspace $p_t\mathbf{B}\neq\{\theta\}$ such that the restriction T to $p_t\mathbf{B}$ is continuous.

II. There is another version of the above results. Define

$$\|x\|_s:=\|p_s x\|\quad s\in\Lambda;\quad x\in\mathbf{B}$$

and let $\mathbf{B}_e$ be the completion of $\mathbf{B}$ via the locally convex topology generated by the seminorms $\{\|\cdot\|_s;\ s\in\Lambda\}$. Then the Corollary tells us that for a causal operator T mapping $\mathbf{B}$ into $\mathbf{B}$, there is $t\in\Lambda$, such that T is bounded (and hence continuous) with respect to the seminorm $\|\cdot\|_t$ since, by the proof of Theorem 1, if this is *not* the case, then there is $x_0\in\mathbf{B}$ such that

$$Tx_0\notin\mathbf{B}\quad\text{but}\quad Tx_0\in\mathbf{B}_e.$$

III. There is an obvious closed connection between the extended space $\mathbf{B}_e$ of [3] p. 173—180 and our completion $\mathbf{B}_e$. Moreover, by Definition 13 of [3] p. 179, the operator T is stable if there is $M>0$ such that

$$\|Tx\|_s\leqq M\|x\|_s\quad s\in\Lambda;\quad x\in\mathbf{B}.$$

If T is a causal operator on $\mathbf{B}$, then

$$\|Tx\|_s\leqq M\|x\|_s\quad x\in\mathbf{B}$$

is automatically satisfied for *at least one* $s\in\Lambda$. (See also the Corollary of Theorem 2 in the next section.)

3. In this section, let Λ be a partially ordered *group* and for $s,t\in\Lambda$ we define

$$s>t$$

if $s-t>0$ where 0 is the unit in Λ moreover

$$\Lambda_+:=\{s:\ s\in\Lambda,\ s>0\}.$$

Definition 2. The operator U_s $(s\in\Lambda)$ is called *shift-operator* if

$$p^{t+s}U_s=U_s p_t;\quad t\in\Lambda.$$

In the next, we suppose the existence of these shift operators $\{U_\tau;\ \tau\in\Lambda\}$ in $\mathbf{B}$.

Definition 3. The operator T is called *shift-invariant* or *time-invariant* if

$$TU_s=U_sT\quad s\in\Lambda.$$

Now we consider an *inductive limit topology* in $\mathbf{B}$. The sequence $\{x_n\}$ is called convergent and $x_n \to x$ if there exists s such that $\{x_n\} \subset p_s \mathbf{B}$ and $x \in p_s \mathbf{B}$ so that $x_n \to x$ in the seminorm $\|\cdot\|_s$.

Remark. It is easy to check that the locally convex topology generated by the seminorms $\|\cdot\|_s$; $s \in \Lambda$ is *weaker* than the inductive limit topology above considered.

Our main object in this section is to show that shift-invariant causal operators are continuous in the inductive limit topology. First we give examples for the structure described above.

Example 1. If $\mathbf{B}$ is any Banach space of time-functions, i.e. functions on the real line, p^s is the truncation operator

$$p^s f(\tau) = \begin{cases} f(\tau) & \text{if } \tau \leqq s \\ 0 & \text{elsewhere,} \end{cases}$$

and U_s is the right shift by s on the real line then we obtain the causal resp. shift-invariant operators in the common sense.

Example 2. [6] Let $X = X(t)$ be a stochastic process with random variables having finite mean and variance and let $p^t X(\tau)$ be the process predicted from the part $\{X(\tau);\ \tau < t\}$ ("before t"). Then the transition operator T will be called causal if from $p^t X(\tau) = p^t Y(\tau)$ it follows that

$$TX(\tau) = TY(\tau) \quad \text{for} \quad \tau < t$$

and

$$p^t TX(\tau) = p^t TY(\tau) \quad \text{for every } \tau.$$

Example 3. ([3] p. 8—9 and [6].) Let $\mathbf{H} = \mathbf{H}(R)$ be a reproducing kernel Hilbert space with kernel $R = R(\tau, t)$ $(\tau, t \in \Lambda_+)$ and let p^s be the projection onto the subspace generated by

$$\{R(.\,, t);\ t \leqq s\}.$$

Example 4. Let $\mathbf{B} = L^1(G)$ where G is a locally compact group and $h \in L^1(G)$ such that

$$\{h * f;\ f \in L^1(G)\}$$

is a closed subset if $*$ is the convolution product. Let p_h be the projection onto $h * \mathbf{B} := \{h * f;\ f \in L^1(G)\}$ and we define $h > g$ if $h * \mathbf{B} \subset g * \mathbf{B}$. Then the operators T with the property

$$T(h * f) = h * Tf \quad h, f \in \mathbf{B}$$

are causal operators with respect to $\{p^h;\ h \in L^1(G)\}$.

