Realizability and approximate realizability by parabolic differential equations

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

In this paper we shall consider systems described by parabolic partial differential equations. Firstly the transfer functions of these systems will be calculated and some of their fundamental properties will be established. Then we shall deal with the realizability of the analytical functions by control systems governed by prescribed parabolic partial differential equations. The realizing system will depend on the transfer function only in the observation term, while the dynamics of the system is universal for all transfer functions. Moreover, this dynamics is the mathematical model of the physically well realizable heat conduction.

1. Let $V \subset H$ be a pair of Hilbert spaces with continuous and dense inclusion. The Hilbert space H will be identified with its dual space H^*. Let $a\colon V \times V \to \mathbf{R}$ be a continuous bilinear form. The symmetry and the coercivity of a will also be supposed. Then a continuous linear operator $A\colon V \to V^*$ can be defined by

$$\langle Au, v\rangle = a(u, v) \quad (u, v \in V).$$

Consider the abstract ordinary differential equation

$$u' + Au = f, \quad u(0) = g, \tag{1}$$

where $f \in L_2((0, T), V^*)$, $g \in H$. Then it is well-known that problem (1) has a unique solution u belonging to the space

$$W^1([0, T], V, V^*) = \{u \in L_2((0, T), V),\ u' \in L_2((0, T), V^*)\},$$

and the application

$$(g, f) \mapsto u$$

is linear and continuous, that is, there exists a constant K such that

$$\|u\|_{W^1} \leqq K(\|g\|_H^2 + \|f\|_{L^2}^2)^{1/2}. \tag{2}$$

We notice that the constant K depends only on the constant α of the coercivity estimate

$$a(u, u) \geqq \alpha \|u\|_v^2. \tag{3}$$

We shall consider the following special cases.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with piecewise smooth regular boundary. Let $a_0, a_{ij} \in L_\infty(\Omega)$ be functions having the following properties:

a) $a_{ij} = a_{ji}$ $(i, j = 1, 2, \ldots, n)$,

b) there exists a constant $\alpha > 0$ such that

$$a_0(x) \geqq \alpha \quad \text{a.e. in } \Omega,$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geqq \alpha \|\xi\|^2$$

for each $\xi \in \mathbf{R}^n$ a.e. in Ω.

1. Consider the Hilbert spaces $V = H^1(\Omega)$, $H = L_2(\Omega)$, where $H^*(\Omega)$ is the following Sobolev space

$$H^1(\Omega) = \{u \in L_2(\Omega),\ \partial_{x_i} u \in L_2(\Omega),\ i = 1, 2, \ldots, n\}.$$

Define the bilinear form $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbf{R}$ by

$$a(u, v) = \int_{\Omega} \Big(\sum_{i,j=1}^{n} a_{ij} \partial_{x_i} u \partial_{x_j} v + a_0 uv \Big). \tag{4}$$

This bilinear form a corresponds to the differential operator $A\colon H^1(\Omega) \to H^1(\Omega)$ defined by

$$Au = -\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij} \partial_{x_j} u) + a_0 u$$

with the boundary condition $\left.\dfrac{du}{dv_A}\right|_{\partial\Omega} = 0.$

2. Now consider the Hilbert spaces $V = H_0^1(\Omega)$, $H = L_2(\Omega)$. The bilinear form $a\colon H_0^1(\Omega) \times H_0^1(\Omega)$ will be defined in the same way. This restricted bilinear form a has the symmetry and coercitivity properties. Finally, we mention that this corresponds to the same differential operator A with the boundary condition $u|_{\partial\Omega} = 0$.

In either of these cases, the coercitivity estimate (3) holds.

