Comparison theorems for double summability methods

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Abstract. We establish sufficient conditions under which (i) a double Nörlund matrix (N, q_{jk}) is stronger than a double weighted mean matrix $(\bar{N}, q_{jk})$, (ii) $(\bar{N}, p_{jk})$ is stronger than $(\bar{N}, q_{jk})$, (iii) (N, q_{jk}) is stronger than $(C, 1, 1)$, the double Cesàro matrix of order $(1, 1)$. As special cases of (ii) we obtain sufficient conditions under which (iv) $(C, 1, 1)$ is stronger than $(\bar{N}, q_{jk})$, and (v) $(\bar{N}, q_{jk})$ is stronger than $(C, 1, 1)$.

Key Words and Phrases: comparison theorems, doubly infinite matrices, Cesàro, Nörlund, summability weighted means.

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1. Introduction

Among others, the previous works of ISHIGURO [1] and the second author, jointly with KUTTNER [2], contain conditions under which Nörlund matrices are stronger than weighted mean matrices generated by the same single sequence $\{q_n: n=0, 1, \ldots\}$. In this paper we obtain analogous conditions for the double Cesàro matrix of order $(1, 1)$ to be stronger than double weighted mean matrices, etc. In the proofs we cannot rely on the tools of matrix theory for summability methods for single series. Our results are of interest not only in their own right, but for possible application to double Fourier series and orthogonal series.

Let $A=(a_{mnjk}: m, n, j, k=0, 1, \ldots)$ be a doubly infinite matrix, $\{s_{mn}: m, n=0, 1, \ldots\}$ a double sequence, and set

$$t_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} s_{jk} \quad (m, n = 0, 1, \ldots) \tag{1}$$

where the convergence of the double series is meant in the Pringsheim sense. The matrix A is called *regular* if, for each sequence $\{s_{mn}\}$ which is bounded and converges to a limit s, the sequence $\{t_{mn}\}$ defined by (1) is also bounded and converges to the same limit s, where m and n tend to infinity independently of one another in both limits.

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As is known (see [6]), necessary and sufficient conditions for a matrix A to be regular are the following:

$$\lim_{m,n\to\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk} = 1, \tag{2}$$

$$\lim_{m,n\to\infty} \sum_{j=0}^{\infty} |a_{mnjk}| = 0 \quad (k = 0, 1, \ldots), \tag{3}$$

$$\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |a_{mnjk}| = 0 \quad (j = 0, 1, \ldots), \tag{4}$$

$$\|A\| = \sup_{m,n\geqq 0} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |a_{mnjk}| < \infty. \tag{5}$$

Either of conditions (3) and (4) implies

$$\lim_{m,n\to\infty} a_{mnjk} = 0 \quad (j, k = 0, 1, \ldots).$$

Let $\{q_{jk}: j, k=0, 1, \ldots\}$ be a double sequence of positive real numbers and set

$$Q_{mn} = \sum_{j=0}^{m}\sum_{k=0}^{n} q_{jk} \quad (m, n = 0, 1, \ldots).$$

Given a double sequence $\{s_{mn}\}$, its double Nörlund and double weighted means are defined as follows:

$$t_{mn} = \frac{1}{Q_{mn}} \sum_{j=0}^{m}\sum_{k=0}^{n} q_{m-j,n-k} s_{jk} \tag{6}$$

and

$$u_{mn} = \frac{1}{Q_{mn}} \sum_{j=0}^{m}\sum_{k=0}^{n} q_{jk} s_{jk}. \tag{7}$$

The induced double Nörlund and double weighted mean methods of summability are denoted by (N, q_{jk}) and $(\overline{N}, q_{jk})$, respectively.

The double Cesàro means of order $(1, 1)$, or briefly the $(C, 1, 1)$-means, of a sequence $\{s_{mn}\}$ are defined by

$$\sigma_{mn} = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m}\sum_{k=0}^{n} s_{jk}.$$

We say that a sequence $\{q_{jk}\}$ is nondecreasing if

$$q_{jk} \leqq \min\{q_{j+1,k}, q_{j,k+1}\},$$

and nonincreasing if

$$q_{jk} \geqq \max\{q_{j+1,k}, q_{j,k+1}\}.$$

In the sequel, we use the notations

$$\Delta_{10} q_{jk} = q_{jk} - q_{j+1,k},$$
$$\Delta_{01} q_{jk} = q_{jk} - q_{j,k+1},$$
$$\Delta_{11} q_{jk} = q_{jk} - q_{j+1,k} - q_{j,k+1} + q_{j+1,k+1}.$$

2. Comparison of (N, q_{jk}) and $(\overline{N}, q_{jk})$

Given two doubly infinite matrices A and B and a double sequence $\{s_{mn}\}$, denote by $t_{mn}(A)$ and $t_{mn}(B)$ the corresponding means defined by (1). We say that the method of summability generated by A is *stronger* than the one generated by B if, for any double sequence $\{s_{mn}\}$, whenever $\{t_{mn}(B)\}$ is bounded and converges to a limit t, $\{t_{mn}(A)\}$ is also bounded and converges to the same limit t as $m, n \to \infty$. Equivalently, the method of summability generated by B *is included in* the one generated by A.