Theorem 2. *If T is causal with respect to $\{p^s;\ s \in \Lambda\}$, $\|U_\tau x\| = \|x\|$ for every $x \in B$, $\tau \in \Lambda_+$ and*

$$TU_\tau = U_\tau T \quad \tau \in \Lambda_+,$$

then for every $t \in \Lambda_+$ *there is* $M_t > 0$ *such that*

$$\|p^s T p_t x\| \leqq K_t \|p_t x\| \quad s \in \Lambda_+.$$

From [2] II.2.7 and the properties I. and II. for $\{p^s;\ s \in \Lambda\}$ we get the following

Corollary. *A causal and shift-invariant operator* T $(\mathbf{D}(T) = \mathbf{B}!)$ *is bounded (and hence continuous) in the inductive limit topology introduced at the beginning of this section.*

THE PROOF OF THEOREM 2. Indirect. We suppose that there exists $t \in \Lambda_+$ such that $p^s T$ is unbounded on $p_t \mathbf{B}$. Then there is a sequence $\{x_k\}$ with $\|p_t x_k\| = 1$ such that

$$\|p^s T p_t x_n\| > 2^n \left(n + \sum_{k=1}^{n-1} \frac{1}{2^k} \|T p_t x_k\| \right).$$

Now it is obvious that

$$x_0 = \sum_{k=1}^{\infty} \frac{1}{2^k} U_{(2k-1)s} p_t x_k \in \mathbf{B}$$

i.e. the infinite series on the right hand side is convergent. We shall show that for any integer $N > 0$

$$\|T x_0\| < N$$

and hence $x_0 \notin \mathbf{D}(T)$ contradicting the supposition $\mathbf{D}(T) = \mathbf{B}$. In fact

$$\|T x_0\| \geqq \|p^{2Ns} T x_0\| = \left\| \sum_{k=1}^{N-1} \frac{1}{2^k} p^{2Ns} T U_{(2k-1)s} p_t x_k + \right.$$

$$\left. + \frac{1}{2^N} p^{2Ns} T U_{(2_N-1)s} p_t x_N + p^{2Ns} T \sum_{k=N+1}^{\infty} \frac{1}{2^k} U_{(2k-1)s} p_t x_k \right\| :$$

considering the right hand side of the equality, *for the first term* we have

$$\left\| \sum_{k=1}^{N-1} \frac{1}{2^k} p^{2Ns} T U_{(2k-1)s} p_t x_k \right\| \leqq \sum_{k=1}^{N-1} \frac{1}{2^k} \|T p_t x_k\|;$$

and *for the third term*

$$p^{2Ns} T \sum_{k=N+1}^{\infty} \frac{1}{2^k} U_{(2k-1)s} p_t x_k = p^{2Ns} T \sum_{k=N+1}^{\infty} \frac{1}{2^k} p^{2Ns} \cdot U_{(2k-1)s} p_t x_k = \theta$$

since T is causal, p^{2Ns} is bounded and

$$p^{2Ns} U_{(2k-1)s} p_t = p^{2Ns} p_{t+(2k-1)s} U_{(2k-1)s},$$

$$p^{2Ns} p_{t+(2k-1)s} = p^{2Ns}(I - p^{t+(2k-1)s}) = 0 \quad \text{for} \quad k \geqq N+1$$

since $2Ns < t + (2k-1)s$ in this case.

Finally, *considering the second term* of the right hand side of the epuality

$$\frac{1}{2^N} p^{2Ns} T U_{(2N-1)s} p_t x_N = \frac{1}{2^N} U_{(2N-1)s} p^s T p_t x_N$$

since

$$TU_{(2N-1)s} = U_{(2N-1)s}T \quad \text{and} \quad p^{2Ns}U_{(2N-1)s} = U_{(2N-1)s}p^s$$

and hence

$$\left\| \frac{1}{2^N} p^{2Ns} TU_{(2N-1)s} p_t x_N \right\| = \frac{1}{2^N} \|p^s T p_t x_N\|.$$