Hereafter suppose that the immersion $V \subset H$ is a compact operator. Then the operator $A\colon V^* \to V^*$ defined in the dense subspace $V \subset V^*$ has a selfadjoint compact inverse operator $G\colon V^* \to V^*$. In fact, the operator

$$\begin{array}{ccc} G: V^* & \longrightarrow & V^* \\ & {\scriptstyle A^{-1}} \searrow & \cup \\ & & V \subset H \end{array}$$

is compact. Thus there exists a nondecreasing sequence $(\lambda_k)\colon \mathbf{N} \to \mathbf{R}_+$ of eigenvalues of the operator A with the corresponding complete orthonormed sequence $(\varphi_k)\colon \mathbf{N} \to V$ of eigenvectors. Then the spaces V, H, V^* can be characterized by the Fourier coefficients $\hat{u}(k)$ $(k \in \mathbf{N})$ of their elements u with respect to the orthonormed sequence (φ_k). Thus

a) $u \in V$ if and only if $\displaystyle\sum_{k=1}^{\infty} \hat{u}(k)^2 \lambda_k < \infty$,

b) $u \in H$ if and only if $\sum_{k=1}^{\infty} \hat{u}(k)^2 < \infty$,

c) $u \in V^*$ if and only if $\sum_{k=1}^{\infty} \hat{u}(k)^2 \lambda_k^{-1} < \infty$.

2. Systems governed by differential equations (1).

Now we turn to defining control systems with observation equation. For this consider the following space of right-hand side functions

$$\left\{f \colon \mathbf{R}_+ \to V^* \colon \|f\|^2 = \int_0^{\infty} \|f(t)\|_{V^*}^2 \exp(-2Rt)\, dt < \infty\right\},$$

where $R \in \mathbf{R}_+$ is a given number. Then the estimate

$$\|f\|_{L_2((0,T),V^*)} \leqq \|f\| \exp RT$$

holds for each $T \in \mathbf{R}_+$. Therefore, from the inequality (2) and our remark we get the estimate

$$\|u\|_{W^1([0,T],V,V^*)} \leqq K(\|g\|_H^2 + \|f\|^2 \exp 2RT)^{1/2} \leqq$$
$$\leqq K(\|g\|_H^2 + \|f\|^2) \exp RT.$$

Define the following space of solutions

$$S_R = \left\{u \colon \mathbf{R}_+ \to V \colon u_{s_R} = \sup_{T>0} \|u\|_{W^1([\varrho,T],V,V^*)} \exp(-RT) < \infty\right\}.$$

Thus the previous estimate shows that

$$\|u\|_{s_R} \leqq K(\|g\|_H^2 + \|f\|^2)^{1/2}. \tag{5}$$

Now we turn to defining a single input single output system. Let the space C_R of controls be defined by the following way:

$$C_R = \left\{v \colon \mathbf{R}_+ \to \mathbf{R} \colon \|v\|_{C_R}^2 = \int_0^{\infty} v(t)^2 \exp(-2Rt)\, dt < \infty\right\}.$$

Then we shall define the systems with the elements $b \in V^*$, $c \in V^*$ by the equations

$$\begin{aligned} u' + Au &= bv, \quad u(0) = g, \\ y &= \langle c, u \rangle, \end{aligned} \tag{6}$$

where $\langle \cdot, \cdot \rangle$ is the dual pair map of V, V^*. From the inequality (5) we can prove the following estimate:

$$\int_0^T y(t)^2\, dt \exp(-2RT) \leqq \|c\|_{V^*}^2 \|u\|_{W^1([0,T],V,V^*)}^2 \exp(-2TR) \leqq$$
$$\leqq K^2 \|c\|_{V^*}^2 (\|g\|_H^2 + \|b\|_{V^*}^2 \|v\|_{C_R}^2).$$

Now define the space O_R of observation by

$$O_R = \Big\{y\colon \mathbf{R}_+\mathbf{R}\colon \|y\|^2_{O_R} = \sup_{T>0} \int_0^T y(t)^2\,dt \exp(-2RT) < \infty\Big\}.$$

Therefore the input-output map

$$(g, v) \to y,$$

as an application $H \times C_R \to O_R$, is a bounded linear operator.

