On the non-existence of certain Riemannian connections with torsion and of constant curvature

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold, and ∇ an arbitrary linear connection on M for which $\nabla g=0$ (i.e. ∇ is supposed to be metrical).

$$T(X,Y):=\nabla_X Y-\nabla_Y X-[X,Y] \tag{1}$$

$$R(X,Y)Z:=\nabla_X\circ\nabla_Y Z-\nabla_Y\circ\nabla_X Z-\nabla_{[X,Y]}Z \quad X,Y,Z\in\mathfrak{X}(M) \tag{2}$$

are torsion and curvature tensors of ∇. Let $p\in M$, X and Y be independent vectors at p, and denote by γ the plane-position of X and Y. Then the sectional curvature of this connection is defined by

$$K^{\nabla}(p,\gamma):=\frac{\langle R(X,Y)X,Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2}, \tag{3}$$

where $\langle U,V\rangle\equiv g(U,V)$. If $K^{\nabla}(p,\gamma)\equiv k_0$(const.), then the connection is said to be of constant curvature. It is well known that in case of the torsion free Levi—Civita connection $\overset{*}{\nabla}$ and for $\dim M>2$ the independence of $K^{\overset{*}{\nabla}}$ from γ yields also the independence from p, and thus in this case $K^{\overset{*}{\nabla}}\equiv$const. (Schur's theorem).

In this paper we want to investigate the question: Do metrical connections exist with torsion and with constant curvature? We want to show that for $\dim M\geqq 4$, and with the property $\underset{(X,Y,Z)}{\sigma} R(X,Y)Z=0$ only the Levi—Civita connection can be metrical and of constant curvature.

2. A "strong" Schur's theorem

Let us denote

$$(4)\quad \begin{aligned} &\text{a)}\quad {}^4R(X,Y,Z,V) := \langle R(X,Y)Z, V\rangle \\ &\text{b)}\quad {}^4S(X,Y,Z,V) := \langle X,Z\rangle\langle Y,V\rangle - \langle Y,Z\rangle\langle X,V\rangle \\ &\text{c)}\quad R(X,Y,Z) := R(X,Y)Z \\ &\text{d)}\quad S(X,Y,Z) := \langle X,Z\rangle Y - \langle Y,Z\rangle X \end{aligned}$$

$$X,Y,Z,V \in \mathfrak{X}(M).$$

The following two propositions are known (see e.g. [2] Vol. I. Chap. V., § 1.).

Proposition 1. *Let* $Q\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to \mathbf{R}$ *(the reals) be a quadrilinear mapping. If it satisfies the conditions*

$$\text{(A)}\qquad Q(X_1, X_2, X_3, X_4) = -Q(X_2, X_1, X_3, X_4)$$

$$\text{(B)}\qquad Q(X_1, X_2, X_3, X_4) = -Q(X_1, X_2, X_4, X_3)$$

$$\text{(C)}\qquad Q(X_1, X_2, X_3, X_4)+Q(X_1, X_3, X_4, X_2)+Q(X_1, X_4, X_2, X_3) = 0,$$

then

$$\text{(D)}\qquad Q(X_1, X_2, X_3, X_4) = Q(X_3, X_4, X_1, X_2).$$

Proposition 2. *If* Q *and* $\bar{Q}$ *satisfy* (A), (B), (C) *and*

$$\text{(E)}\qquad Q(X_1, X_2, X_1, X_2) = \bar{Q}(X_1, X_2, X_1, X_2)$$

then

$$Q(X_1, X_2, X_3, X_4) \equiv \bar{Q}(X_1, X_2, X_3, X_4).$$

As well known, ${}^4R$ satisfies (A) and (B). If we suppose

$$(5)\qquad \underset{(X,Y,Z)}{\sigma} R(X,Y,Z) = 0,$$

where $\underset{(X,Y,Z)}{\sigma}$ means the sum formed from $R(X, Y, Z)$ by cyclic permutation of the vectors in its argument, then ${}^4R$ satisfies (C) too. It is easy to check that $k\,{}^4S$ also satisfies (A), (B) and (C).

Suppose that K^∇ depends on p only. Then $K^\nabla \equiv k(p) \in C^\infty(M)$ and

$$(6)\qquad \langle R(X,Y)X,Y\rangle = k(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2).$$

In this case ${}^4R$ and $k\,{}^4S$ satisfy (E) and (C). Thus from Proposition 2 we obtain that

$$(7)\qquad {}^4R(X,Y,Z,V) \equiv k\,{}^4S(X,Y,Z,V).$$

Remark 1. (A "strong" Schur's theorem.) We can extend the definition of $K^\nabla(p,\gamma)$ and define

$$\overset{\square}{K}{}^\nabla(p;\ \gamma^2) := \frac{\langle R(X,Y)Z,V\rangle}{\langle X,Z\rangle\langle Y,V\rangle - \langle Y,Z\rangle\langle X,V\rangle}\quad (\gamma^2 \equiv X,Y;\ Z,V).$$

This is an extension of $K^\nabla(p;\ X,Y)$, for $K^\nabla(p;\ X,Y) \equiv \overset{\square}{K}{}^\nabla(p;\ X,Y;\ X,Y)$.