We call a matrix A *doubly triangular* if, for all m and n,

$$a_{mnjk} = 0 \quad \text{for} \quad j > m \quad \text{or} \quad k > n.$$

Each of the matrices (N, q_{jk}), $(\overline{N}, q_{jk})$, and $(C, 1, 1)$ is doubly triangular.

Lemma. *Let A be a doubly triangular matrix, $\{q_{jk}\}$ a double sequence of positive numbers, and t_{mn} and u_{mn} defined by* (1) *and* (7), *respectively. Then*

$$t_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} b_{mnjk} u_{jk} \quad (m, n = 0, 1, \dots)$$

where

(8)
$$b_{mnjk} = Q_{jk} \Delta_{11} \frac{a_{mnjk}}{q_{jk}},$$

$$b_{mnjn} = Q_{jn} \Delta_{10} \frac{a_{mnjn}}{q_{jn}} \quad (j = 0, 1, \dots, m-1;$$

$$b_{mnmk} = Q_{mk} \Delta_{01} \frac{a_{mnmk}}{q_{mk}} \quad k = 0, 1, \dots, n-1),$$

$$b_{mnmn} = Q_{mn} \frac{a_{mnmn}}{q_{mn}}.$$

Moreover,

(9)
$$\sum_{j=0}^{m} \sum_{k=0}^{n} b_{mnjk} = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{mnjk} \quad (m, n = 0, 1, \dots).$$

PROOF. Equation (7) can be solved for s_{jk} to obtain

$$q_{jk} s_{jk} = \Delta_{11} (Q_{j-1,k-1} u_{j-1,k-1}).$$

Substituting this into (1) and performing a double Abel transformation (see, e.g. [5]) yields

$$t_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{a_{mnjk}}{q_{jk}} \Delta_{11} (Q_{j-1,k-1} u_{j-1,k-1}) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} Q_{jk} u_{jk} \Delta_{11} \frac{a_{mnjk}}{q_{jk}} +$$

$$+ \sum_{j=0}^{m-1} Q_{jn} u_{jn} \Delta_{10} \frac{a_{mnjn}}{q_{jn}} + \sum_{k=0}^{n-1} Q_{mk} u_{mk} \Delta_{01} \frac{a_{mnmk}}{q_{mk}} + Q_{mn} u_{mn} \frac{a_{mnmn}}{q_{mn}} = \sum_{j=0}^{m} \sum_{k=0}^{n} b_{mnjk} u_{jk}$$

where the b_{mnjk} are those as indicated in (8).

To prove (9), take $s_{jk}=1$ for all j and k. Then $u_{mn}=1$ for all m and n and

$$t_{mn}=\sum_{j=0}^{m}\sum_{k=0}^{n}b_{mnjk} \quad\text{and}\quad t_{mn}=\sum_{j=0}^{m}\sum_{k=0}^{n}a_{mnjk},$$

yielding (9).

Theorem 1. *If* $\{q_{jk}\}$ *is nondecreasing,* $\Delta_{11}(q_{m-j,n-k}/q_{jk})$ *is of constant sign,*

$$\lim_{m,n\to\infty}\frac{1}{Q_{mn}}\sum_{j=0}^{m}q_{jn}=0, \tag{10}$$

and

$$\lim_{m,n\to\infty}\frac{1}{Q_{mn}}\sum_{k=0}^{n}q_{mk}=0, \tag{11}$$

then (N, q_{jk}) *is stronger than* $(\overline{N}, q_{jk})$.

Theorem 2. *If* $\{q_{jk}\}$ *is nonincreasing,* $\Delta_{11}(q_{m-j,n-k}/q_{jk})$ *is of constant sign, and*

$$\inf_{m,n\geqq 0}q_{mn}=L>0, \tag{12}$$

then (N, q_{jk}) *is stronger than* $(\overline{N}, q_{jk})$.

Proof of Theorem 1. We apply the Lemma to the matrices $A=(N, q_{jk})$ and $(\overline{N}, q_{jk})$. In this case equations (8) take the form

$$b_{mnjk}=\frac{Q_{jk}}{Q_{mn}}\Delta_{11}\frac{q_{m-j,n-k}}{q_{jk}}, \tag{13}$$

$$b_{mnjn}=\frac{Q_{jn}}{Q_{mn}}\Delta_{10}\frac{q_{m-j,0}}{q_{jn}} \quad (j=0,1,\ldots,m-1;$$

$$b_{mnmk}=\frac{Q_{mk}}{Q_{mn}}\Delta_{01}\frac{q_{0,n-k}}{q_{mk}} \quad k=0,1,\ldots,n-1),$$

$$b_{mnmn}=\frac{q_{00}}{q_{mn}}.$$

To prove Theorem 1, it is sufficient to show that the matrix $B=(b_{mnjk})$ is regular.

By (9),

$$\sum_{j=0}^{m}\sum_{k=0}^{n}b_{mnjk}=\frac{1}{Q_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n}q_{mnjk}=1,$$

and (2) is satisfied.

Now we check condition (3). To this end, let k be fixed. We distinguish two cases.

