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Associativity equation revisited

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Abstract. We present an alternative way to find the general solution of the associativity equation on connected and unlimited totally ordered sets that are representable by utility functions.

1. Introduction

Let X be a nonempty set. A map $F : X \times X \to X$ is a solution of the *associativity equation* if it verifies $F(F(x,y),z) = F(x, F(y,z))$ for every $x, y, z \in X$, so defining on X an algebraic structure of *semigroup* if we interpret F as a *binary operation* "+", just writing $x + y = F(x,y)$, $(x, y \in X)$.

A crucial question is to *compare* the semigroups so obtained to well-known semigroups, as for instance the additive real line $(\mathbb{R}, +)$, looking for homomorphisms $u : X \to \mathbb{R}$ such that $u(F(x,y)) = u(x)+u(y)$, $(x, y \in X)$. In the particular case in which the set X is endowed with a total ordering "$\precsim$", it is natural to look for *utility homomorphisms* into the additive real line endowed with its natural ordering "$\leq$", that is, $u : (X, \precsim) \to (\mathbb{R}, \leq)$ such that $u(F(x,y)) = u(x) + u(y)$ and also $x \precsim y \iff u(x) \leq u(y)$, $(x, y \in X)$.

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In classical works the general solution of the associative equation was studied for maps $F : I \times I \to I$, where I is *a proper interval of the real line*. (The original ideas appeared in Aczél [1] and Tamari [2], and were extended in Clifford [3] and Ling [4]). The key result in this approach is the following, and appears, e.g., in Aczél [5], p. 107, or else in Craigen and Páles [6].

Theorem 1. *Let I be a proper interval of the real line and $F{:}I\times I \to I$ a continuous and associative function that is cancellative (i.e.: $F(x,z) = F(y,z) \iff x = y \iff F(z,x) = F(z,y)$, $(x,y,z \in I)$). Then there exists a continuous and strictly monotonic function $\phi : J \to I$ such that $F(x,y) = \phi(\phi^{-1}(x) + \phi^{-1}(y))$, $x,y \in I$ where J is an interval of the real line closed for usual addition.*

Remark 1. (a) Theorem 1 means, in particular, that I can be given through F a structure of *semigroup*, which, in addition, is representable on the additive real line through an *additive utility function* (namely: ϕ^{-1}). In this paper we shall introduce an alternative proof of this fact by extending Aczél's approach, since we do not need to start with intervals of the real line. We shall also prove that the possibility of defining a structure of additively representable semigroup is inherent to connected totally ordered sets. Actually: "*Every totally ordered set that is connected as regards the order topology and representable by a utility function, can be endowed with a structure of topological totally ordered semigroup that is cancellative*". Moreover, the converse is also true: "*Every topological totally ordered semigroup that is connected and cancellative can be represented through a continuous additive utility function*". This last result essentially appears, without proof, in Clifford [3] (comment just before Theorem 7 on p. 313).

(b) Roughly speaking, if we consider the operation "F" in the statement of Theorem 1 as being analogous to the multiplication of positive reals, then in his original solution in 1948, Aczél constructed the *exponential* function. The shorter proof provided by Craigen and Páles [6], worked with the *logarithm* function instead. However, in both works the authors start from intervals of the real line.

2. Preliminaries

Let X be a set. Let $\precsim$ be a total order defined on X (i.e.: $\precsim$ is an antisymmetric transitive and complete binary relation). Associated to $\precsim$ we define the strict preference relation $\prec$ as: $x \prec y \iff \neg\{y \precsim x\}$ $(x,y \in X)$, and the indifference relation $\sim$ is given by: $x \sim y \iff \{x \precsim y\} \wedge \{ y \precsim x\}$. Given $x,y \in X$ we shall denote: $(x,y) = \{z \in X : x \prec z \prec y\}$;

$[x, y] = \{z \in X : x \precsim z \precsim y\}$; $(\leftarrow, x) = \{z \in X : z \prec x\}$ and $(x, \rightarrow) = \{z \in X : x \prec z\}$.