We show that:

(I) the independence of $K^{\overset{*}{\nabla}}(p, \gamma)$ from γ is equivalent to

(II) the independence of $\overset{\square}{K}{}^{\overset{*}{\nabla}}(p; \gamma^2)$ from γ^2:

$$K^{\overset{*}{\nabla}}(p) \Leftrightarrow \overset{\square}{K}{}^{\overset{*}{\nabla}}(p) \quad \text{and} \quad K^{\overset{*}{\nabla}}(p) = \overset{\square}{K}{}^{\overset{*}{\nabla}}(p).$$

Namely for the Levi—Civita connection $\overset{*}{\nabla}$ (5) is always fulfilled (the first Bianchi identity), and thus by Proposition 2 we get (7), and this means that $\overset{\square}{K}{}^{\overset{*}{\nabla}}(p; \gamma^2)$ is independent of γ^2. The reverse statement is trivial, since K is a restriction of $\overset{\square}{K}$. — Thus the assertion that (I) implies the constantness of $K^{\overset{*}{\nabla}}$: $K^{\overset{*}{\nabla}}(p) = k_0$ (i.e. Schur's theorem) is equivalent to the assertion that (II) implies the constantness of $\overset{\square}{K}{}^{\overset{*}{\nabla}}$: $\overset{\square}{K}{}^{\overset{*}{\nabla}}(p) = \overset{\square}{k}_0$

$$K^{\overset{*}{\nabla}}(p) = k_0 \Leftrightarrow \overset{\square}{K}{}^{\overset{*}{\nabla}}(p) = \overset{\square}{k}_0 (= k_0).$$

Therefore it seems to be reasonable to call the following statement: "The independence of $\overset{\square}{K}{}^{\nabla}(p; \gamma^2)$ of γ^2 implies the independence of $\overset{\square}{K}{}^{\nabla}$ also of p" as a strong Schur's theorem. The above considerations show the coincidence of the two (normal and strong) Schur's theorems in case of the Levi—Civita connection.

We show the equivalence of the normal and of the strong Schur's theorems also in those cases, when on the connection only (5) is assumed:

$$K^{\nabla}(p) = k_0 \Leftrightarrow \overset{\square}{K}{}^{\nabla}(p) = \overset{\square}{k}_0 (= k_0) \quad \text{if} \quad \underset{(X,Y,Z)}{\sigma} R(X, Y, Z) = 0,$$

and it is not necessarily a Levi—Civita connection (i.e. if one of them is true, then so is the other too). — Indeed, the premise of the strong Schur's theorem is a consequence of the premise of the normal Schur's theorem and of (5). Conversely, the premise of the normal Schur's theorem is a part of the premise of the strong Schur's theorem. Thus, under assumption of (5), they are equivalent. Also the conclusions are consequences of each other, for $\overset{\square}{K}{}^{\nabla}$ is an extension of K^{∇}.

We also can see that in case of the independence of $\overset{\square}{K}{}^{\nabla}$ from γ^2 we get $\underset{(X,Y,Z)}{\sigma} R(X, Y, Z) = 0$. Namely in this case ${}^4R = k\,{}^4S$ and hence $R = kS$, from which $\underset{(X,Y,Z)}{\sigma} R(X, Y, Z) = k \underset{(X,Y,Z)}{\sigma} S(X, Y, Z)$, and because of $\underset{(X,Y,Z)}{\sigma} S(X, Y, Z) = 0$ this yields (5).

3. The non-existence theorem

Suppose that K^{∇} depends on p only: $K^{\nabla}(p, \gamma) = k(p)$. Then from (7) we obtain

(8) $$R(X, Y, Z) = kS(X, Y, Z)$$

and conversely. Covariant derivation of (8) gives

(9) $$(\nabla_U R)(X, Y, Z) = U(k) S(X, Y, Z) + k(\nabla_U S)(X, Y, Z).$$

Here

$$(\nabla_U S)(X,Y,Z) = \nabla_U(S(X,Y,Z)) - S(\nabla_U X,Y,Z) - S(X,\nabla_U Y,Z) - S(X,Y,\nabla_U Z).$$

