Defining nets for integration

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Introduction

An ordered pair $(\Omega, \mathcal{S})$ consisting of a set Ω and a collection $\mathcal{S}$ of subsets of Ω such that $A \setminus B = \bigcup_{i \in I} C_i$ for some finite disjoint family $(C_i)_{i \in I}$ in $\mathcal{S}$ whenever $A, B \in \mathcal{S}$ is called a measurable space.

A net $\mathfrak{N} = ((\sigma_\alpha, \tau_\alpha))_{\alpha \in \Gamma}$ such that $\sigma_\alpha = (\sigma_{\alpha i})_{i \in I_\alpha}$ and $\tau_\alpha = (\tau_{\alpha i})_{i \in I_\alpha}$ are finite families in $\mathcal{S}$ and Ω, respectively, with σ_α being disjoint for all $\alpha \in \Gamma$, is called a disjoint defining net for integration over $(\Omega, \mathcal{S})$.

An ordered triple (X, Y, Z) of Banach spaces, over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, equipped with a bilinear map $(x, y) \to xy$ from $X \times Y$ into Z such that $|xy| \leqq |x|\,|y|$ for all $x \in X$ and $y \in Y$ is called a multiplication system.

For functions f from Ω into X and μ from $\mathcal{S}$ into Y, the limit

$$\int f \, d\mu = \lim_{\alpha} \sum_{i \in I_\alpha} f(\tau_{\alpha i})\, \mu(\sigma_{\alpha i}),$$

whenever it exists, is called the $\mathfrak{N}$-integral of f with respect to μ.

The above concept has been introduced in our former papers [31] and [32], where we showed that the most important basic linearity and continuity properties of the usual integrals remain true for this more general integral.

For a preliminary illustration, we have considered there the important particular case when $\Omega = \mathbf{R}$ and

$$\mathcal{S} = \{[r, s[: \ -\infty < r \leqq s < +\infty\},$$

and $\mathfrak{N} = ((\sigma_\alpha, \tau_\alpha))_{\alpha \in \Gamma}$ such that

$$\sigma_{\alpha i} = [i2^{1-\alpha}, (i+1)2^{1-\alpha}[\quad \text{and} \quad \tau_{\alpha i} = i2^{1-\alpha}$$

for all $\alpha \in \Gamma = \mathbf{N}$ and $i \in I_\alpha = \mathbf{Z} \cap [-\alpha 2^{\alpha-1}, \alpha 2^{\alpha-1}[$.

In the present paper, to include or extend several existing integrals, we provide a systematic list of defining nets for integration. For this, we also need the concept of topological measurable spaces and several important division properties of those.

The exact relationships among the corresponding net integrals, and also the standard integrals, and the characteristic properties of the single net integrals will not be established here. This may only be the subject of some further investigations carried out not necessarily by the present authors.

1. Measurable spaces

Definition 1.1. A collection $\mathscr{S}$ of subsets of a set Ω will be called a measurable system in Ω if for any $A, B \in \mathscr{S}$ there exists a finite disjoint family $(C_i)_{i \in I}$ in $\mathscr{S}$ such that $A \setminus B = \bigcup_{i \in I} C_i$.

An ordered pair $\Omega(\mathscr{S}) = (\Omega, \mathscr{S})$ consisting of a set Ω and a measurable system $\mathscr{S}$ in Ω will be called a measurable space.

Remark 1.2. This definition of a measurable space is more general than the usual ones. (See, for instance, [11, p. 73], [2, p. 35] and [14, p. 149].)

It has mainly been suggested to us by the following examples and some recent definitions of ALIPRANTIS—BURKINSHAW [1, p. 78] and JANSSEN—VAN DER STEEN [16, p. 98].

Example 1.3. If $\Omega = \mathbf{R}$ and

$$\mathscr{S} = \{[r, s[: \; -\infty < r \leqq s < +\infty\},$$

then $\Omega(\mathscr{S})$ is a measurable space.

Remark 1.4. The higher dimensional case can be easily derived hence with the help of a simple product construction for measurable spaces.

Example 1.5. If $\Omega = \mathbf{R}^2$ and $\mathscr{S}$ is the collection of all simplexes in $\mathbf{R}^2$ in the sense of Weir [34, p. 87], then $\Omega(\mathscr{S})$ is a measurable space.

Remark 1.6. Some authors (see [29] and [28], for instance) consider simplexes to be more suitable building blocks for integration on Euclidean spaces than intervals.

Example 1.7. If $\mathscr{C}$ is a lattice of subsets of a set Ω and

$$\mathscr{S} = \{A \setminus B : \; A, B \in \mathscr{C}\}$$

then $\Omega(\mathscr{S})$ is a measurable space.

Remark 1.8. The above general example comes from HALMOS [11, pp. 25—26]. (See also DINCULEANU [7, p. 7] and GÜNZLER [10, p. 24].)

To briefly formulate several useful properties of measurable spaces, it seems convenient to introduce the following

Definition 1.9. Let $\Omega(\mathscr{S})$ be a measurable space. A disjoint family $(A_i)_{i \in I}$ in $\mathscr{S}$ will be called an $\mathscr{S}$-division of a subset A of Ω if $A = \bigcup_{i \in I} A_i$.

Moreover, a subset A of Ω having a finite (countable) $\mathscr{S}$-division will be called finitely (countably) $\mathscr{S}$-divisible.

Theorem 1.10. *If $\Omega(\mathscr{S})$ is a measurable space and $A\in\mathscr{S}$, then the set $A\setminus\bigcup_{i\in I} A_i$ is finitely $\mathscr{S}$-divisible for any finite family $(A_i)_{i\in I}$ in $\mathscr{S}$.*

PROOF. By using induction on the cardinality of I, a similar argument as in the proof of [1, Theorem 9.2, 1] can be applied.

Corollary 1.11. *If $\Omega(\mathscr{S})$ is a measurable space, then the set $\bigcap_{i\in I} A_i$ is finitely $\mathscr{S}$-divisible for any nonvoid finite family $(A_i)_{i\in I}$ in $\mathscr{S}$.*

PROOF. Note that $\bigcap_{i\in I} A_i=A_{i_0}\setminus\bigcup_{i\in I}(A_{i_0}\setminus A_i)$ for some $i_0\in I$, and thus Theorem 1.10 can be appiied.

Remark 1.12. It is a striking fact that Corollary 1.11 has been overlooked by several authors. (See, for instance, [4, § 4], [34, p. 85] and [16, p. 96].)

Example 1.13. If $\Omega(\mathscr{S})$ is as in Example 1.3, then the set

$$[0,1[\setminus\bigcup_{i=1}^{\infty}[i^{-1},1[\;=\bigcap_{i=1}^{\infty}[0,i^{-1}[\;=\{0\}$$

is not $\mathscr{S}$-divisible, and thus Theorem 1.10 and Corollary 1.11 cannot be extended to countable families.