Case (i): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \geqq 0$. By a single Abel transformation (with the agreement, here and in the sequel, that $Q_{-1,k}=0$ for any k), we get that

$$\text{(14)} \quad \sum_{j=0}^{m} |b_{mnjk}| = \sum_{j=0}^{m} b_{mnjk} = \frac{1}{Q_{mn}} \left[\sum_{j=0}^{m-1} Q_{jk} \Delta_{11} \frac{q_{m-j,n-k}}{q_{jk}} + Q_{mk} \Delta_{01} \frac{q_{0,n-k}}{q_{mk}} \right] =$$

$$= \frac{1}{Q_{mn}} \sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \Delta_{01} \frac{q_{m-j,n-k}}{q_{jk}}.$$

Hence,

$$\sum_{j=0}^{m} |b_{mnjk}| \leqq \frac{1}{Q_{mn}} \sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \frac{q_{m-j,n-k}}{q_{jk}} =$$

$$= \frac{1}{Q_{mn}} \sum_{j=0}^{m} \frac{q_{m-j,n-k}}{q_{jk}} \sum_{s=0}^{k} q_{js} \leqq \frac{k+1}{Q_{mn}} \sum_{j=0}^{m} q_{m-j,n-k} \leqq \frac{k+1}{Q_{mn}} \sum_{j=0}^{m} q_{m-j,n},$$

which tends to zero as $m, n \to \infty$, by (10).

Case (ii): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \leqq 0$. Then, using (14),

$$\text{(15)} \quad \sum_{j=0}^{m} |b_{mnjk}| = -\sum_{j=0}^{m-1} b_{mnjk} + b_{mnmk} = -\sum_{j=0}^{m} b_{mnjk} + 2b_{mnmk} =$$

$$= \frac{1}{Q_{mn}} \left[-\sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \Delta_{01} \frac{q_{m-j,n-k}}{q_{jk}} + 2Q_{mk} \Delta_{01} \frac{q_{0,n-k}}{q_{mk}} \right].$$

Since $\Delta_{01}(q_{m-j,n-k}/q_{jk}) \geqq 0$ for each j, it follows from (10) that

$$\sum_{j=0}^{m} |b_{mnjk}| \leqq \frac{2Q_{mk} q_{0,n-k}}{Q_{mn} q_{mk}} \leqq \frac{2(m+1)(k+1) q_{0,n-k}}{Q_{mn}} \leqq$$

$$\leqq \frac{2(k+1)}{Q_{mn}} \sum_{j=0}^{m} q_{j,n-k} \leqq \frac{2(k+1)}{Q_{mn}} \sum_{j=0}^{m} q_{jn},$$

which tends to zero as $m, n \to \infty$.

Thus, we have proved (3) in either case.

The proof of (4) is similar, by using (11) instead of (10).

Proving (5), we distinguish the same two cases as above.

Case (i): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \geqq 0$. Then $b_{mnjk} \geqq 0$, $\sum_{j=0}^{m} \sum_{k=0}^{n} b_{mnjk} = 1$, and $\|B\| = 1$.

Case (ii): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \leqq 0$. Now

$$\text{(16)} \quad \sum_{j=0}^{m} \sum_{k=0}^{n} |b_{mnjk}| = -\sum_{j=0}^{m-1} \sum_{k=0}^{n-1} b_{mnjk} + \sum_{j=0}^{m-1} b_{mnjn} + \sum_{k=0}^{n-1} b_{mnmk} + b_{mnmn} =$$

$$= 2\left[\sum_{j=0}^{m} b_{mnjn} + \sum_{k=0}^{n-1} b_{mnmk} \right] - 1.$$

Performing a single Abel transformation, by (13),

$$\sum_{j=0}^{m} b_{mnjn} = \frac{1}{Q_{mn}} \sum_{j=1}^{m-1} Q_{jn} \Delta_{10} \frac{q_{m-j,0}}{q_{jn}} + \frac{q_{00}}{q_{mn}} = \tag{17}$$

$$= \frac{1}{Q_{mn}} \sum_{j=0}^{m} (Q_{jn} - Q_{j-1,n}) \frac{q_{m-j,0}}{q_{jn}} = \frac{1}{Q_{mn}} \sum_{j=0}^{m} \frac{q_{m-j,0}}{q_{jn}} \sum_{s=0}^{n} q_{js} \leqq$$

$$\leqq \frac{n+1}{Q_{mn}} \sum_{j=0}^{m} q_{m-j,0} \leqq \frac{1}{Q_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} q_{m-j,k} = 1.$$

Similarly,

$$\sum_{k=0}^{n-1} b_{mnmk} \leqq \sum_{k=0}^{n} b_{mnmk} \leqq 1.$$

Collecting (16), (17) and this together yields $\|B\| \leqq 3$.

Proof of Theorem 2. It is again a systematic verification of conditions (2)—(5) for the matrix B. In the proof of Theorem 1 we have checked the fulfillment of (2) without using any condition on the monotonicity of $\{q_{jk}\}$.

To check the fulfillment of (3), we fix k and again distinguish two cases.

