On completely additive functions

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Let $1<q<2$ and $L:=\sum_{n=1}^{\infty}\frac{1}{q^n}=\frac{1}{q-1}$. For every $x\in[0, L]$ let

$$\varepsilon_n(x):=\begin{cases}1 & \text{if } \sum_{i=1}^{n-1}\frac{\varepsilon_i(x)}{q^i}+\frac{1}{q^n}\leqq x\\ 0 & \text{if } \sum_{i=1}^{n-1}\frac{\varepsilon_i(x)}{q^i}+\frac{1}{q^n}> x,\end{cases} \tag{1.1}$$

be defined by induction on n. Then we have (see [1], [2], [3])

$$x=\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{q^n} \tag{1.2}$$

and we call the representation (1.2) the *regular* expansion of x. By the definition (1.1), if $n\in\mathbf{N}$ such that $\varepsilon_n(x)=0$, then

$$\sum_{i=1}^{n-1}\frac{\varepsilon_i(x)}{q^i}+\frac{1}{q^n}>x=\sum_{i=1}^{\infty}\frac{\varepsilon_i(x)}{q^i}$$

whence

$$\frac{1}{q^n}>\sum_{i=n+1}^{\infty}\frac{\varepsilon_i(x)}{q^i}. \tag{1.3}$$

Since $1<L$, the expansion

$$1=\sum_{n=1}^{\infty}\frac{\varepsilon_n(1)}{q^n} \tag{1.4}$$

exists. If $\varepsilon_n(1)=1$ for infinitely many n, then let

$$l_n:=\varepsilon_n(1)\quad (n\in\mathbf{N}). \tag{1.5}$$

If $\varepsilon_n(1)=1$ holds only for finitely many n, then let $s>1$ be the largest index for which $\varepsilon_s(1)=1$. Then $\varepsilon_n(1)=0$ for every $n>s$. In this case let

$$l_n:=\begin{cases}\varepsilon_i(1) & \text{if } n=k\cdot s+i \quad (1\leqq i\leqq s-1,\ k=0,1,\ldots)\\ 0 & \text{if } n=k\cdot s \qquad (k=1,2,\ldots).\end{cases} \tag{1.6}$$

It follows from (1.4), (1.6) that

$$1 = \sum_{n=1}^{\infty} \frac{l_n}{q^n} \tag{1.7}$$

where $l_n = 1$ for infinitely many n. We call the right-hand side of (1.7) the *infinite* expansion of the number 1.

The number $x \in [0, L]$ is called *uniquely determined* if there exists only one sequence $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$ for which

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n}. \tag{1.8}$$

Remark. The numbers 0, L are always uniquely determined.

Let $a_n \in \mathbf{R}$ and $\sum_{n=1}^{\infty} |a_n| < \infty$. Then we call the function

$$F(x) := \sum_{n=1}^{\infty} \varepsilon_n(x) \cdot a_n \quad (x \in [0, L]) \tag{1.9}$$

additive, where $\varepsilon_n(x)$ stands for the digits 0, 1 in the regular expansion of x.

The function $F\colon [0, L] \to \mathbf{R}$ is called *completely additive,* if

$$F\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n}\right) = \sum_{n=1}^{\infty} \varepsilon_n \cdot F\left(\frac{1}{q^n}\right) \tag{1.10}$$

holds for every sequence $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$. By the definition of additive functions a completely additive function is additive. In our investigations we use the following result proved in [4].

Theorem 1.1. *If $F\colon [0, L] \to \mathbf{R}$ is a completely additive function and $F(x) \geqq 0$ holds for all $x \in [0, L]$ then there exists $\alpha \in \mathbf{R}$ such that $F(x) = \alpha \cdot x$ for all $x \in [0, L]$.*

In this paper we will prove that a completely additive function is linear.

2. Let $1 < q < 2$ and $L := \sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{q-1}$. For every $k > 1$, $k \in \mathbf{N}$ denote by $q^*(k)$ the unique number in the interval $(1, 2)$ for which

$$L = \sum_{n=1}^{k} \frac{1}{q^n} \tag{2.1}$$

holds. The following theorem has been proved in [1].