Theorem 1.14. *If $\Omega(\mathscr{S})$ is a measurable space, then for any finite family $(A_i)_{i\in I}$ in $\mathscr{S}$ there exists a finite $\mathscr{S}$-division $(B_j)_{j\in J}$ of $\bigcup_{i\in I} A_i$ such that each A_i is a union of some B_j's.*

PROOF. The assertion trivially holds for the void family. Suppose now that $(A_i)_{i\in I}$ is an arbitrary family in $\mathscr{S}$ for which the assertion holds, and $A_{i_0}\in\mathscr{S}$ with $i_0\notin I$. Then, by Corollary 1.11 and Theorem 1.10, the sets $A_{i_0}\cap B_j$, $B_j\setminus A_{i_0}$ and $A_{i_0}\setminus\bigcup_{j\in J} B_j$ have finite $\mathscr{S}$-divisions $(C_{jk})_{k\in K_j}$, $(D_{jl})_{l\in L_j}$ and $(E_m)_{m\in M}$, respectively. Here, the index sets K_j, L_j and M may clearly be choosen so that the sets $K=\bigcup_{j\in J}\{j\}\times K_j$, $L=\bigcup_{j\in J}\{j\}\times L_j$ and M be pairwise disjoint. Now, by considering the family $(C_k)_{k\in K}\cup(D_l)_{l\in L}\cup(E_m)_{m\in M}$, it is clear that the assertion holds also for the enlarged family $(A_i)_{i\in I\cup\{i_0\}}$. And thus by the principle of induction the proof is complete.

Remark 1.15. The above important theorem comes from BERBERIAN [2, Lemma 22.1]. (See also DINCULEANU [7, Lemma 6.1] and GÜNZLER [10, Lemma 1.2].)

For the reader's convenience, we included here a more natural inductive proof suggested by Günzler. Note that now we cannot say anything about the cardinality of J.

Example 1.16. If $\Omega(\mathscr{S})$ is as in Example 1.3 and $A_i=[0,i^{-1}[$ for all positive integer i, then the set $\bigcup_{i=1}^{\infty} A_i$ fails to have an $\mathscr{S}$-division $(B_j)_{j\in J}$ such that each A_i is the union of some B_j's, and thus Theorem 1.14 cannot be extended to countable families.

To check the above assertion, note that $0\in\bigcap_{i=1}^{\infty} A_i$, but the set $\bigcap_{i=1}^{\infty} A_i=\{0\}$ fails to contain an element B of $\mathcal{S}$ with $0\in B$.

Remark 1.17. In connection with Theorem 1.14, it is also worth mentioning that if $(A_i)_{i\in I}$ is a nonvoid family in $\mathcal{S}$ and $(B_j)_{j\in J}$ is an $\mathcal{S}$-division of $\bigcup_{i\in I} A_i$ such that each A_i is the union of some B_j's, then each B_j is necessarily contained in some A_i.

Theorem 1.18. *If $\Omega(\mathcal{S})$ is a measurable space, then for any countable family $(A_i)_{i\in I}$ in $\mathcal{S}$, there exists a countable $\mathcal{S}$-division $(B_j)_{j\in J}$ of $\bigcup_{i\in I} A_i$ such that each B_j is contained in some A_i.*

PROOF. By using that $I=\{i_n\}_{n=1}^{\infty}$ for some sequence $(i_n)_{n=1}^{\infty}$ in I whenever $I\neq\emptyset$, the argument given in the proof of [1, Theorem 9.2, 2] can be repeated.

As an immediate consequence of Theorems 1.14 and 1.18, we have:

Corollary 1.19. *If $\Omega(\mathcal{S})$ is a measurable space, then the set $\bigcup_{i\in I} A_i$ is finitely (countably) $\mathcal{S}$-divisible for any finite (countable) family $(A_i)_{i\in I}$ in $\mathcal{S}$.*

For a preliminary classification of measurable spaces, it seems convenient to introduce the following

Definition 1.20. A measurable space $\Omega(\mathcal{S})$ will be called

(i) additive if $\mathcal{S}$ contains all finitely $\mathcal{S}$-divisible sets;
(ii) σ-additive if it is additive and $\mathcal{S}$ contains all countably $\mathcal{S}$-divisible subsets of its members;
(iii) finite (σ-finite) if Ω is finitely (countably) $\mathcal{S}$-divisible.

Remark 1.21. The apparently very strange terminology of (ii) and (iii) is mainly motivated by the fact that we are unwilling to consider infinite-valued measures on $\mathcal{S}$.

By simple applications of Theorem 1.10 and Corollaries 1.11 and 1.19, one can easily prove the next useful propositions.

Proposition 1.22. *If $\Omega(\mathcal{S})$ is an additive measurable space, then $\mathcal{S}$ is closed under subtractions, finite intersections and finite unions.*

Proposition 1.23. *If $\Omega(\mathcal{S})$ is an arbitrary measurable space and $\tilde{\mathcal{S}}$ is the collection of all finitely $\mathcal{S}$-divisible subsets of Ω, then $\Omega(\tilde{\mathcal{S}})$ is an additive measurable space.*

Proposition 1.24. *An additive measurable space $\Omega(\mathcal{S})$ is σ-additive if and only if $\mathcal{S}$ is closed under countable intersections.*

Remark 1.25. Additive (σ-additive) measurable systems are usually called rings or clans (δ-rings or δ-clans).

Propositions 1.23 and 1.24 are essentially equivalent to Propositions 1.13 and 1.8 of Dinculeanu [7].

Proposition 1.26. *A measurable space $\Omega(\mathcal{S})$ is finite (σ-finite) if and only if Ω is a finite (countable) union of elements of $\mathcal{S}$.*

Remark 1.27. In this respect, it is also worth mentioning that an additive measurable space $\Omega(\mathscr{S})$ is finite if and only if $\Omega \in \mathscr{S}$.

On the other hand, due to the terminology of (ii) in Definition 1.20, a σ-additive σ-finite measurable space need not be finite.

2. Topological measurable spaces

Definition 2.1. An ordered triple $\Omega(\mathscr{S}, \mathscr{T})=(\Omega, \mathscr{S}, \mathscr{T})$ consisting of a set Ω, a measurable system $\mathscr{S}$ in Ω and a topology $\mathscr{T}$ on Ω will be called a topological measurable space if $\mathscr{S}$ and $\mathscr{T}$ are compatible in the sense that for each $x \in \Omega$ and $V \in \mathscr{T}$ with $x \in V$ there exists $A \in \mathscr{S}$ such that $x \in A^0$ and $A \subset V$.

This possibly unusual definition has mainly been suggested to us by the following examples.

Example 2.2. If $\Omega(\mathscr{S})$ is as in Example 1.3 and $\mathscr{T}$ is the usual topology of Ω, then $\Omega(\mathscr{S}, \mathscr{T})$ is a topological measurable space.

Example 2.3. If $\Omega(\mathscr{S})$ is as in Example 1.5 and $\mathscr{T}$ is the usual topology of Ω, then $\Omega(\mathscr{S}, \mathscr{T})$ is a topological measurable space.

Example 2.4. If $\Omega(\mathscr{T})$ is a locally compact Hausdorff space, $\mathscr{C}$ is the collection of all compact subsets of $\Omega(\mathscr{T})$ and $\mathscr{S}$ is as in Example 1.7, then $\Omega(\mathscr{S}, \mathscr{T})$ is a topological measurable space.