Case (i): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \geqq 0$. Then, by (15),

$$\sum_{j=0}^{m} |b_{mnjk}| = \sum_{j=0}^{m-1} b_{mnjk} - b_{mnmk} =$$

$$= \frac{1}{Q_{mn}} \left[\sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \Delta_{01} \frac{q_{m-j,n-k}}{q_{jk}} - 2Q_{mk} \Delta_{01} \frac{q_{0,n-k}}{q_{mk}} \right].$$

Since this time $\Delta_{01}(q_{m-j,n-k}/q_{jk}) \leqq 0$ for each j, it follows from (12) that

$$\sum_{j=0}^{m} |b_{mnjk}| \leqq \frac{2Q_{mk} q_{0,n-k-1}}{Q_{mn} q_{m,k+1}} \leqq \frac{2(m+1)(k+1) q_{00}^2}{(m+1)(n+1) L^2},$$

which tends to zero as $m, n \to \infty$.

Case (ii): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \leqq 0$. Then, by (14) and (12),

$$\sum_{j=0}^{m} |b_{mnjk}| = -\sum_{j=0}^{m} b_{mnjk} = -\frac{1}{Q_{mn}} \sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \Delta_{01} \frac{q_{m-j,n-k}}{q_{jk}} \leqq$$

$$\leqq \frac{1}{Q_{mn}} \sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \frac{q_{m-j,n-k-1}}{q_{j,k+1}} \leqq$$

$$\leqq \frac{1}{(m+1)(n+1)L^2} \sum_{j=0}^{m} (k+1) q_{00}^2 = \frac{(k+1) q_{00}^2}{(n+1) L^2},$$

which tends to zero as $m, n \to \infty$, and (3) is proved in either case.

Condition (4) is proved in an analogous way.

Finally, we show that (5) is satisfied.

Case (i): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \geqq 0$. Then

$$(18) \qquad \sum_{j=0}^{m}\sum_{k=0}^{n}|b_{mnjk}| = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} b_{mnjk} - \sum_{j=0}^{m-1} b_{mnjn} - \sum_{k=0}^{n-1} b_{mnmk} + b_{mnmn} =$$

$$= 1 - 2\Big[\sum_{j=0}^{m-1} b_{mnjn} + \sum_{k=0}^{n-1} b_{mnmk}\Big].$$

Hence, by (13) and (12),

$$\sum_{j=0}^{m}\sum_{k=0}^{n}|b_{mnjk}| \leqq 1 - \frac{2}{Q_{mn}}\left[Q_{m-1,n}\sum_{j=0}^{m-1}\Delta_{10}\frac{q_{m-j,0}}{q_{jn}} + Q_{m,n-1}\sum_{k=0}^{n-1}\Delta_{01}\frac{q_{0,n-k}}{q_{mk}}\right] =$$

$$= 1 + \frac{2}{Q_{mn}}\left[Q_{m-1,n}\left(\frac{q_{00}}{q_{mn}} - \frac{q_{m0}}{q_{0n}}\right) + Q_{m,n-1}\left(\frac{q_{00}}{q_{mn}} - \frac{q_{0n}}{q_{m0}}\right)\right] \leqq 1 + \frac{4q_{00}}{L}.$$

Case (ii): $\Delta_{11}(q_{m-j,n-k}/q_{jk}) \leqq 0$. Then

$$(19) \quad \sum_{j=0}^{m}\sum_{k=0}^{n}|b_{mnjk}| = -\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} b_{mnjk} - \sum_{j=0}^{m-1} b_{mnjn} - \sum_{k=0}^{n-1} b_{mnmk} + b_{mnmn} = -1 + 2b_{mnmn},$$

whence, by (13) and (12), $\|B\| \leqq 2q_{00}/L$.

3. Comparison of $(\overline{N}, p_{jk})$ and $(\overline{N}, q_{jk})$

Let $\{p_{jk}: j, k = 0, 1, \ldots\}$ be another double sequence of positive numbers and set

$$P_{mn} = \sum_{j=0}^{m}\sum_{k=0}^{n} p_{jk} \quad (m, n = 0, 1, \ldots).$$

Theorem 3. *If* $\{p_{jk}/q_{jk}\}$ *is nonincreasing,* $\Delta_{11}(p_{jk}/q_{jk})$ *is of constant sign, and*

$$(20) \qquad \lim_{m,n\to\infty}\frac{1}{P_{mn}}\sum_{j=0}^{m}\frac{p_{jk}}{q_{jk}}\sum_{s=0}^{k} q_{js} = 0 \quad (k = 0, 1, \ldots),$$

$$(21) \qquad \lim_{m,n\to\infty}\frac{1}{P_{mn}}\sum_{k=0}^{m}\frac{p_{jk}}{q_{jk}}\sum_{r=0}^{j} q_{rk} = 0 \quad (j = 0, 1, \ldots),$$

$$(22) \qquad \sup_{m,n\geqq 0}\frac{1}{P_{mn}}\sum_{j=0}^{m}\frac{p_{jn}}{q_{jn}}\sum_{s=0}^{n} q_{js} < \infty,$$

$$(23) \qquad \sup_{m,n\geqq 0}\frac{1}{P_{mn}}\sum_{k=0}^{n}\frac{p_{mk}}{q_{mk}}\sum_{r=0}^{m} q_{rk} < \infty,$$

then $(\overline{N}, p_{jk})$ *is stronger than* $(\overline{N}, q_{jk})$.