Developing each term of the right-hand side with respect to $(4, d)$, and taking into account that ∇ is a metrical connection, and hence

$$0 = (\nabla_U g)(X,Z) = U\langle X,Z\rangle - \langle \nabla_U X, Z\rangle - \langle X, \nabla_U Z\rangle$$

holds, we obtain $(\nabla_U S)(X,Y,Z)=0 \;\; \forall X,Y,Z,U \in \mathfrak{X}(M)$. Then (9) reduces to

$$(\nabla_U R)(X,Y,Z) = U(k)S(X,Y,Z),$$

and in view of this

(10)
$$\underset{(X,Y,Z)}{\sigma}\{(\nabla_X R)(Y,Z,V)\} = \underset{(X,Y,Z)}{\sigma}\{X(k)S(Y,Z,V)\} =$$
$$= X(k)(\langle Y,V\rangle Z - \langle Z,V\rangle Y) + Y(k)(\langle Z,V\rangle X - \langle X,V\rangle Y) +$$
$$+ Z(k)(\langle X,V\rangle Y - \langle Y,V\rangle X).$$

In a neighbourhood $\mathcal{U}$ of p let $V=Z$ and X, Y, Z an orthonormal system. Then (10) yields

(11)
$$\underset{(X,Y,Z)}{\sigma}\{(\nabla_X R)(Y,Z,V)\}\big|_{V=Z} = Y(k)X - X(k)Y.$$

According to the second Bianchi identity ([2] Vol. I., Ch. III., § 5.)

(12)
$$\underset{(X,Y,Z)}{\sigma}\{(\nabla_X R)(Y,Z,V) + R(T(X,Y),Z)V\} = 0.$$

In view of (8)

$$\underset{(X,Y,Z)}{\sigma}\{R(T(X,Y),Z)V\} = \underset{(X,Y,Z)}{\sigma}\{kS(T(X,Y),Z,V)\} =$$
$$= k(\langle T(X,Y),V\rangle Z - \langle Z,V\rangle T(X,Y) + \langle T(Y,Z),V\rangle X -$$
$$- \langle X,V\rangle T(Y,Z) + \langle T(Z,X),V\rangle Y - \langle Y,V\rangle T(Z,X)).$$

Taking again in $\mathcal{U}$ $V=Z$ and X,Y,Z for an orthonormal system, we get

$$\underset{(X,Y,Z)}{\sigma}\{R(T(X,Y),Z)V\}\big|_{V=Z} =$$
$$= k(\langle T(X,Y),Z\rangle Z - T(X,Y) + \langle T(Y,Z),Z\rangle X + \langle T(Z,X),Z\rangle Y).$$

Substituting this and (11) into the Bianchi identity (12), in which Z is taken in place of V, and X, Y, Z is an orthonormal system in $\mathcal{U}$, so we obtain

(13)
$$k(\langle T(X,Y),Z\rangle Z - T(X,Y) + \langle T(Y,Z),Z\rangle X + \langle T(Z,X),Z\rangle Y) + (Yk)X - (Xk)Y = 0.$$

If K^∇ depends neither on p, then $K^\nabla = k_0 = \text{const.}$, and hence $Y(k_0) = X(k_0) = 0$. Therefore it follows from (13) that either $k_0 = 0$, or

$$\langle T(X,Y),Z\rangle Z - T(X,Y) + \langle T(Y,Z),Z\rangle X + \langle T(Z,X),Z\rangle Y = 0$$

i.e.

(14)
$$T(X,Y) = \langle T(X,Y),Z\rangle Z + \langle T(Y,Z),Z\rangle X + \langle T(Z,X),Z\rangle Y$$
$$\forall \text{ orthonormal } X,Y,Z \in \mathfrak{X}(M).$$

We consider the case $k_0 \neq 0$. Suppose that dim $M \geqq 4$. Then we can choose at p two orthonormal triads X, Y, Z and X, Y, Z_1; $Z \perp Z_1$, since dim $M \geqq 4$. Establishing (14) for these two triads, their difference yields

$$\begin{aligned}(15)\qquad &\langle T(X,Y),Z\rangle Z-\langle T(X,Y),Z_1\rangle Z_1+(\langle T(Y,Z),Z\rangle-\langle T(Y,Z_1),Z_1\rangle)X+\\ &+(\langle T(Z,X),Z\rangle-\langle T(Z_1,X),Z_1\rangle)Y=0.\end{aligned}$$

The scalar coefficients of Z, Z_1, X, Y in (15) vanish, for they are independent. By taking into account the vanishing of the coefficients in (15), we obtain from (14)

$$(16)\qquad \begin{aligned}&\text{a) } T(X,Y)=\langle T(Y,Z),Z\rangle X+\langle T(Z,X),Z\rangle Y\\ &\text{b) } T(X,Y)=\langle T(Y,Z_1),Z_1\rangle X+\langle T(Z_1,X),Z_1\rangle Y.\end{aligned}$$