On a totally ordered set $(X, \precsim)$ we shall consider the *order topology* θ of X, which has the following subbasis: $\mathcal{S} = \{(\leftarrow, x), (y, \rightarrow) : x, y \in X\}$. This topology θ is Hausdorff.

The total order $\precsim$ is said to be:

(i) *unlimited* if either there is no $a \in X$ such that $a \precsim x$, $(x \in X)$, or else there is no $b \in X$ such that $x \precsim b$, $(x \in X)$,

(ii) *perfectly separable* if there exists a countable subset $D \subseteq X$ such that, for every $x, y \in X$ with $x \prec y$, there exists $d \in [x, y] \cap D$. The subset D is said to be *order-dense* in X;

(iii) *representable by a utility function* if there exists a real function $u : X \longrightarrow \mathbb{R}$ such that $x \precsim y \iff u(x) \leq u(y)$ $(x, y \in X)$. If, in addition, there exists a utility function u that is continuous as regards the order topology θ on X and the usual Euclidean topology τ on $\mathbb{R}$, $(X, \precsim)$ is said to be *continuously representable*.

It is well-known that: *"A totally ordered set $(X, \precsim)$ is continuously representable if and only if it is perfectly separable."* (See, e.g., Ch. 4 in Bridges and Mehta [7].)

A *semigroup* $(S, +)$ is a nonempty set S endowed with an associative binary operation which we shall denote by "+".

A semigroup S with a null element e such that $x + e = x = e + x$ for every $x \in S$ is said to be a *monoid*. If each element x in a monoid S has a converse $-x$ such that $x + (-x) = (-x) + x = e$ then S is said to be a *group*.

A semigroup $(S, +)$ endowed with a total ordering $\precsim$ is said to be a *totally ordered semigroup*. If $+ : S \times S \rightarrow S$ is continuous as regards the topology θ, S is said to be a *topological totally ordered semigroup*.

A totally ordered semigroup $(S, +, \precsim)$ is called *translation-invariant* if $x \precsim y \iff x + z \precsim y + z \iff z + x \precsim z + y$ for every $x, y, z \in S$. (In particular, a translation-invariant semigroup is always *cancellative*, i.e.: $x = y \iff x + z = y + z \iff z + x = z + y$ for every $x, y, z \in S$).

Given a totally ordered semigroup $(S, +, \precsim)$, an element $x \in S$ is said to be *positive*, (respectively: *negative*) if $x \prec x + x$, (respectively: if $x + x \prec x$). (Notice that when $(S, +, \precsim)$ is translation-invariant, an element $x \in S$ is positive, (respectively: negative) if and only if $y \prec x + y$, (respectively: $x + y \prec y$) for every $y \in S$.)

The subset of all the positive elements, (respectively: negative elements) in S constitutes the *positive cone*, (respectively: the *negative cone*) of S, which we shall denote by S^+, (respectively: by S^-). When $(S, +, \precsim)$ is translation-invariant, the positive and negative cones are indeed semigroups.

A totally ordered semigroup $(S, +, \precsim)$ is said to be:

(i) *positive*, (respectively: *negative*) if it only contains positive elements, (respectively: negative elements),

(ii) *additively representable* if there exists a utility function $u : S \to \mathbb{R}$ verifying $u(x + y) = u(x) + u(y)$, for every $x, y \in S$.

A positive semigroup $(S, +, \precsim)$ is said to be:

(i) *Archimedean* if for every $x, y \in S$ with $x \prec y$, there exists $n \in \mathbb{N}$ such that $y \prec n.x$, $(n.x = x + \overset{n \text{ times}}{\cdots} + x)$,

(ii) *super-Archimedean* if for every $x, y \in S$ with $x \prec y$ there exists $n \in \mathbb{N}$ such that $(n + 1).x \prec n.y$.