Theorem 4. *If* $\{p_{jk}/q_{jk}\}$ *is nondecreasing,* $\Delta_{11}(p_{jk}/q_{jk})$ *is of constant sign, and*

(24)
$$\lim_{m,n\to\infty} \frac{p_{m,k+1}Q_{mk}}{q_{m,k+1}P_{mn}} = 0 \quad (k = 0, 1, \ldots),$$

(25)
$$\lim_{m,n\to\infty} \frac{p_{j+1,n}Q_{jn}}{q_{j+1,n}P_{mn}} = 0 \quad (j = 0, 1, \ldots),$$

(26)
$$\sup_{m,n\geqq 0} \frac{p_{mn}Q_{mn}}{q_{mn}P_{mn}} < \infty,$$

then $(\overline{N}, p_{jk})$ *is stronger than* $(\overline{N}, q_{jk})$.

Proof of Theorem 3. We apply the Lemma to the weighted mean matrices $A=(\overline{N}, p_{jk})$ and $(\overline{N}, q_{jk})$. So, this time the t_{mn} are the $(\overline{N}, p_{jk})$ means, while the u_{jk} are the $(\overline{N}, q_{jk})$ means of the sequence $\{s_{jk}\}$. Equations (8) now take the form

(27)
$$b_{mnjk} = \frac{Q_{jk}}{P_{mn}} \Delta_{11} \frac{p_{jk}}{q_{jk}},$$

$$b_{mnjn} = \frac{Q_{jn}}{P_{mn}} \Delta_{10} \frac{p_{jn}}{q_{jn}} \quad (j = 0, 1, \ldots, m-1;$$

$$b_{mnmk} = \frac{Q_{mk}}{P_{mn}} \Delta_{01} \frac{p_{mk}}{q_{mk}} \quad k = 0, 1, \ldots, n-1),$$

$$b_{mnmn} = \frac{Q_{mn}p_{mn}}{P_{mn}q_{mn}}.$$

Our goal is to prove that the matrix $B=(b_{mnjk})$ is regular. From (9),

$$\sum_{j=0}^{m}\sum_{k=0}^{n} b_{mnjk} = \frac{1}{P_{mn}} \sum_{j=0}^{m}\sum_{k=0}^{n} p_{jk} = 1,$$

which is (2). To prove (3), we distinguish two cases.

Case (i): $\Delta_{11}(p_{jk}/q_{jk})\geqq 0$. Applying a single Abel transformation,

$$\sum_{j=0}^{m} |b_{mnjk}| = \sum_{j=0}^{m} b_{mnjk} = \frac{1}{P_{mn}} \sum_{j=0}^{m} (Q_{jk}-Q_{j-1,k})\Delta_{01}\frac{p_{jk}}{q_{jk}} \leqq$$

$$\leqq \frac{1}{P_{mn}} \sum_{j=0}^{m} (Q_{jk}-Q_{j-1,k})\frac{p_{jk}}{q_{jk}} = \frac{1}{P_{mn}} \sum_{j=0}^{m} \frac{p_{jk}}{q_{jk}} \sum_{s=0}^{k} q_{js}.$$

For fixed k, this tends to zero as $m, n\to\infty$, due to (20).

Case (ii): $\Delta_{11}(p_{jk}/q_{jk}) \leqq 0$. From (15) and (27),

$$\sum_{j=0}^{m} |b_{mnjk}| = -\sum_{j=0}^{m} b_{mnjk} + 2b_{mnmk} =$$

$$= -\frac{1}{P_{mn}} \sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \Delta_{01} \frac{p_{jk}}{q_{jk}} + 2\frac{Q_{mk}}{P_{mn}} \Delta_{01} \frac{p_{mk}}{q_{mk}}.$$

Since $\Delta_{01}(p_{jk}/q_{jk}) \geqq 0$ for each j, we have

$$\sum_{j=0}^{m} |b_{mnjk}| \leqq 2\frac{Q_{mk}\, p_{mk}}{P_{mn}\, q_{mk}} = \frac{2p_{mk}}{P_{mn}\, q_{mk}} \sum_{j=0}^{m} \sum_{s=0}^{k} q_{js} \leqq \frac{2}{P_{mn}} \sum_{j=0}^{m} \frac{p_{jk}}{q_{jk}} \sum_{s=0}^{m} q_{js}$$

which tends to zero as $m, n \to \infty$, for any fixed k, by (20).

Thus, we have proved (3) in either case.

Relation (4) can be proved in a similar way, using (21).

To prove (5) we distinguish the usual two cases.

Case (i): $\Delta_{11}(p_{jk}/q_{jk}) \geqq 0$. Then $b_{mnjk} \geqq 0$, so $\|B\| = 1$.