To check this, recall that in a locally compact Hausdorff space the compact neighbourhoods of a point form a base for its neighbourhood system [18, p. 146].

Remark 2.5. The above important example has also been considered by Bogdanowicz [3, p. 220].

Its particular case when $\Omega(\mathscr{T})$ is a discrete space will allow us to regard various sums as particular integrals.

Some extreme examples for topological measurable spaces can also be obtained from the next simple

Theorem 2.6. *If $\Omega(\mathscr{S})$ is a measurable space such that $\Omega=\bigcup \mathscr{S}$, then $\mathscr{S}$ is a base for a topology $\mathscr{T}_{\mathscr{S}}$ on Ω, and $\mathscr{T}_{\mathscr{S}}$ is the largest topology on Ω such that $\Omega(\mathscr{S}, \mathscr{T}_{\mathscr{S}})$ is a topological measurable space.*

Proof. Corollary 1.11 shows that $\mathscr{S}$ is a base for a topology $\mathscr{T}_{\mathscr{S}}$ on Ω, and hence it is clear that $\mathscr{S}$ and $\mathscr{T}_{\mathscr{S}}$ are compatible in the sense of Definition 2.1.

Suppose now that $\mathscr{T}$ is another topology on Ω which is also compatible with $\mathscr{S}$. If $V \in \mathscr{T}$, then by Definition 2.1, for any $x \in V$ there exists $A \in \mathscr{S}$ such that $x \in A \subset V$, and thus $V \in \mathscr{T}_{\mathscr{S}}$. Consequently, $\mathscr{T} \subset \mathscr{T}_{\mathscr{S}}$.

Remark 2.7. This theorem shows that we could at once have started with topological measurable spaces instead of the measurable ones without an essential restriction of generality.

The appropriateness of our former definitions is also well-shown by the following theorem and its subsequent consequences.

Theorem 2.8. *If $\Omega(\mathscr{S}, \mathscr{T})$ is a hereditarily Lindelöf topological measurable space, then $\mathscr{T}$ consists of countably $\mathscr{S}$-divisible sets.*

Proof. If $V\in\mathcal{T}$, then by Definition 2.1, for each $x\in V$ there exists $A_x\in\mathcal{S}$ such that $x\in A_x^0$ and $A_x\subset V$. Thus, in particular, we have $V=\bigcup_{x\in V} A_x^0$. Hence, by the hereditary Lindelöf property of $\Omega(\mathcal{T})$, there exists a countable family $(x_i)_{i\in I}$ in V such that $V=\bigcup_{i\in I} A_{x_i}^0$. Thus, since $A_{x_i}\subset V$ for all $i\in I$, we also have $V=\bigcup_{i\in I} A_i$. Hence, by Corollary 1.19, it is clear that V is countably $\mathcal{S}$-divisible.

Corollary 2.9. *If $\Omega(\mathcal{S},\mathcal{T})$ is a hereditarily Lindelöf σ-additive topological measurable space, then $A\cap V\in\mathcal{S}$ for all $A\in\mathcal{S}$ and $V\in\mathcal{T}$.*

Proof. If $V\in\mathcal{T}$, then by Theorem 2.8, V has a countable $\mathcal{S}$-division $(A_i)_{i\in I}$. Thus, in particular, we have $A=\bigcup_{i\in I} A\cap A_i$ for any $A\in\mathcal{S}$. Hence, by Corollary 1.11 and the σ-additivity of it is clear that $A\cap V\in\mathcal{S}$ for any $A\in\mathcal{S}$.

Corollary 2.10. *If $\Omega(\mathcal{S},\mathcal{T})$ is a hereditarily Lindelöf σ-additive finite topological measurable space, then $\mathcal{T}\subset\mathcal{S}$.*

Proof. By Remark 1,27, now we have $\Omega\in\mathcal{S}$. Thus, by Corollary 2.9, $V=\Omega\cap\cap V\in\mathcal{S}$ for all $V\in\mathcal{T}$.

Remark 2.11. In the proof of Theorem 2.8, we have only used that each open subspace of $\Omega(\mathcal{T})$ is Lindelöf. However, this already implies that $\Omega(\mathcal{T})$ is hereditarily Lindelöf [8, Theorem III.3.2].

By the Lindelöf theorem [8, Lemma II.12.4] each second countable space is hereditarily Lindelöf. On the other hand, a semi-metric space is hereditarily Lindelöf if and only if it is separable (second countable) [8, Theorem II.12.1, Lemma II.12.4 and Exercise II.12.10].

Thus, in particular, the topological measurable spaces given in Examples 2.2 and 2.3 are hereditarily Lindelöf. In this respect, it is also worth mentioning that if $\Omega(\mathcal{S})$ is as in Example 1.3 and $\mathcal{T}_{\mathcal{S}}$ is as in Theorem 2.6, then $\Omega(\mathcal{S},\mathcal{T}_{\mathcal{S}})$ is hereditarily Lindelöf [8, Example III.4] and separable, but not second countable, and hence not semi-metrizable.

3. Divisions

Definition 3.1. Let $\Omega(\mathcal{S})$ be a measurable space. A disjoint family $\sigma=(\sigma_i)_{i\in I}$ in $\mathcal{S}$ will be called a division in $\Omega(\mathcal{S})$. The collection of all divisions in $\Omega(\mathcal{S})$ will be denoted by $\mathcal{D}(\Omega,\mathcal{S})$.

A division $\sigma=(\sigma_i)_{i\in I}$ in $\Omega(\mathcal{S})$ is said to be a division of $\Omega(\mathcal{S})$ if $\Omega=\bigcup\sigma=\bigcup_{i\in I}\sigma_i$. The collection of all divisions of $\Omega(\mathcal{S})$ will be denoted by $\mathcal{D}^*(\Omega,\mathcal{S})$.

Moreover, the collections of all finite (countable) members of $\mathcal{D}(\Omega,\mathcal{S})$ and $\mathcal{D}^*(\Omega,\mathcal{S})$ will be denoted by $\mathcal{D}_0(\Omega,\mathcal{S})$ $(\mathcal{D}_1(\Omega,\mathcal{S}))$ and $\mathcal{D}_0^*(\Omega,\mathcal{S})$ $(\mathcal{D}_1^*(\Omega,\mathcal{S}))$, respectively.

Remark 3.2. Examples 1.3, 1.5 and 1.7 and the fact that we shall mainly be interested in finite divisions explain why a division in $\Omega(\mathcal{S})$ cannot at once be required to be a division of $\Omega(\mathcal{S})$.

Definition 3.3. For any two elements $\sigma=(\sigma_i)_{i\in I}$ and $\varrho=(\varrho_j)_{j\in J}$ of $\mathcal{D}(\Omega,\mathcal{S})$, we write $\sigma\leqq\varrho$ if each σ_i is a union of some ϱ_j's.

In this case, we say that ϱ is a refinement of σ, and the relation $\leqq$ will be called the natural refinement relation of $\mathscr{D}(\Omega, \mathscr{S})$.

