Note on polynomial mappings

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Abstract. In this note the following theorem is proved: if a polynomial mapping between locally convex linear topological spaces is continuous in one point, then it is continuous everywhere.

It is a wellknown and trivial property of linear mappings between linear topological spaces, that continuity in one point implies continuity everywhere (see e.g. KELLEY—NAMIOKA [3]). The aim of this work is to prove the same property for polynomial mappings between locally convex linear topological spaces.

Let E, F be linear topological spaces. Following the terminology of NACHBIN [4], we say that the mapping $P\colon E\to F$ is a polynomial, if it has a representation of the form

$$P(x) = P_n(x)+\ldots+P_1(x)+P_0$$

where $P_k\colon E\to F$ is a k-homogeneous polynomial, that is

$$P_k(x) = A_k(x, \ldots, x)$$

holds for all x in E with some k-linear and symmetric operator $A_k\colon E^k\to F$ $(k=1, \ldots, n)$, and P_0 is an element of F (which can be called a 0-homogeneous polynomial). Polynomial mappings play an important role in the theory of functional equations, as all solutions of a wide class of functional equations can be expressed by polynomial mappings (see ACZÉL [1], SZÉKELYHIDI [5]).

We note, that similar results concerning regularity properties of polynomial mappings on topological groups have appeared recently in SZÉKELYHIDI [6], but here we apply completely different technique using the observation that a multilinear mapping can always be identified with a linear mapping on some tensor product space. It would be interesting to know, whether a similar method can be applied to polynomial mappings on topological groups using some tensor product of topological groups. If so, then some results of SZÉKELYHIDI [6] could be generalized.

Theorem 1. *Let E, F be locally convex linear topological spaces, n a positive integer, and $A\colon E^n\to F$ an n-linear mapping, which is continuous at the origin. Then A is continuous everywhere.*

PROOF. We denote by T the tensor product of n copies of E, equipped with the locally convex topology, which has the local base formed by the convex circled texensions of the sets $U_1 \otimes \ldots \otimes U_n$, where U_i runs through a given local base of E $(i=1, \ldots, n)$ (see KELLEY—NAMIOKA [3], p. 152). Further, we denote by Φ the canonical homomorphism $(x_1, \ldots, x_n) \to x_1 \otimes \ldots \otimes x_n$ of E^n into T, which is obviously continuous. Let $a: T \to F$ be the linear mapping defined by the condition

$$A = a \circ \Phi.$$

Obviously, a is uniquely determined by A (see KELLEY—NAMIOKA [3], p. 152). We show, that a is continuous at the origin. Let $W \subset F$ be a convex circled neighborhood of the origin, then there exists a convex circled neighborhood $U \subset E$ of the origin such that $A(U \times \ldots \times U) \subset W$. Obviously $U \times \ldots \times U$ is convex and circled. By the definition of the topology of T, the convex circled extension of $U \otimes \ldots \otimes U$ is a neighborhood of the origin in T. On the other hand, the convex circled extension of $U \otimes \ldots \otimes U$ is the set of all elements of the form $\sum_{i=1}^{m} \lambda_i z_i$, where z_i belongs to $U \otimes \ldots \otimes U$, and $\sum_{i=1}^{m} |\lambda_i| \leqq 1$ (see KELLEY—NAMIOKA [3], p. 14). It means, that $z_i = x_{i_1} \otimes \ldots \otimes x_{i_n}$, where x_{i_j} belongs to U $(i=1, \ldots, m,\ j=1, \ldots, n)$, and hence

$$a\Big(\sum_{i=1}^{m} \lambda_i z_i\Big) = \sum_{i=1}^{m} \lambda_i a(z_i) = \sum_{i=1}^{m} \lambda_i a(x_{i_1} \otimes \ldots \otimes x_{i_n}) = \sum_{i=1}^{m} \lambda_i A(x_{i_1}, \ldots, x_{i_n})$$

and this element belongs to W, as W is convex and circled. Hence, a is continuous at the origin. Then by linearity it follows that a is continuous everywhere (see KELLEY—NAMIOKA [3], p. 37), and then the continuity of a and Φ implies that A is continuous everywhere.

Theorem 2. *Let E, F be locally convex linear topological spaces and $P: E \to F$ be a polynomial. If P is continuous at one point, then it is continuous everywhere.*

PROOF. As any translate of a polynomial is a polynomial again, we may sup pose that P is continuous at the origin. Let

$$P(x) = A_n(x, \ldots, x) + \ldots + A_1(x) + A_0$$

where A_k is k-additive and symmetric $(k=0, 1, \ldots, n)$. By the obvious identity (see e.g. HOSSZÚ [2], SZÉKELYHIDI [5])

$$A_n(x, \ldots, x_n) = \frac{1}{n!} \Delta^n_{x_1, \ldots, x_n} P(0)$$

where $\Delta^n_{x_1, \ldots, x_n}$ is the usual difference operator (see e.g. SZÉKELYHIDI [5]), we have that A_n is continuous at the origin of E^n. Using the previous theorem, and then applying the same argument for the function $x \to P(x) - A_n(x, \ldots, x)$ we get the statement by induction.

By the same method as in SZÉKELYHIDI [6] we infer from Theorem 2 the following

Corollary 3. *Let E, F be locally convex linear topological spaces and $P\colon E\to F$ be a polynomial. If P is bounded on some non-void open set, then it is continuous everywhere.*

References

[1] J. ACZÉL, Lectures on Functional Equations and Their Applications, *Academic Press, New York, London,* 1966.

[2] M. HOSSZÚ, On the Fréchet's functional equation, *Bul. Inst. Polit. Iasi* **10** (1964), 15—16.

[3] J. L. KELLEY and J. NAMIOKA, Linear Topological Spaces, Van Nostrand, Princeton, *New Jersey, Toronto, New York, London,* 1963.

[4] L. NACHBIN, Topology on Spaces of Holomorphic Mappings, *Springer-Verlag, Berlin, Heidelberg, New York,* 1969.

[5] L. SZÉKELYHIDI, On a class of linear functional equations, *Publ. Math. Debrecen* **29** (1982), 19—28.

[6] L. SZÉKELYHIDI, Regularity properties of polynomials on groups, *Acta Math. Hung.* **45** (1—2), (1985), 15—19.

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