Case (ii): $\Delta_{11}(p_{jk}/q_{jk}) \leqq 0$. By (16) and (27), while perfoı ming two single Abel transformations, we get

$$\sum_{j=0}^{m} \sum_{k=0}^{n} |b_{mnjk}| = 2\Big[\sum_{j=0}^{m} b_{mnjn} + \sum_{k=0}^{n-1} b_{mnmk}\Big] - 1 =$$

$$= \frac{2}{P_{mn}} \left[\sum_{j=0}^{m} (Q_{jn} - Q_{j-1,n}) \frac{p_{jn}}{q_{jn}} + \sum_{k=0}^{n} (Q_{mk} - Q_{m,k-1}) \frac{p_{mk}}{q_{mk}} - Q_{mn} \frac{p_{mn}}{q_{mn}}\right] - 1$$

whence by (22) and (23),

$$\sum_{j=0}^{m} \sum_{k=0}^{n} |b_{mnjk}| \leqq \frac{2}{P_{mn}} \left[\sum_{j=0}^{m} \frac{p_{jn}}{q_{jn}} \sum_{s=0}^{n} q_{js} + \sum_{k=0}^{n} \frac{p_{mk}}{q_{mk}} \sum_{r=0}^{m} q_{rk}\right] = \mathcal{O}(1).$$

Proof of Theorem 4. Applying the Lemma, equations (8) again take the form (27). We will prove again that $B = (b_{mnjk})$ is regular. Relation (2) is a consequence of (9).

Now check (3). To this end, let k be fixed and distinguish two cases again.

Case (i): $\Delta_{11}(p_{jk}/q_{jk}) \geqq 0$. From (15) and (27),

$$\sum_{j=0}^{m} |b_{mnjk}| = \sum_{j=0}^{m} b_{mnjk} - 2b_{mnmk} =$$

$$= \frac{1}{P_{mn}} \left[\sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \Delta_{01} \frac{p_{jk}}{q_{jk}} - 2Q_{mk} \Delta_{01} \frac{p_{mk}}{q_{mk}}\right].$$

Since this time $\Delta_{01}(p_{jk}/q_{jk}) \leqq 0$ for each j, we have

$$\sum_{j=0}^{m} |b_{mnjk}| \leqq 2 \frac{Q_{mk} p_{m,k+1}}{P_{mn} q_{m,k+1}}$$

which tends to zero as $m, n \to \infty$, by (24).

Case (ii): $\Delta_{11}(p_{jk}/q_{jk}) \leqq 0$. By (27),

$$\sum_{j=0}^{m} |b_{mnjk}| = -\sum_{j=0}^{m} b_{mnjk} = -\frac{1}{P_{mn}} \sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \Delta_{01} \frac{p_{jk}}{q_{jk}} \leqq$$

$$\leqq \frac{1}{P_{mn}} \sum_{j=0}^{m} (Q_{jk} - Q_{j-1,k}) \frac{p_{j,k+1}}{q_{j,k+1}} \leqq \frac{p_{m,k+1} Q_{mk}}{q_{m,k+1} P_{mn}}$$

which tends to zero as $m, n \to \infty$, again by (24).

Thus, we have proved (3) in either case.

Relation (4) can be proved in an analogous manner, using (25).

Finally, we check (5).

Case (i): $\Delta_{11}(p_{jk}/q_{jk}) \geqq 0$. By (18), (27), and (26),

$$\sum_{j=0}^{m} \sum_{k=0}^{n} |b_{mnjk}| = 1 - 2\left[\sum_{j=0}^{m-1} b_{mnjn} + \sum_{k=0}^{n-1} b_{mnmk}\right] =$$

$$= 1 - \frac{2}{P_{mn}} \left[\sum_{j=0}^{m} (Q_{jn} - Q_{j-1,n}) \frac{p_{jn}}{q_{jn}} + \sum_{k=0}^{n} (Q_{mk} - Q_{m,k-1}) \frac{p_{mk}}{q_{mk}} - 2Q_{mn} \frac{p_{mn}}{q_{mn}}\right] \leqq$$

$$\leqq 1 + \frac{4 p_{mn} Q_{mn}}{q_{mn} P_{mn}} = \mathcal{O}(1).$$

Case (ii): $\Delta_{11}(p_{jk}/q_{jk}) \leqq 0$. By (19) and (27),

$$\sum_{j=0}^{m} \sum_{k=0}^{n} |b_{mnjk}| = -1 + 2b_{mnmn} \leqq \frac{2 p_{mn} Q_{mn}}{q_{mn} P_{mn}} = \mathcal{O}(1).$$

4. Comparison of $(C, 1, 1)$ and $(\overline{N}, q_{jk})$

Since $(C, 1, 1)$ is also a weighted mean matrix, where

$$p_{jk} = 1 \quad \text{and} \quad P_{mn} = (m+1)(n+1) \quad (j, k, m, n = 0, 1, \ldots),$$

we can deduce the following four comparison theorems as corollaries.

Corollary 1. *If* $\{q_{jk}\}$ *is nondecreasing and* $\Delta_{11}(1/q_{jk})$ *is of constant sign, then* $(C, 1, 1)$ *is stronger than* $(\overline{N}, q_{jk})$.