Theorem 3.4. *The natural refinement relation of* $\mathscr{D}(\Omega, \mathscr{S})$ *is a partial order on* $\mathscr{D}(\Omega, \mathscr{S})$ *which turns* $\mathscr{D}_0(\Omega, \mathscr{S})$, $\mathscr{D}_0^*(\Omega, \mathscr{S})$, $\mathscr{D}_1^*(\Omega, \mathscr{S})$ *and* $\mathscr{D}^*(\Omega, \mathscr{S})$ *into directed sets.*

PROOF. It is easy to check that the natural refinement relation $\leqq$ is a reflexive, antisymmetric and transitive relation on $\mathscr{D}(\Omega, \mathscr{S})$.

Moreover, the fact that this relation directs $\mathscr{D}_0(\Omega, \mathscr{S})$ follows immediately from Theorem 1.14. (Actually, Theorem 1.14 gives a common refinement $\delta \in \mathscr{D}_0(\Omega, \mathscr{S})$ of any $\sigma, \varrho \in \mathscr{D}_0(\Omega, \mathscr{S})$ with the additional property $\bigcup \delta = (\bigcup \sigma) \cup (\bigcup \varrho)$.)

Finally, to complete the proof, note that if $\sigma = (\sigma_i)_{i \in I}$ and $\varrho = (\varrho_j)_{j \in J}$ are arbitrary members of $\mathscr{D}^*(\Omega, \mathscr{S})$, then by Corollary 1.11, the set $\sigma_i \cap \varrho_j$ has a finite $\mathscr{S}$-division $(\delta_{ijk})_{k \in K_{ij}}$ for any $i \in I$ and $j \in J$, and we may consider the family $\delta = (\delta_{ijk}\colon i \in I, j \in J, k \in K_{ij})$ which is clearly a member of $\mathscr{D}^*(\Omega, \mathscr{S})$ such that $\sigma \leqq \delta$ and $\varrho \leqq \delta$.

Note that if in particular σ and ϱ belong to $\mathscr{D}_0^*(\Omega, \mathscr{S})$ $(\mathscr{D}_1^*(\Omega, \mathscr{S}))$, then δ also belongs to $\mathscr{D}_0^*(\Omega, \mathscr{S})$ $(\mathscr{D}_1^*(\Omega, \mathscr{S}))$.

To have a straightforward development hence to the topological case, we first provide a useful reformulation of the natural refinement relation for divisions of fixed sets.

Definition 3.5. For $\sigma = (\sigma_i)_{i \in I} \in \mathscr{D}(\Omega, \mathscr{S})$, the relation

$$R_\sigma = \bigcup_{i \in I} \sigma_i \times \sigma_i$$

will be called the equivalence relation induced by σ.

Proposition 3.6. *If* $\sigma = (\sigma_i)_{i \in I}$ *and* $\varrho = (\varrho_j)_{j \in J}$ *are members of* $\mathscr{D}(\Omega, \mathscr{S})$ *such that* $\bigcup \sigma = \bigcup \varrho$, *then the following properties are equivalent:*

(i) $\sigma \leqq \varrho$;

(ii) *each* ϱ_j *is contained in some* σ_i;

(iii) $R_\varrho \subset R_\sigma$.

This simple proposition and the fact that R_σ is an open neighbourhood of the diagonal of $\Omega(\mathscr{T}_{\mathscr{S}})$ for all $\sigma \in \mathscr{D}^*(\Omega, \mathscr{S})$ provide the necessary motivation for the next

Definition 3.7. If $\Omega(\mathscr{S}, \mathscr{T})$ is a topological measurable space and R is a neighbourhood of the diagonal of $\Omega(\mathscr{T})$, then a division σ in $\Omega(\mathscr{S})$ will be called R-fine if $R_\sigma \subset R$.

Remark 3.8. Note that if in particular $\mathscr{T}$ is induced by a semi-metric d on Ω, then the relation B_r is defined on Ω by

$$B_r(x) = \{y\colon d(x, y) \leqq r\}$$

is a neighbourhood of the diagonal of $\Omega(\mathscr{T})$ for all $r > 0$, and the B_r-fineness of a division $\sigma = (\sigma_i)_{i \in I}$ in $\Omega(\mathscr{S})$ means simply that

$$|\sigma| = \sup\,\{\operatorname{diam} \sigma_i\}_{i \in I} \leqq r.$$

Theorem 3.9. *If $\Omega(\mathcal{S}, \mathcal{T})$ is a topological measurable space, then for any compact (Lindelöf) subspace Q of $\Omega(\mathcal{T})$ and any neighbourhood R of the diagonal of $\Omega(\mathcal{T})$ there exists an R-fine, finite (countable) division σ in $\Omega(\mathcal{S})$ such that $Q \subset \bigcup \sigma$.*

PROOF. By the definition of the product topology, for each $x \in Q$, there exists $V_x \in \mathcal{T}$ such that $x \in V_x$ and $V_x \times V_x \subset R$. Moreover, by Definition 2.1, for each $x \in Q$, there exists $A_x \in \mathcal{S}$ such that $x \in A_x^0$ and $A_x \subset V_x$. Thus, in particular, we have $Q \subset \bigcup_{x \in Q} A_x^0$. Now, since Q is compact (Lindelöf), there exists a finite (countable) family $(x_i)_{i \in I}$ in Q such that $Q \subset \bigcup_{i \in I} A_{x_i}^0$, and hence $Q \subset \bigcup_{i \in I} A_{x_i}$. Hence, by applying Theorem 1.14 (1.18), we can get a finite (countable) $\mathcal{S}$-division $\sigma = (\sigma_j)_{j \in J}$ of $\bigcup_{i \in I} A_{x_i}$ such that each σ_j is contained in some A_{x_i}. Thus, we have $Q \subset \bigcup_{j \in J} \sigma_j$. Moreover, since $A_{x_i} \subset V_{x_i}$ and $V_{x_i} \times V_{x_i} \subset R$ for all $i \in I$, it is clear that $R_\sigma \subset R$.

Corollary 3.10. *If $\Omega(\mathcal{S}, \mathcal{T})$ is a compact (Lindelöf) topological measurable space, then for any neighbourhood R of the diagonal of $\Omega(\mathcal{T})$ there exists an R-fine finite (countable) division σ of $\Omega(\mathcal{S})$.*

Corollary 3.11. *A compact (Lindelöf) topological measurable space $\Omega(\mathcal{S}, \mathcal{T})$ is finite (σ-finite).*

Remark 3.12. Corollary 3.11 and the simple fact that σ-compact spaces are Lindelöf show that the topological measurable space given in Example 2.4 is Lindelöf if and only if it is σ-finite.

To briefly formulate a further useful consequence of Theorem 3.9, it seems convenient to introduce the next

Definition 3.13. A topological measurable space will be called almost compact (almost Lindelöf) if each element of $\mathcal{S}$ is contained in a compact (Lindelöf) subspace of $\Omega(\mathcal{T})$.

Remark 3.14. Note that the topological measurable spaces given in Examples 2.2, 2.3 and 2.4 are almost compact, and thus the next theorem can be applied to them.