Theorem 2.1. *Let $1 < q \leqq q^*(2)$ and let $F\colon [0, L] \to \mathbf{R}$ be a completely additive function. Then there exists an $\alpha \in \mathbf{R}$ such that $F(x) = \alpha \cdot x$ $(x \in [0, L])$.*

We will prove the following theorems.

Theorem 2.2. *Let $q = q^*(k)$ for some fixed number $k > 2$, $k \in \mathbf{N}$. Let $F\colon [0, L] \to \mathbf{R}$ be a completely additive function. Then there exists an $\alpha \in \mathbf{R}$ such that $F(x) = \alpha \cdot x$ for all $x \in [0, L]$.*

PROOF. Let

$$F^*(x) = F(x) - \frac{F(L)\cdot x}{L} \tag{2.2}$$

for every $x\in[0, L]$. Then F^* is a completely additive function and $F^*(0)=F^*(L)=0$. Let $a_n := F^*\left(\frac{1}{q^n}\right)$ $(n\in\mathbf{N})$. Then

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sum_{n=1}^{\infty} a_n = 0. \tag{2.3}$$

By $q=q^*(k)$

$$1 = \sum_{i=1}^{k} \frac{1}{q^i} \tag{2.4}$$

consequently for every $n\in\mathbf{N}$

$$\frac{1}{q^n} = \sum_{i=1}^{k} \frac{1}{q^{n+i}}.$$

Since F^* is completely additive we have for every $n\in\mathbf{N}$

$$a_n = \sum_{i=1}^{k} a_{n+i}$$

and

$$a_{n+1} = \sum_{i=1}^{k} a_{n+1+i}.$$

By these

$$a_{n+1} - a_n = a_{n+k+1} - a_{n+1}$$

namely

$$a_{n+k+1} - 2\cdot a_{n+1} + a_n = 0 \tag{2.5}$$

holds for every $n\in\mathbf{N}$.

Let

$$f(z) := \sum_{i=1}^{\infty} a_i \cdot z^i. \tag{2.6}$$

Then if $|z|\leqq 1$ the power series $f(z)$ is convergent, and $f(1)=f(0)=0$. Multiply the equality (2.5) by z^{n+k+1} $(n=1, 2, \ldots)$ and sum up for n. Then by (2.6) we have

$$f(z)\cdot(z^{k+1} - 2\cdot z^k + 1) = \sum_{i=1}^{k+1} a_i \cdot z^i - 2\cdot z^{k+1}\cdot a_1.$$

But $f(1)=f(0)=0$ so

$$f(z)\cdot(z^{k+1} - 2\cdot z^k + 1) = z\cdot(z-1)\cdot Q_{k-1}(z) \tag{2.7}$$

where $|z|\leqq 1$, and the degree of the polynomial $Q_{k-1}(z)$ is not greater than $k-1$. Let

$$p(z) := z^{k+1} + 1, \quad g(z) := -2\cdot z^k.$$

By $k\geqq 3$

$$\infty > |g(z)| > |p(z)| > 0$$

if $|z|=1,1$. Hence, by the theorem of Rouche the functions $p+g, g$ have the same number of zeros on the disk $|z|<1,1$. But if $1<|z|<1,1$ then

$$|p(z)+g(z)|>0.$$

As the function g has k roots on the closed disc $|z|\leqq 1$, the function $p+g$ has exactly k roots on it. Thus, by (2.6) the polynomial $Q_{k-1}(z)$ has k roots, but in this case $Q_{k-1}(z)\equiv 0$. That is $f(z)\equiv 0$, $(|z|\leqq 1)$ hence

$$a_n = 0 \tag{2.8}$$

for every $n\in\mathbf{N}$. So we have

$$F\left(\frac{1}{q^n}\right) = \frac{F(L)}{L}\cdot\frac{1}{q^n} \tag{2.9}$$

for every $n\in\mathbf{N}$. That is by $c:=\dfrac{F(L)}{L}$

$$F(x) = c\cdot x \tag{2.10}$$

for all $x\in[0, L]$. □

Theorem 2.3. *Let* $q^*(2)<q<2$ *be a fixed number for which*

$$1 \neq \frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^n} \tag{2.11}$$

holds for every $n>2$, $n\in\mathbf{N}$. *Moreover let* $F\colon [0, L]\to\mathbf{R}$ $\left(L:=\sum_{n=1}^{\infty}\frac{1}{q^n}=\frac{1}{q-1}\right)$ *be a completely additive function defined by the sequence* $a_n\in\mathbf{R}$, $\left(\sum_{n=1}^{\infty}|a_n|<\infty\right)$. *Then F does not change sign.*