Corollary 2. *If* $\{q_{jk}\}$ *is nonincreasing,* $\Delta_{11}(1/q_{jk})$ *is of constant sign, and condition* (12) *is satisfied, then* $(C, 1, 1)$ *is stronger than* $(\overline{N}, q_{jk})$.

PROOF OF COROLLARY 1. In order to apply Theorem 3 to the weighted mean matrices $(\overline{N}, p_{jk})=(C, 1, 1)$ and $(\overline{N}, q_{jk})$, we have to check the fulfillment of conditions (20)—(23). For example,

$$(28)\qquad \frac{1}{P_{mn}}\sum_{j=0}^{m}\frac{p_{jk}}{q_{jk}}\sum_{s=0}^{k}q_{js}=\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\frac{1}{q_{jk}}\sum_{s=0}^{k}q_{js}\leqq$$

$$\leqq\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}(k+1)=\frac{k+1}{n+1}$$

which tends to zero as $m, n\to\infty$, for fixed k. This is (20). Estimate (28) for $k=n$ also shows that

$$0<\frac{1}{P_{mn}}\sum_{j=0}^{m}\frac{p_{jn}}{q_{jn}}\sum_{s=0}^{n}q_{js}=\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\frac{1}{q_{jn}}\sum_{s=1}^{n}q_{js}\leqq 1$$

which is (22).

Similarly, conditions (21) and (23) are also satisfied. Now the conclusion of Theorem 3 in this special case and that of Corollary 1 coincide.

PROOF OF COROLLARY 2. This time we are going to apply Theorem 4. So, we have to verify the fulfillment of conditions (24)—(26). For example, by (12),

$$\frac{Q_{mk}}{(m+1)(n+1)\,q_{m,k+1}}\leqq\frac{(k+1)\,q_{00}}{(n+1)L}.$$

For fixed k, this tends to zero as $m, n\to\infty$, that is (24) is satisfied. Analogous estimates verify (25) and (26), too.

Corollary 3. *If $\{q_{jk}\}$ is nonincreasing, $\Delta_{11}q_{jk}$ is of constant sign,*

$$(29)\qquad \lim_{m,n\to\infty}\frac{1}{Q_{mn}}\sum_{j=0}^{m}q_{j0}=0,$$

and

$$(30)\qquad \lim_{m,n\to\infty}\frac{1}{Q_{mn}}\sum_{k=0}^{n}q_{0k}=0,$$

then $(\overline{N}, q_{jk})$ is stronger than $(C, 1, 1)$.

Corollary 4. *If $\{q_{jk}\}$ is nondecreasing, $\Delta_{11}q_{jk}$ is of constant sign, and*

$$\lim_{m,n\to\infty}\frac{(m+1)\,q_{m,k+1}}{Q_{mn}}=0\quad(k=0,1,\ldots),$$

$$\lim_{m,n\to\infty}\frac{(n+1)\,q_{j+1,n}}{Q_{mn}}=0\quad(j=0,1,\ldots),$$

$$\sup_{m,n\geqq 0}\frac{(m+1)(n+1)\,q_{mn}}{Q_{mn}}<\infty,$$

then $(\overline{N}, q_{jk})$ is stronger than $(C, 1, 1)$.

PROOF OF COROLLARY 3. We will apply Theorem 3, while interchanging the role of p_{jk} and q_{jk}. By (12),

$$\frac{1}{Q_{mn}}\sum_{j=0}^{m}\frac{q_{jk}}{p_{jk}}\sum_{s=0}^{k}p_{js}=\frac{1}{Q_{mn}}\sum_{j=0}^{m}(k+1)q_{jk}.$$

Since $\{q_{nk}\}$ is nonincreasing, (20) is implied by (29). Similarly, (30) implies (21). Condition (22) becomes

$$\sup_{m,n\geqq 0}\frac{1}{Q_{mn}}(n+1)q_{jn}<\infty.$$

Since $\{q_{jk}\}$ is nonincreasing,

$$\frac{1}{Q_{mn}}\sum_{j=0}^{m}(n+1)q_{jn}\leqq\frac{1}{Q_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n}q_{jk}=1,$$

and (22) is satisfied. Again (23) is similarly proved.

PROOF OF COROLLARY 4. We apply Theorem 4, while interchanging q_{jk} for p_{jk} and then setting $p_{jk}=1$. After these modifications, conditions (24)—(26) coincide with the conditions occurring in Corollary 4, and the conclusion to be proved is immediate.

5. Comparison of (N, q_{jk}) and $(C, 1, 1)$

In the theory of summability of single sequences, it is a well-known result (see, e.g. [3, p. 67]) that if $\{q_n\colon n=0, 1, \ldots\}$ is a nondecreasing sequence of positive numbers and the Nörlund method (N, q_n) is regular, then (N, q_n) is stronger than $(C, 1)$. As our final result, we establish the corresponding analogue in the case of double sequences.

Theorem 5. *If* $\{q_{jk}\}$ *is nondecreasing, satisfies conditions* (10) *and* (11), *and* $\Delta_{11}q_{jk}$ *is of constant sign, then* (N, q_{jk}) *is stronger than* $(C, 1, 1)$.