Theorem 3.15. *If $\Omega(\mathcal{S}, \mathcal{T})$ is an almost compact (almost Lindelöf) topological measurable space, then for any $A \in \mathcal{S}$ and any neighbourhood R of the diagonal of $\Omega(\mathcal{T})$ there exists an R-fine finite (countable) division ϱ in $\Omega(\mathcal{S})$ such that $A = \bigcup \varrho$.*

PROOF. By the assumptions, there exists a compact (Lindelöf) subspace Q of $\Omega(\mathcal{T})$ such that $A \subset Q$. Moreover, by Theorem 3.9, there exists an R-fine, finite (countable) division $\sigma = (\sigma_i)_{i \in I}$ in $\Omega(\mathcal{S})$ such that $Q \subset \bigcup_{i \in I} \sigma_i$. Thus, we have $A = \bigcup_{i \in I} A \cap \sigma_i$. On the other hand, by Corollary 1.11, for each $i \in I$, there exists a finite $\mathcal{S}$-division $(\varrho_{ik})_{k \in K_i}$ of $A \cap \sigma_i$. Hence, it is clear that the family $\varrho = (\varrho_{ik}\colon i \in I, k \in K_i)$ is a division in $\Omega(\mathcal{S})$ with the required properties.

Corollary 3.16. *If $\Omega(\mathcal{S}, \mathcal{T})$ is an almost compact (almost Lindelöf) topological measurable space, then for any finite (countable) division σ in $\Omega(\mathcal{S})$ and any neigh-*

bourhood R of the diagonal of $\Omega(\mathcal{T})$, there exists an R-fine finite (countable) division ϱ in $\Omega(\mathcal{S})$ such that $\sigma \leqq \varrho$ and $\bigcup \sigma = \bigcup \varrho$.

PROOF. If $\sigma=(\sigma_i)_{i\in I}$, then by Theorem 3.15, for each $i\in I$, there exists an R-fine finite (countable) division $(\varrho_{ij})_{j\in J_i}$ in $\Omega(\mathcal{S})$ such that $\sigma_i=\bigcup_{j\in J_i} \varrho_{ij}$. And thus, we can define $\varrho=(\varrho_{ij}\colon i\in I, j\in J_i)$.

4. Tagged divisions

Definition 4.1. Let $\Omega(\mathcal{S})$ be a measurable space. A family $\tau=(\tau_i)_{i\in I}$ in Ω will be called a tag (choice) for a division $\sigma=(\sigma_i)_{i\in I}$ in $\Omega(\mathcal{S})$ (if $\tau_i\in\sigma_i$ whenever $\sigma_i\neq\emptyset$). The collection of all tags (choices) for σ will be denoted by $\mathcal{T}_\Omega(\sigma)$ $(\mathcal{C}_\Omega(\sigma))$.

An ordered pair (σ,τ) consisting of a division σ in $\Omega(\mathcal{S})$ and a tag (choice) τ for σ will be called a tagged (choiced) division in $\Omega(\mathcal{S})$. The collection of all tagged (choiced) divisions in $\Omega(\mathcal{S})$ will be denoted by $\mathcal{D}\mathcal{T}(\Omega,\mathcal{S})$ $\big(\mathcal{D}\mathcal{C}(\Omega,\mathcal{S})\big)$.

Moreover, in connection with the collections $\mathcal{D}\mathcal{T}(\Omega,\mathcal{S})$ and $\mathcal{D}\mathcal{C}(\Omega,\mathcal{S})$ we shall use the same kind of notation as in Definition 3.1. Thus, for instance $\mathcal{D}_0\mathcal{T}(\Omega,\mathcal{S})$ $\big(\mathcal{D}_0^*\mathcal{T}(\Omega,\mathcal{S})\big)$ will denote the collection of all finite members (σ,τ) of $\mathcal{D}\mathcal{T}(\Omega,\mathcal{S})$ (such that $\Omega=\bigcup\sigma$).

Remark 4.2. The above definition has mainly been suggested to us by McLeod [24, p. 17] and McShane [25, p. 14].

To include their ideas of fine tagged divisions in our more general setting, it seems convenient to introduce the following

Definition 4.3. A relation R on a topological space $\Omega(\mathcal{T})$ will be called a semi-neighbourhood of the diagonal of $\Omega(\mathcal{T})$ if $x\in R(x)^0$ for all $x\in\Omega$.

Remark 4.4. Recall that R is a neighbourhood of the diagonal iff $(x,x)\in R^0$ for all $x\in\Omega$.

Moreover, note that the above definition is a little more general than the one suggested by ČECH [5, p. 306].

Definition 4.5. If $\Omega(\mathcal{S},\mathcal{T})$ is a topological measurable space and R is a semi-neighbourhood of the diagonal of $\Omega(\mathcal{T})$, then a tagged division $(\sigma,\tau)=\big((\sigma_i)_{i\in I},(\tau_i)_{i\in I}\big)$ in $\Omega(\mathcal{S})$ will be called R-fine if $\sigma_i\subset R(\tau_i)$ for all $i\in I$.

Remark 4.6. The above crucial definition has its origin in a 1957 paper of KURZWEIL [21].

HENSTOCK [12] working independently came to the same idea around 1960. (See also MCSHANE [25].)

Analogously to Theorem 3.9, now we have:

Theorem 4.7. *If $\Omega(\mathcal{S},\mathcal{T})$ is a topological measurable space, then for any compact (Lindelöf) subspace Q of $\Omega(\mathcal{T})$ and any semi-neighbourhood R of the diagonal of $\Omega(\mathcal{T})$ there exists an R-fine finite (countable) tagged division (σ,τ) in $\Omega(\mathcal{S})$ such that $Q\subset\bigcup\sigma$ and the members of τ are in Q.*

PROOF. A similar argument as in the proof of Theorem 3.9 shows that there are finite (countable) families $(x_i)_{i\in I}$ and $(A_i)_{i\in I}$ in Q and $\mathcal{S}$, respectively, such that $x_i \in A_i \subset R(x_i)$ for all $i \in I$, and $Q \subset \bigcup_{i \in I} A_i$. Hence, by applying Theorem 1.14 (1.18), we can get a finite (countable) $\mathscr{S}$-division $\sigma = (\sigma_j)_{j \in J}$ of $\bigcup_{i \in I} A_i$ such that $\sigma_j \subset A_{i_j}$ for some $i_j \in I$ whenever $j \in J$. Now, by defining $\tau_j = x_{i_j}$ for all $j \in J$, we can add a tag $\tau = (\tau_j)_{j \in J}$ to σ such that the tagged division (σ, τ) have the required properties.

Corollary 4.8. *If $\Omega(\mathscr{S}, \mathscr{T})$ is a compact (Lindelöf) topological measurable space, then for any semi-neighbourhood R of the diagonal of $\Omega(\mathscr{T})$ there exists an R-fine finite (countable) tagged division (σ, τ) of $\Omega(\mathscr{S})$.*

The proof of the next theorem is also quite similar to that of Theorem 3.15.