Proof. Indirect. Suppose that F changes sign, and let

$$P:=\{n|n\in\mathbf{N}, a_n>0\}. \tag{2.12}$$

Then $P\neq\mathbf{N}$, $P\neq\emptyset$. Let

$$x:=\sum_{n\in P}\frac{1}{q^n} \tag{2.13}$$

$$y:=\sum_{m\notin P}\frac{1}{q^m}. \tag{2.14}$$

So $x, y\in(0, L)$ and these numbers are uniquely determined (see in [1]). Thus the regular expansions of x, y are just the right-hand sides of (2.13), (2.14). Moreover the sets $\mathbf{N}\setminus P$, P contain infinitely many numbers.

Denote by $n_0\in\mathbf{N}$ that unique number for which

$$\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n_0-1}}<1<\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^{n_0}} \tag{2.15}$$

holds, then by $q^*(2)<q$, and (2.11) this number exists and $n_0 \geqq 3$. Hence we have

$$(2.16) \qquad 1 = \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^{n_0-1}} + \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^i}$$

where l_i denotes the digits 0, 1 in the infinite expansion of the number 1.

Let

$$(2.17) \qquad t := \min \{n | n \in \mathrm{N},\ n > n_0,\ n \notin P,\ n+1 \in P\}.$$

Then by (1.3), (2.13), (2.14), we have

$$(2.18) \qquad \sum_{n>t, n \in P} \frac{1}{q^n} < \frac{1}{q^t}$$

$$(2.19) \qquad \sum_{m>t+1, m \notin P} \frac{1}{q^m} < \frac{1}{q^{t+1}}.$$

We distinguish the following cases:

$$(2.20) \qquad \text{I.} \quad \sum_{n>t, n \in P} \frac{1}{q^n} = \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}}$$

$$(2.21) \qquad \text{II.} \quad \sum_{n>t, n \in P} \frac{1}{q^n} < \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}}$$

$$(2.22) \qquad \text{III.} \quad \sum_{n>t, n \in P} \frac{1}{q^n} > \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}}.$$

It is easy to see that there are no other cases.

In the case I. by (2.16), (2.20) we have

$$(2.23) \qquad \frac{1}{q^{t-n_0}} = \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \sum_{n>t, n \in P} \frac{1}{q^n}.$$

Hence, since F is completely additive

$$F\left(\frac{1}{q^{t-n_0}}\right) = a_{t-n_0}$$

$$F\left(\frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \sum_{n>t, n \in P} \frac{1}{q^n}\right) = a_{t-n_0+1} + \dots + a_{t-1} + \sum_{n>t, n \in P} a_n$$

that is

$$a_{t-n_0} = a_{t-n_0+1} + \dots + a_{t-1} + \sum_{n>t, n \in P} a_n$$

and by the definition of the set P we have

$$(2.24) \qquad a_{t-n_0} > a_{t-n_0+1} + \dots + a_{t-1}.$$

Now we consider the following number

$$(2.25) \qquad x := \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{m>t+1, m \notin P} \frac{1}{q^m}.$$

Then $x \in [0, L]$ and by (2.15)

$$\frac{1}{q^{t-n_0}} < x. \tag{2.26}$$

That is, by (1.3) the regular expansion of x, $(\varepsilon_n(x))$ is different from the right-hand side of (2.25). Moreover, by $t+1 \in P$, (2.19), (2.20), (2.16) we have

$$x < \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{n>t, n\in P} \frac{1}{q^n} = \frac{1}{q^{t-n_0}} + \frac{1}{q^t},$$

whence

$$x = \frac{1}{q^{t-n_0}} + \sum_{i=t+1}^{\infty} \frac{\varepsilon_i(x)}{q^i}. \tag{2.27}$$