Remark. If $\{q_{jk}\}$ is nondecreasing, then conditions (10) and (11) are equivalent to (2)—(5): i.e. the double Nörlund method (N, q_{jk}) is regular.

PROOF. We apply the Lemma to $A=(N, q_{jk})$ and $(C, 1, 1)$. As a result, equations (8) for this case are

$$b_{mnjk}=\frac{(j+1)(k+1)}{Q_{mn}}\Delta_{11}q_{m-j,n-k},\tag{31}$$

$$b_{mnjk}=\frac{(j+1)(n+1)}{Q_{mn}}\Delta_{10}q_{m-j,0}\quad (j=0, 1, \ldots, m-1;$$

$$b_{mnmk}=\frac{(m+1)(k+1)}{Q_{mn}}\Delta_{01}q_{0,n-k}\quad k=0, 1, \ldots, n-1),$$

$$b_{mnmn}=\frac{(m+1)(n+1)}{Q_{mn}}q_{00}.$$

We will prove that $B=(b_{mnjk})$ is regular. By (9),

$$\sum_{j=0}^{m}\sum_{k=0}^{n} b_{mnjk}=\frac{1}{Q_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n} q_{m-j,n-k}=1,$$

so condition (2) is satisfied.

To prove (3), we distinguish two cases again.

Case (i): $\Delta_{11}q_{jk}\geqq 0$. By (31),

$$\sum_{j=0}^{m}|b_{mnjk}|=\sum_{j=0}^{m} b_{mnjk}=\frac{k+1}{Q_{mn}}\sum_{j=0}^{m}\Delta_{01}q_{j,n-k}\leqq\frac{k+1}{Q_{mn}}\sum_{j=0}^{m} q_{j,n-k}.$$

Since k is fixed, this tends to zero as $m,n\to\infty$, due to (10).

Case (ii): $\Delta_{11}q_{jk}\leqq 0$. By (15) and (31),

$$\sum_{j=0}^{m}|b_{mnjk}|=-\sum_{j=0}^{m} b_{mnjk}+2b_{mnmk}=\frac{k+1}{Q_{mn}}\Big[-\sum_{j=0}^{m}\Delta_{01}q_{j,n-k}+2(m+1)\Delta_{01}q_{0,n-k}\Big]\leqq$$

$$\leqq\frac{2(m+1)(k+1)}{Q_{mn}}\Delta_{01}q_{0,n-k}\leqq\frac{2(k+1)}{Q_{mn}}\sum_{j=0}^{m} q_{j,n-k}.$$

If k is fixed, this tends to zero as $m,n\to\infty$, again due to (10).

We can check condition (4) in a similar way, while using (11) instead of (10).

It remains to verify condition (5).

Case (i): $\Delta_{11}q_{jk}\geqq 0$. Then $b_{mnjk}\geqq 0$ and $\|B\|=1$.

Case (ii): $\Delta_{11}q_{jk}\leqq 0$. By (16),

$$\sum_{j=0}^{m}\sum_{k=0}^{n}|b_{mnjk}|=2\Big[\sum_{j=0}^{m} b_{mnjn}+\sum_{k=0}^{n-1} b_{mnmk}\Big]-1.$$

By (31),

$$\sum_{j=0}^{m} b_{mnjn}=\frac{n+1}{Q_{mn}}\sum_{j=0}^{m} q_{j0}\leqq\frac{1}{Q_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n} q_{jk}=1.$$

Likewise,

$$\sum_{k=0}^{n-1} b_{mnmk}\leqq 1,$$

so $\|B\|\leqq 3$.

6. Prospects for further research

Based on the results of this paper, it remains to obtain sufficient conditions under which

(i) (N,p_{jk}) is stronger than (N,q_{jk}),
(ii) $(\overline{N},q_{jk})$ is stronger than (N,q_{jk}),
(iii) $(C,1,1)$ is stronger than (N,q_{jk}).

Question (i) was solved formally by Moore [4], who obtained the doubly infinite analogue of Theorem 19 of [3]. Question (ii) is not known even in the case of single sequences. In any of the cases, to express s_{mn} from (6) in terms of the Nörlund means t_{jk} and q_{jk} is virtually impossible due to the lack of matrix techniques available for doubly infinite matrices, and the structure of double Nörlund matrices.

References

[1] K. Ishiguro, The relation between (N, p_n) and $(\bar{N}, p_n)$ summability, *Proc. Japan Acad.* **41** (1965), 120—122.

[2] B. Kuttner and B. E. Rhoades, Relations between (N, p_n) and $(\bar{N}, p_n)$ summability, *Proc. Edinburgh Math. Soc.* **16** (1968), 109—116.

[3] G. H. Hardy, Divergent Series, *Oxford Univ. Press* (1949).

[4] C. N. Moore, On relationships between Nörlund means for double series, *Proc. Amer. Math. Soc.* **5** (1954), 957—963.

[5] F. Móricz, The Kronecker lemmas for multiple series and some applications, *Acta Math. Acad. Sci. Hungar.* **37** (1981), 39—50.

[6] G. M. Robinson, Divergent double sequences and series, *Trans. Amer. Math. Soc.* **28** (1926), 50—73.

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