A totally ordered semigroup $(S, +, \precsim)$ is said to be *Archimedean*, (respectively: *super-Archimedean*) if its positive cone S^+ is Archimedean, (respectively: super-Archimedean) and also its negative cone S^- is Archimedean, (respectively: super-Archimedean) as regards the *dual order* $\precsim_d$ defined by $x \precsim_d y \iff y \precsim x$ $(x, y \in S)$.

For the case of translation-invariant totally ordered semigroups, the existence of an additive utility function was characterized in Alimov [8]. (See also Fuchs [9], pp. 230 and ff., Holman [10] or else De Miguel et al. [11].)

A key result in this context is stated in the following

Theorem 2.

(a) The following assertions are equivalent for a positive translation-invariant semigroup $(S, +, \precsim)$:

(i) $(S, +, \precsim)$ is additively representable,

(ii) $(S, +, \precsim)$ is super-Archimedean.

(b) In the case of translation-invariant totally ordered groups we can add the validity of the Archimedean property as being equivalent to the conditions in (a).

(c) A translation-invariant totally ordered semigroup $(S, +, \precsim)$ is additively representable if and only if its positive and negative cones are both additively representable.

Proof. See Lemma 5, Theorem 2 and Theorem 4 in De Miguel et al. [11]. □

Remark 2. With respect to the *continuity* of the *additive* utility representations of totally ordered semigroups, it holds that in the case of a totally ordered *topological* semigroup $(S, +, \precsim)$, every *additive* utility function is *continuous* with respect to the order topology θ on S and the usual Euclidean topology τ on $\mathbb{R}$. Moreover, a totally ordered semigroup representable by a continous additive utility function must be *topological*. (See Candeal et al. [12].)

3. Solving the associativity equation

In order to introduce the main result, we need some preparatory lemmas.

Lemma 1. *Let* $(S,+,\precsim)$ *be a translation-invariant totally ordered semigroup and let* $a,b\in S$ *be such that* $a+b\precsim b+a$. *Then, for every* $n\in\mathbb{N}$ *it holds that* $n.a+n.b\precsim n.(a+b)\precsim n.(b+a)\precsim n.b+n.a$.

Proof. It is a straigthforward exercise to prove this by induction. This lemma has already been used in the literature. See for instance Conrad [13], or else Fuchs [9], p. 162. □

Lemma 2. *Let* $(S,+,\precsim)$ *be a connected and translation-invariant topological totally ordered semigroup. Then, for every nonempty subset* $A\subset S$ *bounded from above, (respectively: from below) and for every* $a\in S$, *there exists* $\sup(a+A)$ *and* $\sup(a+A)=a+\sup A$, (*respectively: there exists* $\inf(a+A)$ *and* $a+\inf A=\inf(a+A)$).

Proof. See Clifford [14], or Fuchs [9], pp. 176–177. □

Lemma 3. *Let* $(S,+,\precsim)$ *be a positive, connected and translation-invariant totally ordered topological semigroup. Let* $a,b,x\in S$ *be such that* $a\prec b$ *and* $a+x\precsim b$. *Then there exists an element* $y\in S$ *such that* $a+y=b$. *Similarly, if there exists* $t\in S$ *such that* $t+a\precsim b$ *then we can find an element* $z\in S$ *such that* $z+a=b$.

Proof. (We only prove the existence of the element y, the proof of the existence of z being entirely analogous.) Suppose that the assertion is not true for the elements $a,b,x\in S$ with $a\prec b$ and $a+x\precsim b$. Consider the following subsets of S:

$$X=\{x\in S;\ a+x\prec b\}$$

and

$$Y=\{y\in S;\ b\prec a+y\}.$$

The subset Y is nonempty because, S being positive, we have that $b\prec a+b$. The subset X is nonempty by hypothesis. Therefore $\{X,Y\}$ is a partition of S. By translation-invariance, it follows that $x\prec y$ for every $x\in X$ and $y\in Y$. Thus X is bounded from above, and Y is bounded from below, so by connectedness, there exist $\sup X$, $\inf Y$, and $\sup X\precsim\inf Y$. Let us prove that $\sup X=\inf Y$: Indeed, if $\sup X\prec\inf Y$, the partition $S=\{X,Y\}$ would consist of nonempty open subsets, in contradiction with the hypothesis of the connectedness of S. Now by Lemma 2 we have that $a+\sup X=\sup(a+X)$ and also $a+\inf Y=\inf(a+Y)$. Therefore $\sup(a+X)=\inf(a+Y)=b\implies a+\sup X=a+\inf Y=b$. □

Lemma 4. *Let $(S, +, \precsim)$ be a totally ordered topological semigroup which is connected and cancellative. Then, it holds that:*

(i) *For every $x \in S$ the maps $R_x : S \to S$ and $L_x : S \to S$ defined by $R_x(t) = t + x$, $L_x(t) = x + t$ are both monotonic.*

(ii) *Given $a, b \in S$ with $a \prec b$, either $a + x \prec b + x$ for every $x \in S$ or $b + x \prec a + x$ for every $x \in S$. Similarly, either $t + a \prec t + b$ for every $t \in S$ or $t + b \prec t + a$ for every $t \in S$.*

(iii) *$(S, +, \precsim)$ is translation-invariant.*

PROOF. (i) This follows from the fact that S is connected and R_x, L_x are continuous and injective. Actually, *every continuous and injective map $F : S \to S$ must be monotonic*. To see this, let us prove that given $a \in S$ either $F((a, \to)) \subseteq (F(a), \to)$, or else $F(a, \to) \subseteq (\leftarrow, F(a))$: Since $(a, \to)$ is connected and F continuous, $F(a, \to)$ must be connected. In addition, $F(a) \notin F(a, \to)$ because F is injective. Therefore $F(a, \to) \subseteq (\leftarrow, F(a)) \cup (F(a), \to)$. We conclude using a standard argument of connectedness.

(ii) First notice that the sets $A = \{x \in S : a + x \prec b + x\}$ and $B = \{x \in S : b + x \prec a + x\}$ are *open*. Let us prove this for the set A: If A is empty, it is trivially open, so let us assume that there exists an element $x_0 \in A \implies a + x_0 \prec b + x_0$. Because θ is Hausdorff, we can find neigbourhoods U, V of $a + x_0$ and $b + x_0$, respectively, such that $u \prec v$ for every $u \in U, v \in V$. By continuity of "+" we can find neighbourhoods U_1, V_1 of x_0 such that $a + U_1 \subseteq U$ and $b + V_1 \subseteq V$. Let $W = U_1 \cap V_1$. W is a neigbourhood of x_0 such that $a + W \subseteq U$ and also $b + W \subseteq V$. Thus $z \prec t$ for every $z \in a + W, t \in b + W$. Thus, A must be open.

Now observe that A is disjoint from B, and $A \cup B = S$. Therefore, by connectedness of S, either A or else B is empty. Similarly, one of the sets $C = \{x \in S : x + a \prec x + b\}$ and $D = \{x \in S : x + b \prec x + a\}$ must be *empty*.

(iii) (This generalizes a result in TAMARI [2].) Consider $x, y, z \in X$ with $x \prec y$ and let us show that $x + z \prec y + z$. If $y + z \prec x + z$, by part (ii) it would follow that $y + t \prec x + t$ for every $t \in S$. In particular, $y + (z + z) \prec x + (z + z)$. However, since R_z is monotonic by part (i), $y + z \prec x + z \implies (x + z) + z \prec (y + z) + z$, and we arrive to a contradiction. Similarly we could prove that $z + x \prec z + y$. □

Lemma 5. *Let $(S, +, \precsim)$ be a totally ordered topological semigroup that is connected and translation-invariant. Then it is super-Archimedean, hence additively representable.*