Replacing in (16, b)) X by Z we get

$$(17)\qquad T(Z,Y)=\langle T(Y,Z_1),Z_1\rangle Z+\langle T(Z_1,Z),Z_1\rangle Y.$$

Then from (16, b)): (I) $\langle T(X,Y),X\rangle=\langle T(Y,Z_1),Z_1\rangle$, and similarly from (17): (II) $\langle T(Y,Z_1),Z_1\rangle=\langle T(Z,Y),Z\rangle$. Then from the alternation property of $T(Z,Y)$: (III) $\langle T(Z,Y),Z\rangle=-\langle T(Y,Z),Z\rangle$. Finally from the scalar product of (16, a)) with X: (IV) —$\langle T(Y,Z),Z\rangle=-\langle T(X,Y),X\rangle$. (I)—(IV) yield that $2\langle T(X,Y),X\rangle=0$. Thus $T(X,Y)\perp X$. — Interchanging the role of X and Y, we get $Y\perp T(Y,X)= =-T(X,Y)$. — However, according to (16, a)) $T(X,Y)$ is a linear combination of X and Y. Hence $T(X,Y)=0\ \forall p$, and for any orthonormal X and Y. Thus $T\equiv 0$, and $\nabla=\overset{*}{\nabla}$. — Our result is expressed by the following

Theorem. *If in a Riemannian manifold (M,g) with* dim $M\geqq 4$ *∇ is a metrical linear connection of nonvanishing constant curvature, for which* $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$, *then ∇ is the Levi—Civita connection.*

This means that $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$ (i.e. the fulfilment of the first Bianchi identity) characterizes the Levi—Civita connection among the metrical connections having nonvanishing constant curvature.

Remark 2. In the proof of our Theorem we used $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$ just to obtain ${}^4R=k\,{}^4S$. If we assume ∇ to be of constant curvature in the sense of $\overset{\square}{K}$, then this immediately gives ${}^4R=k\,{}^4S$, and we do not need the assumption of $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$ for our result. That is if in an (M,g) with dim $M\geqq 4$ ∇ is of nonvanishing constant "extended curvature": $\overset{\square}{K}{}^{\nabla}(p;\gamma^2)\equiv\overset{\square}{k}_0(=\text{const.})\neq 0$, then ∇ is the Levi—Civita connection.

Corollary. *For a given Riemannian manifold (M,g)* dim $M\geqq 4$ *there exists in general no metrical connection ∇ for which* $\overset{\square}{K}{}^{\nabla}\equiv\text{const.}\neq 0$. *However if such a connection exists, then it is unique, and it is the Levi—Civita connection.* — This means the incompatibility of torsion with constant extended curvature.

Remark 3. Condition $K^\nabla = k_0(\text{const.})$ is in our Theorem essential, i.e. $T=0$ cannot be concluded from the remaining assumptions: $\dim M \geqq 4$, $\nabla g=0$, and $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$. — We have shown ([1] Prop. 1) that for a Riemannian manifold (M,g) and for a linear one-form π on it there exists a unique linear connection ∇ with the properties:

$$\nabla g = 0 \quad (\nabla \text{ is metrical})$$

and

$$T(X,Y) = \pi(X)Y - \pi(Y)X \quad (\nabla \text{ is } \pi\text{-semi-symmetric})$$

Moreover, for a π-semi-symmetric connection ∇, $d\pi=0$ and $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$ are equivalent ([1] Theorem 1). — If we consider now a nonvanishing closed one-form π on (M,g), then there exists a π-semi-symmetric and metrical linear connection ∇ on M, for which $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$, and yet $T\neq 0$, since ∇ is semi-symmetric only.

We return to the case $k_0=0$. From this and from $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$ we cannot conclude $T=0$. We show this on an example: Using a local coordinate system (x) we put $X_i := \frac{\partial}{\partial x^i}$. Let g and ∇ be given by

$$\langle X_i, X_j\rangle \equiv g_{ij}(x) := \delta_{ij}, \quad \nabla_{X_j} X_i \equiv \Gamma_i{}^k{}_j X_k, \quad \Gamma_2{}^1{}_1(x) = -\Gamma_1{}^2{}_1(x) := A = \text{const.} \neq 0;$$
$$\Gamma_2{}^1{}_2(x) = -\Gamma_1{}^k{}_2(x) := B = \text{const.} \neq 0,$$

the other $\Gamma_i{}^k{}_j(x):=0$. Then $\nabla g=0$, $R=0$, hence $K^\nabla=0=k_0$, however $T\neq 0$.

We guess the existence of simple examples for (M,g), $\dim M=3, 2$ and ∇, for which $\nabla g=0$, $\underset{(X,Y,Z)}{\sigma} R(X,Y,Z)=0$, $K^\nabla=k_0(\text{const.})\neq 0$, yet $T\neq 0$.

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