By the definition of the set P, and since F is completely additive we have

$$F(x) = a_{t-n_0} + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i = a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{m>t+1, m\notin P} a_m \leqq$$

$$\leqq a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i.$$

That is

$$a_{t-n_0} \leqq a_{t-n_0+1} + \dots + a_{t-1} + a_t$$

and, by $t \notin P$ $a_t \leqq 0$, so we have

$$a_{t-n_0} \leqq a_{t-n_0+1} + \dots + a_{t-1} \tag{2.28}$$

which contradicts (2.24).

In the case II. we first consider the following number

$$x := \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{n>t, n\in P} \frac{1}{q^n}. \tag{2.29}$$

By (2.15)

$$\frac{1}{q^{t-n_0}} < x, \tag{2.30}$$

that is, by (1.3) the regular expansion of x, $(\varepsilon_n(x))$ is different from the right-hand side of (2.29). Moreover, by (2.21), (2.16) we have

$$x < \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}} = \frac{1}{q^{t-n_0}} + \frac{1}{q^t}.$$

Consequently

$$x = \frac{1}{q^{t-n_0}} + \sum_{i=t+1}^{\infty} \frac{\varepsilon_i(x)}{q^i}. \tag{2.31}$$

Hence we have

$$a_{t-n_0}+\sum_{i=t+1}^{\infty}\varepsilon_i(x)\cdot a_i = a_{t-n_0+1}+\ldots+a_{t-1}+a_t+\sum_{n>t,\,n\in P}a_n \geqq$$

$$\geqq a_{t-n_0+1}+\ldots+a_{t-1}+a_t+\sum_{i=t+1}^{\infty}\varepsilon_i(x)\cdot a_i,$$

(2.32) $$a_{t-n_0}\geqq a_{t-n_0+1}+\ldots+a_{t-1}+a_t.$$

Now we consider the following number

(2.33) $$y:=\frac{1}{q^{t-n_0+1}}+\ldots+\frac{1}{q^{t-1}}+\frac{1}{q^t}+\sum_{m>t+1,\,m\notin P}\frac{1}{q^m}.$$

By (2.15)

(2.34) $$\frac{1}{q^{t-n_0}}<y,$$

that is, by (1.3) the regular expansion of y, $(\varepsilon_n(y))$ is different from the right-hand side of (2.33). Moreover, by $t+1\in P$, (2.19), (2.21), (2.16) we have

$$y<\frac{1}{q^{t-n_0+1}}+\ldots+\frac{1}{q^{t-1}}+\frac{1}{q^t}+\sum_{n>t,\,n\in P}\frac{1}{q^n}<$$

$$<\frac{1}{q^{t-n_0+1}}+\ldots+\frac{1}{q^{t-1}}+\frac{1}{q^t}+\sum_{i=n_0+1}^{\infty}\frac{l_i}{q^{t-n_0+i}}=\frac{1}{q^{t-n_0}}+\frac{1}{q^t},$$

and by (2.15) we have

(2.35) $$y=\frac{1}{q^{t-n_0}}+\sum_{i=t+1}^{\infty}\frac{\varepsilon_i(y)}{q^i}>\frac{1}{q^{t-n_0}}+\sum_{m>t+1,\,m\notin P}\frac{1}{q^m}.$$

Hence by the definition (1.1) there exist $k\in\mathbf{N}$ such that $\varepsilon_{t+k}(y)=1$ and $t+k\in P$. So we have

$$a_{t-n_0}+\sum_{i=t+1}^{\infty}\varepsilon_i(y)\cdot a_i = a_{t-n_0+1}+\ldots+a_{t-1}+a_t+\sum_{m>t+1,\,m\notin P}a_m<$$

$$<a_{t-n_0+1}+\ldots+a_{t-1}+a_t+\sum_{i=t+1}^{\infty}\varepsilon_i(y)\cdot a_i,$$

(2.36) $$a_{t-n_0}<a_{t-n_0+1}+\ldots+a_{t-1}+a_t$$

which contradicts (2.32).