For convenience system (6) will be considered in a diagonalized form in the basis (φ_k). Thus system (6) can be written by coordinates

$$\hat{u}(k)' + \lambda_k \hat{u}(k) = \hat{b}(k)v, \quad \hat{u}(k)(0) = \hat{g}(k) \quad (k \in \mathbf{N}),$$

$$y = \sum_{k=1}^{\infty} \hat{c}(k)\hat{u}(k).$$

Therefore

$$y(t) = \sum_{k=1}^{\infty} \hat{c}(k)\Big(\exp(-\lambda_k t)\hat{g}(k) + \int_0^t \exp \lambda_k(\tau - t)\hat{b}(k)v(\tau)\,d\tau\Big).$$

The existence in the half plane $\{\operatorname{Re} z > R\}$ of the Laplace transform of the functions $v \in C_R$, $y \in O_R$ follows from the definition of the norms in the spaces C_R, O_R. To simplify the calculation we shall suppose that $g = 0$. Then the Laplace transformed input-output map is the following

$$(Ly)(s) = \sum_{k=1}^{\infty} \hat{c}(k)\hat{b}(k)(s+\lambda_k)^{-1}(Lv)(s) =$$

$$= \sum_{i=0}^{\infty} (-1)^i \Big(\sum_{k=1}^{\infty} \lambda_k^{-i-1}\hat{c}(k)\hat{b}(k)\Big) s^i (Lv)(s).$$

Define the transfer function H by

$$s \mapsto H(s) = \sum_{i=0}^{\infty} (-1)^i \Big(\sum_{k=1}^{\infty} \lambda_k^{-i-1}\hat{c}(k)\hat{b}(k)\Big) s^i. \tag{7}$$

Thus the relation between the Laplace transformed control and observation can be expressed by the equation

$$(Ly)(s) = H(s)(Lv)(s).$$

The realization of an analytical function H consists in giving a system (6) such that its transfer function is H. A function H is said to be realizable, if it has a realization (6).

Now, by the following lemma, we can justify further generalizations of realizability.

Lemma 1. *If* $b, c \in V^*$ *then the function* H *defined in* (7) *is analytical in the open disk* D_α *of radius* α.

PROOF. By the Hadamard formula, the convergence radius r of H satisfies the inequalities

$$r^{-1} = \overline{\lim} \Big| \sum_{k=1}^{\infty} \lambda_k^{-i-1} \hat{c}(k) \hat{b}(k) \Big|^{1/i} \leqq$$

$$\leqq \overline{\lim}\, \alpha^{-1} \Big| \sum_{k=1}^{\infty} \lambda_k^{-1} \hat{c}(k) \hat{b}(k) \Big|^{1/i} \leqq \alpha^{-1} (\|b\|_{V^*} \|c\|_{V^*})^{1/i} = \alpha^{-1}.$$

Therefore, if H has a realization (6) and (3) holds then H is analytical in the open disk D_α.

Let X be a Banach space of certain analytical functions. We shall say that X is approximately realizable if the set of functions H realizable by systems of type (6) is dense in X. Now we turn to the proof of our main theorem.

Theorem 1. *Let $V \subset H$ be a pair of Hilbert spaces with compact and dense inclusion. Suppose that a bilinear form $a\colon V \times V \to \mathbf{R}$ is continuous, symmetrical and satisfies* (3). *Then, for each $0 < r < \alpha$, the Hardy space H_r of analytical functions over D_r with trace belonging to $L_2(\partial D_r)$ is approximately realizable, even if an element $b \in V^*$ is prescribed with the property that infinitely many Fourier coefficients $\hat{b}(k)$ differ from* 0.

PROOF. Let $b \in V^*$ be an element having the property mentioned in Theorem 1. Define the operator $A_b\colon V^* \to H_r$ by

$$A_b c = H$$

where H is defined in (7). It is clear that A_b is well-defined. In fact, by Lemma 1, the function $A_b c$ is analytical in the open disk D_α. Estimate the modul of $H(s) = (A_b c)(s)$.

$$|(A_b c)(s)| \leqq \sum_{i=0}^{\infty} \left(\frac{|s|}{\alpha}\right)^i \Big(\sum_{k=1}^{\infty} \lambda_k^{-1} \hat{c}(k) \hat{b}(k) \Big) \leqq \left(1 - \frac{|s|}{\alpha}\right)^{-1} \|b\|_{V^*} \|c\|_{V^*}.$$