Proof. By Theorem 2 (c), we can assume without loss of generality that S is positive. Let $a, b \in S$, $a \prec b$ and suppose that $n.b \prec (n+1).a$ for every $n \in \mathbb{N}$. Let $A = \{x \in S; n.x \prec (n+1).a$ for every $n \in \mathbb{N}\}$. A is nonempty because $a, b \in A$, and $2.a$ is an upper bound for A, so by connectedness, there exists $c = \sup A$. Observe that $c \in A$: If not, there would exist $n \in \mathbb{N}$ such that $(n+1).a \prec n.c$, and since S is topological, there would exist a neighbourhood U of c such that $n.U = \{n.u;\ u \in U\} \subset ((n+1).a, \rightarrow)$. So we would find an element $m \in A \cap U$, $m \prec c$, such that $n.m \in ((n+1).a, \rightarrow)$. Therefore $m \notin A$, which contradicts $m \in A \cap U \implies m \in A$. Now since $a \prec a+c \prec c+c$ and also $a \prec c+a \prec c+c$, by Lemma 3 there exist $z, t \in S$ such that $a+t = z+a = 2.c$. In particular $c \prec z, c \prec t$. Consequently $z, t \notin A$. Two possibilities may occur: either $t \precsim z$ or else $z \precsim t$. Let us assume first that $t \precsim z$. Then $t + a \precsim z + a = a + t \precsim a + z$. Since $t \notin A$, there exists $k \in \mathbb{N}$ such that $(k+1).a \prec k.t$. Thus, by Lemma 1, $(2k+1).a \prec k.t + k.a \precsim k.(a+t) = (2k).c$. Putting $i = 2k$ we conclude that $(i+1).a \prec i.c \implies c \notin A$. Contradiction. Assume now that $z \precsim t \implies a + z \precsim a + t = z + a \precsim t + a$. Since $z \notin A$ there exists $q \in \mathbb{N}$ such that $(q+1).a \prec q.z$. Thus, again by Lemma 1, it follows that $(2q+1).a \prec q.a + q.z \precsim q.(z+a) = (2q).c$, and finally $c \notin A$, the same contradiction as above. □

We are ready to formulate the main theorem.

Theorem 3. *Let $(X, \precsim)$ be a totally ordered set that is θ-connected and unlimited. The following assertions are equivalent:*

(i) *$(X, \precsim)$ is perfectly separable,*

(ii) *$(X, \precsim)$ is representable through a utility function $u : (X, \theta) \rightarrow (\mathbb{R}, \tau)$,*

(iii) *$(X, \precsim)$ is representable through a continuous utility function $u : (X, \theta) \rightarrow (\mathbb{R}, \tau)$,*

(iv) *There exists a continuous map $F : X \times X \rightarrow X$ that is associative and cancellative (i.e.: $x = y \iff F(x,z) = F(y,z) \iff F(z,x) = F(z,y)$, $(x, y, z \in X)$),*

(v) *There exists a continuous map $G : X \times X \rightarrow X$ that is associative, cancellative, has a null element $e \in X$ such that $G(e,e) = e$, and is also commutative (i.e.: $G(x,y) = G(y,x)$, $(x, y \in X)$) and divisible (i.e.: for every $x \in X$ and every $n \in \mathbb{N}$, there exists a unique $y \in X$ such that $n.y = x$, being, by definition, $2.y = G(y,y)$, $3.y = G(2.y, y)$ and so on).*

Remark 3. Observe that the assertion (iv) states that $(X, \precsim)$ can be given a suitable structure of semigroup, and the assertion v) establishes that $(X, \precsim)$ can be given a suitable structure of monoid. Roughly speaking, such assertions say that the associativity equation has good solutions on any representable totally ordered set.

Proof of Theorem 3. We shall follow the scheme (i) $\iff$ (ii) $\iff$ (iii), (iii) $\implies$ (v) $\implies$ (iv) $\implies$ (iii).