Theorem 4.9. *If $\Omega(\mathscr{S}, \mathscr{T})$ is an almost compact (almost Lindelöf) topological measurable space, then for any $A \in \mathscr{S}$ and any semi-neighbourhood R of the diagonal of $\Omega(\mathscr{T})$ there exists an R-fine finite (countable) tagged division (σ, τ) in $\Omega(\mathscr{S})$ such that $A = \bigcup \sigma$.*

Analogously to Corollary 3.16, now we have:

Corollary 4.10. *If $\Omega(\mathscr{S}, \mathscr{T})$ is an almost compact (almost Lindelöf) topological measurable space, then for any finite (countable) division σ in $\Omega(\mathscr{S})$ and any semi-neighbourhood R of the diagonal of $\Omega(\mathscr{T})$ there exists an R-fine finite (countable) tagged division (ϱ, v) in $\Omega(\mathscr{S})$ such that $\sigma \leqq \varrho$ and $\bigcup \sigma = \bigcup \varrho$.*

Remark 4.11. In the above assertions, we can also make some restrictions on the tags. But, unfortunately, the corresponding assertions do not, in general, hold for choiced divisions.

In connection with choiced divisions, we can only prove an analogue of the compatibility theorem of McLeod [24, p. 38]. For this, it seems convenient to begin with the next useful

Lema 4.12. *For any $\alpha, \beta, \varepsilon \in \mathbf{R}$ with $0 < \varepsilon < \beta - \alpha$ and any semi-neighbourhood S of the diagonal of $\mathbf{R}$ there are points*

$$\alpha = t_0 \leqq \tau_1 < t_1 \leqq \tau_2 < t_2 \leqq \ldots < t_{n-1} \leqq \tau_n < t_n \leqq \beta$$

with $t_n > \beta - \varepsilon$ such that

$$[t_{i-1}, t_i[\subset S(\tau_i)$$

for all $i = 1, 2, \ldots, n$.

Proof. Denote by A the set of all points $x \in [\alpha, \beta]$ for which there are points

$$\alpha = t_0 \leqq \tau_1 < t_1 \leqq \tau_2 < t_2 \leqq \ldots < t_{n-1} \leqq \tau_n < t_n = x$$

such that $[t_{i-1}, t_i[\subset S(\tau_i)$ for all $i = 1, 2, \ldots, n$.

Since $\alpha \in S(\alpha)^0$ and $\alpha < \beta$, there exists $r > 0$ such that $[\alpha, \alpha + r[\subset S(\alpha)$ and $\alpha + r \leqq \beta$. Thus, $\alpha + r \in A$, and hence $\alpha < \gamma = \sup A \leqq \beta$.

Now, it remains only to show that $\gamma = \beta$. For this assume on the contrary $\gamma < \beta$. Since $\gamma \in S(\gamma)^0$, there exists $s > 0$ such that $]\gamma - s, \gamma + s[\subset S(\gamma)$ and $\gamma + s < \beta$. Moreover, since $\gamma = \sup A$, there exists $x \in A$ such that $\gamma - s < x$. For this x, there are points t_i and τ_i as above. Furthermore, we can define $\tau_{n+1}=\gamma$ and $t_{n+1}=\gamma+s$. Hence, it is clear that $\gamma+s\in A$, which is a contradiction.

Remark 4.13. By defining S such that $\beta\notin\overline{S(t)}$ for all $t<\beta$, one can at once see that $\beta\in A$ does not, in general, hold. And thus the above lemma cannot be improved.

Definition 4.14. A division σ in a measurable space $\Omega(\mathscr{S})$ will be called semi-finite if each element of $\mathscr{S}$ intersects only finitely many members of σ.

Theorem 4.15. *If $\Omega(\mathscr{S}, \mathscr{T})$ is as in Example 2.2, then for any semi-finite division σ of $\Omega(\mathscr{S})$ and any semi-neighbourhood R of the diagonal of $\Omega(\mathscr{T})$ there exists an R-fine semi-finite choiced division (ϱ, v) of $\Omega(\mathscr{S})$ such that $\sigma\leqq\varrho$.*

PROOF. Because of the semi-finiteness of σ, we may assume that $\sigma=([t_{k-1}, t_k[)_{k\in\mathbf{Z}}$ such that $t_{k-1}<t_k$ for all $k\in\mathbf{Z}$. Since R is a semi-neighbourhood of the diagonal, for each $k\in\mathbf{Z}$, there exists $r_k>0$ such that

$$]t_k-r_k, t_k+r_k[\subset R(t_k) \quad\text{and}\quad 2r_k<\min\{t_k-t_{k-1}, t_{k+1}-t_k\}.$$

Moreover, by Lemma 4.12, for each $k\in\mathbf{Z}$, there are points

$$t_k+r_k=s_{k0}\leqq v_{k1}<s_{k1}\leqq v_{k2}<s_{k2}\leqq\ldots<s_{kn_k-1}\leqq v_{kn_k}<s_{kn_k}\leqq t_{k+1}$$

with $s_{kn_k}>t_{k+1}-r_{k+1}$ such that

$$\varrho_{kl}=[s_{kl-1}, s_{kl}[\subset R(v_{kl})$$

for all $l=1, 2, \ldots, n_k$. Now, by defining

$$\varrho_{k0}=[s_{k-1n_{k-1}}, s_{k0}[\quad\text{and}\quad \tau_{k0}=t_k$$

for all $k\in\mathbf{Z}$, we can form the required choiced division $(\varrho, v)=((\varrho_j)_{j\in J}, (v_j)_{j\in J})$ with

$$J=\{(k, l)\colon k\in\mathbf{Z},\ l=0, 1, \ldots, n_k\}.$$

Corollary 4.16. *If $\Omega(\mathscr{S}, \mathscr{T})$ is as in Example 2.2, then for any semi-neighbourhood R of the diagonal of $\Omega(\mathscr{T})$ there exists an R-fine semi-finite choiced division (ϱ, v) of $\Omega(\mathscr{S})$.*

PROOF. Defining $\sigma=([k-1, k[)_{k\in\mathbf{Z}}$, Theorem 4.14 can be applied.

To include a reasonable particular case of the compatibility theorem of McShane [26, p. 28] too, it seems appropriate to introduce another special kind of tagged divisions.

Definition 4.17. Let $\Omega(\mathscr{S}, \mathscr{T})$ be a topological measurable space. A family $\tau=(\tau_i)_{i\in I}$ in Ω will be called a semi-choice for a division $\sigma=(\sigma_i)_{i\in I}$ in $\Omega(\mathscr{S}, \mathscr{T})$ if $\tau_i\in\bar{\sigma}_i$ whenever $\sigma_i\neq\emptyset$. The collection of all semi-choices for σ will be denoted by $\overline{\mathscr{C}}_{\Omega(\mathscr{T})}(\sigma)$.

An ordered pair (σ, τ) consisting of a division in $\Omega(\mathscr{S}, \mathscr{T})$ and a semi-choice τ for σ will be called a semi-choiced division in $\Omega(\mathscr{S}, \mathscr{T})$. The collection of all semi-choiced divisions in $\Omega(\mathscr{S}, \mathscr{T})$ will be denoted by $\mathscr{D}\overline{\mathscr{C}}(\Omega, \mathscr{S}, \mathscr{T})$.