Therefore

$$\|A_b c\|_{H_r} \leqq (2r\pi)^{1/2} \left(1 - \frac{r}{\alpha}\right)^{-1} \|b\|_{V^*} \|c\|_{V^*}.$$

The second, improved statement of Theorem 1 follows from the density of the range of A_b or, by using the orthogonality relation

$$H_r = \overline{R(A_b)} \oplus N(A_b^*),$$

from the nullity of the kernel $N(A_b^*)$. We shall prove this last statement. Let $H \in N(A_b^*)$. Identifying the dual space H_r^* with H_r, this means that

$$0 = \langle c, A_b^* H \rangle = \langle A_b c, H \rangle_{H_r} =$$

$$= \frac{1}{2r\pi} \int_0^1 \Big(\sum_{j=0}^{\infty} (-1)^j r^j \exp(-2\pi j i \varphi) H(r \exp 2\pi i \varphi) \cdot \Big(\sum_{k=1}^{\infty} (\lambda_k^{-j-1} \hat{c}(k) \hat{b}(k)) \Big) d\varphi =$$

$$= \sum_{k=1}^{\infty} \hat{c}(k) \lambda_k^{-1} \Big(\sum_{j=0}^{\infty} (-1)^j \hat{H}(j) \lambda_k^{-1} \Big) \hat{b}(k) \quad (c \in V^*),$$

where $\hat{H}(k)$ $(k=0, 1, \ldots)$ are the Fourier coefficients of the function $H \in H_r$. Therefore

$$\left(\sum_{j=0}^{\infty} (-1)^j \frac{\hat{H}(j)}{\lambda_k^j}\right) \hat{b}(k) = 0.$$

By our hypothesis on b, there exists a subsequence $(\lambda_{k_l}^{-1})$: $\mathbf{N} \to \mathbf{R}_+$ such that

$$\sum_{j=0}^{\infty} (-1)^j \hat{H}(j) (\lambda_{k_l}^{-1})^j = 0 \quad (l \in \mathbf{N}).$$

Therefore the analytical function $z \mapsto \sum_{j=0}^{\infty} (-z)^j \hat{H}(j)$ vanishes over a sequence tending to 0. Thus $\hat{H}(j)=0$ $(j=0, 1, \ldots)$, that is, $H=0$.

Remark. If $b, c \in V^*$ then $A_b c$ is analytical over the domain $D_A = \mathbf{C} \setminus \{\lambda_1, \lambda_2, \ldots\}$. Thus H_r has elements having no realization of type (6).

Now we shall deal with the exact realizability of Banach spaces of analytical functions. A Banach space X of certain analytical functions is exactly realizable if each $H \in X$ is realizable.

Let the Banach space X_λ be defined in the following manner

$$X_\lambda = \left\{H(s) = \sum_{k=1}^{\infty} x_k (s+\lambda_k)^{-1} \colon \|H\| = \sum_{k=1}^{\infty} |x_k| \lambda_k^{-1} < \infty\right\}.$$

It is easy to prove that every $H \in X_\lambda$ is analytical over the domain $\mathbf{C} \setminus \{\lambda_1, \lambda_2, \ldots\}$. Now we can prove an other statement.

Theorem 2. *Let H be an analytical function. Then H is realizable by a system (6) if and only if H belongs to X_λ.*

Proof. The necessity follows easily from (7). For the sufficiency, consider a vector x, such that $\sum_{k=1}^{\infty} |x_k| \lambda_k^{-1} < \infty$. Then a required realization will be defined with $b, c \in V^*$ by the following way:

$$\hat{b}(k) = \hat{c}(k) = |x_k|^{1/2}.$$

By modifying the definition of $b, c \in V^*$, we can also obtain realizations with non-vanishing Fourier coefficients $\hat{b}(k)$ $(\hat{c}(k)$ respectively$)$. In fact, let $b_+, c_+ \in V^*$ be defined by

$$\hat{b}_+(k) = \hat{c}_+(k) = \begin{cases} |x_k|^{1/2}, & \text{if } \ x_k \neq 0; \\ k^{-1}, & \text{if } \ x_k = 0. \end{cases}$$