The implications (i) $\iff$ (ii) $\iff$ (iii) are classical results in utility theory. See, e.g., Candeal and Induráin [15] or Ch. 4 in Bridges and Mehta [7].

(iii) $\implies$ (v): Let $u : (X, \precsim, \theta) \to (\mathbb{R}, \leq, \tau)$ be a continuous utility function. Since X is connected and unlimited, and u is continuous, the range $u(X)$ is also a connected and unlimited interval of the real line. Actually, by eventually passing through another continuous and strictly monotonic function from $\mathbb{R}$ to $\mathbb{R}$, there is no loss of generality in assumming that $u(X)$ is of one of the following three types: (i) $u(X) = (-\infty, 0]$, (ii) $u(X) = [0, \infty)$, (iii) $u(X) = \mathbb{R}$. On such intervals, the order topology θ and the induced Euclidean topology τ coincide, so $u^{-1} : (u(X), \tau) \to (X, \theta)$ is an homeomorphism. The map $G : X \times X \to X$ given by $G(x, y) = u^{-1}(u(x) + u(y))$ is by construction continuous, commutative and associative. In addition, being $e \in X$ such that $u(e) = 0$, it follows that $G(e, e) = e$. Now $G(x, z) = G(y, z) \implies u^{-1}(u(x) + u(z)) = u^{-1}(u(y) + u(z)) \implies u(x) + u(z) = u(y) + u(z) \implies u(x) = u(y) \implies x = y$ $(x, y, z \in X)$, so G is cancellative because it is commutative. Finally, given $x \in X$, $n \in \mathbb{N}$, call $y = u^{-1}(\frac{u(x)}{n})$ and notice that $n.y = x$.

The implication (v) $\implies$ (iv) is obvious.

(iv) $\implies$ (iii): By Theorem 2 and Remark 2, it is enough to see that the structure of totally ordered topological semigroup that F defines on $(X, \precsim)$ is translation-invariant and super-Archimedean, but this follows from Lemma 4 and Lemma 5. □

Remarks 4. (i) The equivalence (iii) $\iff$ (iv) for the particular case in which X is an unlimited interval of the real line corresponds to Aczél's classical solution of the associativity equation. (See Aczél [1], [5]). It is noticeable that the proof that appears on pp. 107 and ff. of Aczél [5] is much longer than ours. Another proof of Aczél's result, also given following a simpler approach may be seen in Craigen and Páles [6].

(ii) In the literature, there are also some proofs given for more restrictive situations than Aczél's approach. For instance, in Ch. 6 of Castillo-Ron and Ruiz-Cobo [16] a proof is given for the case of *twice differentiable* functions defined on intervals of the real line.

(iii) The implication (iv) $\Longrightarrow$ (iii) corresponds, essentially, to the comments previous to Theorem 7, p. 313, in Clifford [14].

(iv) By Lemma 4 and Lemma 5 it follows that a totally ordered topological semigroup that is connected and cancellative is additively representable, hence commutative. Moreover, it is straightforward to see that the range of a continuous additive utility function defined on a cancellative totally ordered topological monoid must be one of the following intervals of the real line: $[0, +\infty)$, $(-\infty, 0]$, $\mathbb{R}$, all of which are divisible. Therefore such a monoid must also be divisible. Hence we can replace condition (v) in the statement of Theorem 3 by the apparently weaker, but actually equivalent: "(v*) *There exists a continuous map $G : X \times X \to X$ that is associative, cancellative, and has a null element*."