Moreover, in connection with the collection $\mathscr{D}\overline{\mathscr{C}}(\Omega, \mathscr{S}, \mathscr{T})$ we shall use the same kind of notation as in Definition 3.1. Thus, for instance $\mathscr{D}_1\overline{\mathscr{C}}(\Omega, \mathscr{S}, \mathscr{T})$ $(\mathscr{D}_1^*\overline{\mathscr{C}}(\Omega, \mathscr{S}, \mathscr{T}))$ will denote the collection of all countable members (σ, τ) of $\mathscr{D}\overline{\mathscr{C}}(\Omega, \mathscr{S}, \mathscr{T})$ (such that $\Omega = \bigcup \sigma$).

In contrast to Lemma 4.12 and Remark 4.13, now we have:

Theorem 4.18. *If $\Omega(\mathscr{S}, \mathscr{T})$ is as in Example 2.2, then for any $A \in \mathscr{S}$ and any semi-neighbourhood R of the diagonal of $\Omega(\mathscr{T})$ there exists an R-fine finite semi-choiced division (σ, τ) in $\Omega(\mathscr{S}, \mathscr{T})$ such that $A = \bigcup \sigma$.*

Proof. Since the assertion trivially holds if $A = \emptyset$, we may assume that $A = [\alpha, \beta[$ with $\alpha < \beta$. Now, since $\beta \in R(\beta)^0$, there exists $0 < \varepsilon < \beta - \alpha$ such that $[\beta - \varepsilon, \beta[\subset \subset R(\beta)$. Moreover, by Lemma 4.12, there are points

$$\alpha = t_0 \leqq \tau_1 < t_1 \leqq \tau_2 < t_2 \leqq \ldots < t_{n-1} \leqq \tau_n < t_n \leqq \beta$$

with $t_n > \beta - \varepsilon$ such that $[t_{i-1}, t_i[\subset R(\tau)$ for all $i = 1, 2, \ldots, n$. Hence, by defining $\tau_{n+1} = t_{n+1} = \beta$,

$$\sigma = ([t_{i-1}, t_i[)_{i=1}^{n+1} \quad \text{and} \quad \tau = (\tau_i)_{i=1}^{n+1},$$

we can form the required semi-choiced division (σ, τ).

As an immediate consequence of this theorem, we can state

Corollary 4.19. *If $\Omega(\mathscr{S}, \mathscr{T})$ is as in Example 2.2, then for any semi-finite (countable) division σ in $\Omega(\mathscr{S})$ and any semi-neighbourhood R of the diagonal of $\Omega(\mathscr{T})$ there exists an R-fine semi-finite (countable) semi-choiced division (ϱ, ν) in $\Omega(\mathscr{S}, \mathscr{T})$ such that $\sigma \leqq \varrho$ and $\bigcup \sigma = \bigcup \varrho$.*

Remark 4.20. Theorems 3.9, 3.15, 4.7, 4.9, 4.15 and 4.18 and their corollaries strongly suggest the use of measurable relator spaces instead of the measurable topological ones.

However, at the present, we could not do this generalization since the second author's results on relator spaces have not been published up till now and are still far from being complete.

5. Defining nets for integration

Example 5.1. Let $\Omega(\mathscr{S})$ be a measurable space and $\Gamma = \mathscr{D}_0\mathscr{C}(\Omega, \mathscr{S})$. For any two elements (σ, τ) and (ϱ, ν) of Γ, define $(\sigma, \tau) \leqq (\varrho, \nu)$ if $\sigma \leqq \varrho$. Then the identity function of Γ is a defining net for integration over $\Omega(\mathscr{S})$.

The fact that Γ is a directed set follows at once from Theorem 3.4. Moreover, it is clear that $\Gamma \neq \emptyset$.

Remark 5.2. The above natural net of integration can lead to powerful integrations only in the σ-additive cases.

To obtain powerful integrations also in the more general cases much more difficult constructions are needed.

Definition 5.3. A function φ defined on a subcollection $\mathscr{E}$ of $\mathscr{D}(\Omega, \mathscr{S})$ will be called a finitator on $\mathscr{E}$ if $\varphi(\sigma)$ is a finite subset of I_σ for any $\sigma=(\sigma_i)_{i\in I_\sigma}\in\mathscr{E}$.

The collection of all finitators φ on $\mathscr{E}$ will be denoted by $\Phi(\mathscr{E})$, and for any two $\varphi, \psi\in\Phi(\mathscr{E})$, we write $\varphi\leqq\psi$ if $\varphi(\sigma)\subset\psi(\sigma)$ for all $\sigma\in\mathscr{E}$.

Remark 5.4. Note that the above pointwise partial order turns $\Phi(\mathscr{E})$ into a directed set, since for any $\varphi, \psi\in\Phi(\mathscr{E})$ the function $\varphi\vee\psi$ defined on $\mathscr{E}$ by

$$(\varphi\vee\psi)(\sigma) = \varphi(\sigma)\cup\psi(\sigma)$$

also belongs to $\Phi(\mathscr{E})$.

Example 5.5. Let $\Omega(\mathscr{S})$ be a σ-finite measurable space and

$$\Gamma = \{(\sigma, \tau, \varphi)\colon (\sigma, \tau)\in\mathscr{D}_1^*\mathscr{C}(\Omega, \mathscr{S}),\ \varphi\in\Phi(\mathscr{D}_1^*(\Omega, \mathscr{S}))\}.$$

For any two elements (σ, τ, φ) and (ϱ, ν, ψ) of Γ, define

$$(\sigma, \tau, \varphi) \leqq (\varrho, \nu, \psi) \quad \text{if} \quad \sigma \leqq \varrho \quad \text{and} \quad \varphi \leqq \psi.$$

Then the function $\mathfrak{R}$ defined on Γ by

$$\mathfrak{R}(\sigma, \tau, \varphi) = \big((\sigma_i)_{i\in\varphi(\sigma)}, (\tau_i)_{i\in\varphi(\sigma)}\big)$$

is a defining net for integration over $\Omega(\mathscr{S})$

The fact that Γ is a directed set is immediate from Theorem 3.4 and Remark 5.4. Moreover, the σ-finiteness of $\Omega(\mathscr{S})$ guarantees that $\Gamma\neq\emptyset$.

Remark 5.6. The above net of integration has mainly been suggested to us by Sion [30, p. 17].

The main motivation for it is that, by the iterated limit theorem [18, p. 69], the single limit

$$\int f\,d\mu = \lim_{(\sigma,\tau,\varphi)\in\Gamma}\ \sum_{i\in\varphi(\sigma)} f(\tau_i)\,\mu(\sigma_i)$$

is always equal to the iterated limit

$$\lim_{(\sigma,\tau)\in\mathscr{D}_1^*\mathscr{C}}\ \sum_{i\in I_\sigma} f(\tau_i)\,\mu(\sigma_i) = \lim_{(\sigma,\tau)\in\mathscr{D}_1^*\mathscr{C}}\ \lim_{\substack{J\subset I_\sigma \\ J \text{ is finite}}}\ \sum_{j\in J} f(\tau_j)\,\mu(\sigma_j)$$

whenever this latter exists, and thus the use of infinite approximating sums can be avoided.