In the case III. we first consider the following number

(2.37) $$x:=\frac{1}{q^{t-n_0+1}}+\ldots+\frac{1}{q^{t-1}}+\sum_{n>t,\,n\in P}\frac{1}{q^n}.$$

By (2.16), (2.22)

(2.38) $$\frac{1}{q^{t-n_0}}<x,$$

that is, by (1.3) the regular expansion of x, $(\varepsilon_n(x))$ is different from the right-hand side of (2.37). Furthermore, by (2.15), (2.18)

$$x < \frac{1}{q^{t-n_0}} + \sum_{n>t, n\in P} \frac{1}{q^n} < \frac{1}{q^{t-n_0}} + \frac{1}{q^t}.$$

Consequently

$$x = \frac{1}{q^{t-n_0}} + \sum_{i=t+1}^{\infty} \frac{\varepsilon_i(x)}{q^i} \tag{2.39}$$

and there exists $k\in\mathbf{N}$ such that $\varepsilon_{t+k}(x)=0$ and $t+k\in P$, whence

$$a_{t-n_0} + \sum_{i=t+1}^{\infty} \varepsilon_i(x)\cdot a_i = a_{t-n_0+1} + \ldots + a_{t-1} + \sum_{n>t, n\in P} a_n >$$

$$> a_{t-n_0+1} + \ldots + a_{t-1} + \sum_{i=t+1}^{\infty} \varepsilon_i(x)\cdot a_i.$$

$$a_{t-n_0} > a_{t-n_0+1} + \ldots + a_{t-1}. \tag{2.40}$$

Now consider the following number

$$y := \frac{1}{q^{t-n_0+1}} + \ldots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{m>t+1, m\notin P} \frac{1}{q^m}. \tag{2.41}$$

By (2.15)

$$\frac{1}{q^{t-n_0}} < y, \tag{2.42}$$

that is, by (1.3) the regular expansion of y, $(\varepsilon_n(y))$ is different from the right-hand side of (2.41). Moreover, by (2.15), (2.19) and $n_0 \geqq 3$ we have

$$y < \frac{1}{q^{t-n_0}} + \frac{1}{q^t} + \sum_{m>t+1, m\notin P} \frac{1}{q^m} < \frac{1}{q^{t-n_0}} + \frac{1}{q^t} + \frac{1}{q^{t+1}} < \frac{1}{q^{t-n_0}} + \frac{1}{q^{t-1}}.$$

Thus

$$y = \frac{1}{q^{t-n_0}} + \sum_{i=t}^{\infty} \frac{\varepsilon_i(y)}{q^i} \tag{2.43}$$

and so we have

$$a_{t-n_0} + \sum_{i=t}^{\infty} \varepsilon_i(y)\cdot a_i = a_{t-n_0+1} + \ldots + a_{t-1} + a_t + \sum_{m>t+1, m\notin P} a_m \leqq$$

$$\leqq a_{t-n_0+1} + \ldots + a_{t-1} + a_t + \sum_{i=t+1}^{\infty} \varepsilon_i(y)\cdot a_i,$$

$$a_{t-n_0} < a_{t-n_0+1} + \ldots + a_{t-1} + a_t\cdot(1-\varepsilon_t(y))$$

but $t\notin P$ because $a_t \leqq 0$ hence we have

$$a_{t-n_0} \leqq a_{t-n_0+1} + \ldots + a_{t-1} \tag{2.44}$$

which contradicts (2.40).

So we have a contradiction in each case. Our proof is complete. □

Corollary. *By Theorem* 1.1 *in this case the completely additive function is linear.*

By Theorem 2.1, Theorem 2.2 and the Corollary we have

Theorem 2.3. *Let* $1<q<2$, $L:=\sum_{n=1}^{\infty}\frac{1}{q^n}=\frac{1}{q-1}$ *and let* $F\colon [0, L]\to\mathbf{R}$ *be a completely additive function. Then there exists an* $\alpha\in\mathbf{R}$ *such that* $F(x)=\alpha\cdot x$ *for all* $x\in[0, L]$.

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