We conclude that

$$\|Tx_0\| \geqq \left\| \frac{1}{2^N} p^{2Ns} TU_{(2N-1)s} p_t x_N + \sum_{k=1}^{N-1} \frac{1}{2^k} p^{2Ns} TU_{(2k-1)s} p_t x_k \right\| \geqq$$

$$\geqq \frac{1}{2^N} \|p^s T p_t x_N\| - \sum_{k=1}^{N-1} \frac{1}{2^k} \|Tp_t x_k\| \geqq \left(N + \sum_{k=1}^{N-1} \frac{1}{2^k} \|Tp_t x_k\|\right) - \sum_{k=1}^{N-1} \frac{1}{2^k} \|Tp_t x_k\| = N.$$

4. In the spirit of [3], an operator T on a Hilbert space $\mathbf{H}$ is called *passive* if

$$\mathrm{Re}(Tx|p^t x) \geqq 0 \quad x \in \mathbf{H}$$

where $\{p^t;\ t \in \Lambda\}$ are projection operators of $\mathbf{H}$ which satisfy the same conditions as in Section 2.

First we show that a passive operator is also causal. In fact, if

$$\mathbf{B}_T(f, g) := (Tf|p^t g) + (p^t f|Tg) \quad f, g \in \mathbf{H}$$

then $\mathbf{B}_T$ is a positive bilinear functional and hence the Cauchy inequality

$$(*) \qquad |\mathbf{B}_T(f, g)|^2 \leqq \mathbf{B}_T(f, f) \cdot \mathbf{B}_T(g, g)$$

is valid. So, if $p^t f = \theta$, then $\mathbf{B}_T(f, f) = 0$ and hence $\mathbf{B}_T(f, g) = 0$ for every $g \in \mathbf{H}$ and $p^t Tf = \theta$ by the definition of $\mathbf{B}_T$.

We conclude that $p^t f = \theta$ implies $p^t Tf = \theta$ and this is equivalent to the property c) in Lemma 1 since T is linear.

Our main result in this section is a slight generalization of [5] p. 274—275:

Theorem 3. *Every passive operator* T *is continuous.*

PROOF. Applied II.2.7 from [2] as in the Corollary of Theorem 1, we have only to prove that each of the operators $p^t T$ $(t \in \Lambda)$ is bounded.

It follows from the definition of $\mathbf{B}_T$ that

$$(p^t Tf|g) = \mathbf{B}_T(f, g) - (p^t f|Tg)$$

and

$$\mathbf{B}_T(f, f) = 2\,\mathrm{Re}\,(p^t Tf|f) \geqq 0$$

moreover obviously

$$\mathrm{Re}\,(p^t Tf|f) \leqq |(p^t Tf|f)|.$$

To piece together the above assertions with Cauchy inequalities, particularly with $(*)$, we have

$$|(p^t Tf|g)| \leqq 2\,|(p^t Tf|f)|^{1/2}\,|(p^t Tg|g)|^{1/2} + (f|f)^{1/2} \cdot (Tg|Tg)^{1/2};$$

putting

$$R(f):=\left|(p^tTf|f)\right|^{1/2}+(f|f)^{1/2}$$
$$S(g):=2\left|(p^tTg|g)\right|^{1/2}+(Tg|Tg)^{1/2}$$

we obtain by straightforward calculation

$$(**)\qquad (p^tTf|g)\leqq R(f)\cdot S(g).$$

Now we define

$$F_f(g):=\frac{1}{R(f)}(p^tTf|g)\quad f\neq\theta;$$

it follows from $(**)$ that $F_f(g)$ is bounded for every $g\in\mathbf{H}$, *with bound independent of f*. Hence, by the uniform boundedness principle, $\{\|F_f\|: f\neq\theta\}$ is bounded i.e. there is $M>0$ such that

$$\|F_f\|<M\quad f\neq\theta$$

and hence

$$\|p^tTf\|=\sup\left\{\left|(p^tTf|g)\right|;\ \|g\|=1\right\}\leqq MR(f).$$

We claim that for every operator p^tT there is C such that

$$\|p^tTf\|\leqq C\|f\|.$$

Indeed, from the definition of $R(f)$

$$R(f)\leqq\|p^tTf\|^{1/2}\|f\|^{1/2}+\|f\|$$

hence

$$\|p^tTf\|\leqq MR(f)\leqq\|p^tTf\|^{1/2}M\|f\|^{1/2}+M\|f\|.$$

By completion to a full square for $\|p^tTf\|^{1/2}$ we obtain

$$\|p^tTf\|\leqq\left(\frac{M}{2}+\sqrt{\frac{M^2}{4}+M}\right)^2\|f\|.$$

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