Example 5.7. Let $\Omega(\mathscr{S}, \mathscr{T})$ be an almost compact topological measurable space, and denote by $\mathscr{R}$ the collection of all neighbourhoods (semi-neighbourhoods) of the diagonal of $\Omega(\mathscr{T})$. Moreover, let Γ be the collection of all ordered triples (σ, τ, R) such that (σ, τ) belongs to $\mathscr{D}_0\mathscr{C}(\Omega, \mathscr{S})$ $(\mathscr{D}_0\mathscr{T}(\Omega, \mathscr{S}))$, $R\in\mathscr{R}$ and σ $((\sigma, \tau))$ is R-fine. For any two elements (σ, τ, R) and (ϱ, ν, S) of Γ, define $(\sigma, \tau, R)\leqq\leqq(\varrho, \nu, S)$ if $\sigma\leqq\varrho$ and $R\leqq S$ in the sense that $S\subset R$. Then, the function $\mathfrak{R}$ defined on Γ by

$$\mathfrak{R}(\sigma, \tau, R) = (\sigma, \tau)$$

is a defining net for integration over $\Omega(\mathscr{S})$.

To check that Γ is a directed set note that $R, S\in\mathscr{R}$ implies $R\cap S\in\mathscr{R}$, and apply Theorem 3.4 and Corollary 3.16 (4.10). The fact that $\Gamma\neq\emptyset$ is obvious.

Example 5.8. Let $\Omega(\mathscr{S}, \mathscr{T})$ be a Lindelöf topological measurable space, and denote by $\mathscr{R}$ the collection of all neighbourhoods (semi-neighbourhoods) of the diagonal of $\Omega(\mathscr{T})$. Moreover, let Γ be the collection of all ordered quadruples $(\sigma, \tau, R, \varphi)$ such that (σ, τ) belongs to $\mathscr{D}_1^*\mathscr{C}(\Omega, \mathscr{S})$ $(\mathscr{D}_1^*\mathscr{T}(\Omega, \mathscr{S}))$, $R\in\mathscr{R}$, $\varphi\in\Phi(\mathscr{D}_1^*(\Omega, \mathscr{S}))$ and σ $((\sigma, \tau))$ is R-fine. For any two elements $(\sigma, \tau, R, \varphi)$ and (ϱ, v, S, ψ) of Γ, define $(\sigma, \tau, R, \varphi)\leqq(\varrho, v, S, \psi)$ if $\sigma\leqq\varrho$, $R\leqq S$ and $\varphi\leqq\psi$, where $R\leqq S$ means again $S\subset R$. Then, the function $\mathfrak{N}$ defined on Γ by

$$\mathfrak{N}(\sigma, \tau, R, \varphi) = \big((\sigma_i)_{i\in\varphi(\sigma)}, (\tau_i)_{i\in\varphi(\sigma)}\big)$$

is a defining net for integration over $\Omega(\mathscr{S})$.

The fact that Γ is a directed set follows again from Theorem 3.4 and Corollary 3.16 (4.10) Moreover, Corollary 3.10 (4.8) shows that $\Gamma\neq\emptyset$.

Remark 5.9. The above nets of integration may lead to significant improvements of the classical refinement and norm integrals [15, p. 27] and can include the absolutely convergent gauge integral of McShane [25] which is a Riemann-type equivalent of the classical Lebesgue integral.

However, they seem still to be unsuitable to obtain the non-absolutely convergent gauge integral of Kurzweil [21] and Henstock [12] which is a common, Riemann-type, generalization of the classical Lebesgue and calculus integrals. Therefore, we also need the following particular defining nets for integration whose generalized forms should also be established.

Example 5.10. Let $\Omega(\mathscr{S}, \mathscr{T})$ be as in Example 2.2, and denote by R the collection of all semi-neighbourhoods of the diagonal of $\Omega(\mathscr{T})$. Moreover, let Γ be the collection of all ordered triples (σ, τ, R) such that $(\sigma, \tau)\in\mathscr{D}_0\overline{\mathscr{C}}(\Omega, \mathscr{S}, \mathscr{T})$, $R\in\mathscr{R}$ and (σ, τ) is R-fine. For any two elements (σ, τ, R) and (ϱ, v, S) of Γ, define $(\sigma, \tau, R)\leqq(\varrho, v, S)$ if $\sigma\leqq\varrho$ and $R\leqq S$. Then, the function $\mathfrak{N}$ defined on Γ by

$$\mathfrak{N}(\sigma, \tau, R) = (\sigma, \tau)$$

is a defining net for integration over $\Omega(\mathscr{S})$.

To check that Γ is a directed set, now Theorem 3.4 and Corollary 4.19 can be applied. The fact that $\Gamma\neq\emptyset$ is obvious.

Remark 5.11. A few properties of the integral defined by the above simple, but powerful net of integration has been established by Gulyás [9].

For instance, the fundamental theorem of the calculus holds for this net integral without the assumption of the integrability of the derivative.

Example 5.12. Let $\Omega(\mathscr{S}, \mathscr{T})$ be as in Example 2.2, and denote by $\mathscr{R}$ the collection of all semi-neighbourhoods of the diagonal of $\Omega(\mathscr{T})$. Moreover, let Γ be the collection of all ordered quadruples $(\sigma, \tau, R, \varphi)$ such that (σ, τ) belongs to $\mathscr{D}_1^*\overline{\mathscr{C}}(\Omega, \mathscr{S}, \mathscr{T})$ $(\mathscr{D}_1^*\mathscr{C}(\Omega, \mathscr{S}))$, $R\in\mathscr{R}$ and $\varphi\in\Phi(\mathscr{D}_1^*(\Omega, \mathscr{S}))$ such that (σ, τ) is R-fine (and semi-finite). For any two elements $(\sigma, \tau, R, \varphi)$ and (ϱ, v, S, ψ) of Γ, define $(\sigma, \tau, R, \varphi)\leqq(\varrho, v, S, \psi)$ if $\sigma\leqq\varrho$, $R\leqq S$ and $\varphi\leqq\psi$. Then the function $\mathfrak{N}$ fined on Γ by

$$\mathfrak{N}\,(\sigma, \tau, R, \varphi) = \big((\sigma_i)_{i\in\varphi(\sigma)}, (\tau_i)_{i\in\varphi(\sigma)}\big)$$

is a defining net for integration over $\Omega(\mathscr{S})$.

The fact that Γ is a directed set follows now from Theorem 3.4 and Corollary 4.19 (Theorem 4.15). Moreover, Corollary 4.16 shows that $\Gamma \neq \emptyset$.

The nets of integration listed in this section strongly suggest the investigation of the following

Problems. (1) What are the exact relationships among the corresponding net integrals?

(2) Which are the main characteristic properties of the most important net integrals?

(3) How are these net integrals related to the standard integrals?

Finally, we remark that it is also possible to construct some useful defining nets for integration with the help of a fixed measure.

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Added in proof. Meantime, we learned that B. Riečan [On the Kurzweil integral in compact topological spaces, *Radovi Mat.* **2** (1986) 151—163] and S. I. Ahmed and W. F. Pfeffer [A Riemann integral in a locally compact Hausdorff space, *J. Austral. Math. Soc.* **41** (1986), 115—137] had also arrived at similar ideas.

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