Then the realizations (6) with b_+, c (b, c_+ respectively) satisfy this property. It is obvious that if x_k $(k=1, 2, \ldots)$ are non-vanishing, then the Fourier coefficients $\hat{b}(k)$, $\hat{c}(k)$ are simultaneously non-vanishing. Therefore, in this case, the realizations can be considered canonical. In fact, if all eigenvalues λ_k of A are simple then the approximate controllability and weak observability of the system (6) are equivalent to the non-vanishing of the Fourier coefficients $\hat{b}(k)$ and $\hat{c}(k)$ respectively. This will happen in the one-dimensional parabolic case when the differential equation of the system is

$$\partial_t u - \partial_x(p\partial_x u) + qu = bv$$

over a one-dimensional interval (a, b).

Now we shall apply our results to the special cases mentioned above.

Corollary 1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and $a_0, a_{ij} \in L_\infty(\Omega)$ $(i, j = 1, \dots, n)$ be functions having the properties* a) *and* b). *Then, for each $0 < r < \alpha$, the Hardy space H_r is approximately realizable by systems described by the following parabolic partial differential equation*

$$\partial_t u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j} u) + a_0 u = bv,$$

$$(8) \qquad u(0, \cdot) = 0, \quad u|_{\partial\Omega} = 0,$$

$$y = \langle c, u \rangle,$$

where $b, c \in H_0^1(\Omega)^*$, even if an element $b \in H_0^1(\Omega)^*$ is prescribed with the property that infinitely many Fourier coefficients $\hat{b}(k)$ differ from 0.

We can apply Theorem 1 with $V = H_0^1(\Omega)$, $H = L_2(\Omega)$ and the bilinear function (4). In fact, from properties a), b) it follows that the bilinear form (4) is symmetrical, continuous and satisfies the coercitivity estimate (3).

Corollary 2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and $a_0, a_{ij} \in L_\infty(\Omega)$ $(i, j = 1, \dots, n)$ functions having the properties* a) *and* b). *Then, for each $0 < r < \alpha$ the Hardy space H_r is approximately realizable by systems described by the following parabolic partial differential equation*

$$\partial_t u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j} u) + a_0 u = bv$$

$$(9) \qquad u(0; \cdot) = 0, \quad \left.\frac{du}{d\nu_A}\right|_{\partial\Omega} = 0,$$

$$y = \langle c, u \rangle$$

where $b, c \in H^1(\Omega)^*$, even if an element $b \in H^1(\Omega)^*$ is prescribed with the property that infinitely many Fourier coefficients $\hat{b}(k)$ differ from 0.

Now we can apply Theorem 1 with $V = H^1(\Omega)$, $H = L_2(\Omega)$ and the bilinear function (4).

Consider the eigenvalue problems

$$-\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j} u) + a_0 u = \lambda u, \quad u|_{\partial\Omega} = 0,$$

$$-\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j} u) + a_0 u = \lambda u, \quad \left.\frac{du}{d\nu_A}\right|_{\partial\Omega} = 0$$

and let all their eigenvalues be denoted by

$$\lambda_1 \leqq \lambda_2 \leqq \dots ,$$

$$\mu_1 \leqq \mu_2 \leqq \dots ,$$

respectively.

Corollary 3. *Let H be an analytical function. Then H is realizable by* (8) *(respectively* (9)*) if and only if H belongs to* X_λ *(respectively* X_μ*). We can also obtain realizations with non-vanishing Fourier coefficients* $\hat{b}(k)$ *(*$\hat{c}(k)$*, respectively). Even, if H can be represented by a vector x with non-vanishing coordinates* x_k $(k=1, 2, \dots)$, *then the Fourier coefficients* $\hat{b}(k)$, $\hat{c}(k)$ $(k=1, 2, \dots)$ *are simultaneously non-vanishing, that is, for this fuction H the obtained realization is canonical.*

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