(v) Searching for an *alternative proof* that would look different from the ones existing in the literature to demonstrate Aczél's Theorem 1 or its generalization (iii) $\iff$ (iv) we could think about a shorter version of Theorem 3, based on the equivalences (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (v), proving the "new" implication (v) $\Longrightarrow$ (i). We present here a sketch of the proof of this implication. Observe that the positive cone is given by $X^+ = \{a \in X \text{ such that } a \prec G(a, a)\}$. X^+ is stable because by Lemma 4, X is translation-invariant. Let us show that X^+ is perfectly separable. (The proof for X^- is analogous): Assume that X^+ is nonempty, and fix an element $a \in X^+$. Since X is divisible, given $n \in \mathbb{N}$ there exists an element $c \in X^+$ such that $n.c = a$. By translation-invariance it follows that c is unique. Thus we can use in full sense the notation $c = \frac{a}{n}$. Now it is enough to prove that the countable set $D = \{(m.(\frac{a}{n}))\}_{n,m \in \mathbb{N}}$ is order-dense in X^+. This follows from the fact of X^+ being super-Archimedean, stated in Lemma 5: First notice that given two positive rational numbers $\frac{m}{n}$ and $\frac{m'}{n'}$ with $\frac{m}{n} < \frac{m'}{n'}$ it follows by translation-invariance that $(m.(\frac{a}{n})) < (m'.(\frac{a}{n'}))$. Now take two elements $x, y \in X^+$, with $x \prec y$. Since X^+ is super-Archimedean and translation-invariant, it is Archimedean, and, because it is divisible, we can find an element $m.(\frac{a}{n})$ such that $m.(\frac{a}{n}) \prec x \prec y$. By Archimedeanness again, the sets of real numbers $A = \{\frac{p}{q} \in \mathbb{Q} : p.(\frac{a}{q}) \prec x\}$ and $A' = \{\frac{p'}{q'} \in \mathbb{Q} : p'.(\frac{a}{q'}) \prec y\}$ are bounded. However, by super-Archimedeanness there exists $k \in \mathbb{N}$ such that $x \prec \frac{k}{k+1}.y$. Therefore $\sup A \leq \frac{k}{k+1}.\sup A'$. Thus we can find a rational number $\frac{g}{h}$ $(g, h \in \mathbb{N})$ such that $\sup A < \frac{g}{h} < \sup A'$. Hence $x \precsim g.(\frac{a}{h}) \precsim y$ and the proof is complete. (For more details, see the proof of Theorem 2 in De Miguel et al. [11]).

(vi) About the *uniqueness of the solutions of the associativity equation*, let us observe that if we start with a totally ordered set satisfying the condition (i) of Theorem 3, we can endow it with a structure of semigroup, so finding a solution of the associativity equation on X, for any utility function $u : X \to \mathbb{R}$. Notice also that every multiple $k.u$, ($k \in \mathbb{R}$, $k > 0$) of the utility function u defines the same structure of semigroup in X, or, in other words, leads to the same solution of the associativity equation. Actually, for another utility function w to define the same structure of semigroup as u it is necessary and sufficient that w be a positive multiple of u. This comes from a result that asserts that given two *additive* utility functions u, v defined on a totally ordered semigroup there exists a positive constant α such that $v = \alpha.u$. (See, e.g., Lemma 1 in Marley [17]). Let us give an alternative proof of this result, for the particular case of *positive* semigroups. (The extension to the general case is straightforward): Fix x_0, and consider an element x in the positive cone X^+ of X. Put $k = \frac{u(x)}{u(x_0)}$ and $\alpha = \frac{v(x_0)}{u(x_0)}$. Approximate k by a strictly increasing sequence $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$ of rational numbers ($p_n, q_n \in \mathbb{N}$). We obtain that $p_n.u(x_0) < q_n.u(x) \implies p_n.x_0 \prec q_n.x \implies p_n.v(x_0) < q_n.v(x)$ $(n \in \mathbb{N}) \implies v(x) \geq \lim_{n\to+\infty}\{(\frac{p_n}{q_n}).v(x_0)\} = \frac{u(x)}{u(x_0)}.v(x_0) = \alpha.u(x)$. Changing the roles of u and v it follows in the same way that $u(x) \geq (\frac{1}{\alpha}).v(x)$. Therefore $v(x) = \alpha.u(x)$.

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