A generalization of the fixed effects one-way analysis of the variance model

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Introduction

In his paper [4] BÉLA GYIRES proved the following criterion for the randomized block design ([4], p. 285, Theorem 2).

The expectations of the sample elements can be decomposed into the sum of two quantities corresponding to the block-effect and to the treatment-effect, respectively, if and only if the expectations of the random errors are zero.

The author dealt with the above-mentioned problem in the case of the Latin square design in his papers [7] and [8]. We could not prove the converse of the following theorem in our generalized model: If the expectations of the sample elements decompose into the sum of three quantities corresponding to the row-effect, the column-effect and the effect of treatment, respectively, then the expectations of the random errors are zero. In the paper [8] we gave a counterexample of the reversibility of the previous theorem, using, first of all, the method of minimum dyadical representation of a matrix. This can be found in EGERVÁRY's paper [2].

On the basis of obtained results the author thought this was a consequence of the more complicated restrictions required for the Latin square design. A study of the twentieth paragraph of [6] and H. O. HARTLEY's original paper [5] supported this conception ([6], p. 224).

After this we investigated the reversibility of the next theorem valid for the simplest analysis of the variance model, for the one-way classification with equal numbers of observations at each level of the single factor having systematic effect. If the expectation of a sample element decomposes into the sum of two quantities where the first one is a constant and the second corresponds to the effect of the selected level of the factor, then the expectation of the random error is zero. We were able to reverse this theorem following the method of GYIRES's paper [4]. Naturally these methods are applicable only for fixed effects models. These models involve only such factors which have systematic (not random) effects.

In this paper we will use the following notations: ξ_{jk}, ε_{jk} random variables; $\boldsymbol{\xi}$, $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $\boldsymbol{\zeta}$ matrix-valued random variables with m rows and n columns, that is matrices of dimension $m \times n$ consisting of random variables;

$\boldsymbol{E}$ identity matrix of order m;

$\boldsymbol{O}$ zero matrix of dimension $m \times n$;
$\boldsymbol{B}$ square matrix with complex elements of order n;
$\boldsymbol{X}$ is a matrix of dimension $m \times n$;
$\boldsymbol{S}_1, \boldsymbol{S}_2$ are stochastic matrices of order m and n, respectively;
$\boldsymbol{B}^*$ is the transpose of $\boldsymbol{B}$;
$\boldsymbol{B}^{-1}$ is the inverse matrix of $\boldsymbol{B}$;
$\mathbf{M}(\xi_{jk}), \mathbf{M}(\xi)$ expectations of ξ_{jk} and ξ, respectively. $\mathbf{M}(\xi)$ consists of the expectations of the elements of ξ;
$B=\begin{pmatrix} b_{11}, \dots, b_{1n} \\ \vdots \\ b_{n1}, \dots, b_{nn} \end{pmatrix}$ is a matrix given by its elements;
$\boldsymbol{B}=\|b_{jk}\|_{n\times n}$ or $\boldsymbol{B}=\|b_{jk}\|_{j,k=\overline{1,n}}$ is a square matrix which is given by its general element;
$\boldsymbol{a}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{m-1}, \boldsymbol{\lambda}, \dots$ m-dimensional column-vectors;
$\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-1}, \dots$ n-dimensional column-vectors ($\mathbf{0}$ is the zero vector);
$\boldsymbol{b}_0^*$ is an n-dimensional row-vector;
instead of $j=1, 2, \dots, m$ we use the notation $j=\overline{1, m}$;
$\otimes$ is the operational sign of the direct product. The direct product of the indentity matrix $\boldsymbol{E}$ of order m and of the square matrix $\boldsymbol{B}$ of order n is defined by the equality

$$\boldsymbol{E}\otimes\boldsymbol{B} = \|\boldsymbol{E}\ b_{jk}\|_{j,k=\overline{1,n}};$$

if it is necessary we indicate the dimension of a vector by writing

$$\boldsymbol{\lambda}_{m\times 1}.$$

In Theorem 1 of the second section we give the general solution of a special homogeneous system of linear equations. This is the main theorem of our paper. Theorem 2 and its proof can be found in the third section. Theorem 2 is valid for the generalized model of the one-way analysis of variance. It gives the solution of the earlier mentioned problem in the case of the one-way analysis of the variance model. Theorem 3 is in connection with the testing of a statistical hypothesis according to which the effects of the levels of the single factor are equal to each other.

Theorem 3 contains an equivalent form of this hypothesis. In the fourth section the author gives the minimum dyadical representation of the matrix $\mathbf{M}(\xi)$.

2. The main theorem

Theorem 1. *Let* $\boldsymbol{E}$ *be the identity matrix of order* m *and let* $\boldsymbol{B}$ *be a square matrix with complex elements of order* n. *Necessary and sufficient condition that* $\boldsymbol{B}$ *has* 0 *as an eigenvalue of multiplicity* 1 *with right eigenvector* $\boldsymbol{b}_0$ *is that a nonzero vector* $\boldsymbol{a}$ *of dimension* m *exists such that the general solution of the matrix equation*

$$\text{(1)} \qquad \boldsymbol{E}\boldsymbol{X}_{m\times n}\boldsymbol{B}^* = \boldsymbol{O}_{m\times n}$$

is given by

$$\text{(2)} \qquad \boldsymbol{X} = \gamma\boldsymbol{a}_0\boldsymbol{b}_0^* + \boldsymbol{\lambda}\boldsymbol{b}_0^*,$$

where a_0 *is the right eigenvector of* E *which has all its components equal to* 1, γ *and* λ *are parameters and* λ *runs through all column-vectors satisfying*

$$a^* \lambda = 0. \tag{3}$$

PROOF. We may assume that B is a Hermitian symmetric matrix ([4], p. 279).

Denote by β_0 the zero eigenvalue of B having multiplicity 1. It is known that the matrix E of order m has 1 as eigenvalue with multiplicity m. Let the corresponding linearly independent right eigenvectors be the following ones:

$$a_{0,2k\times 1} = \begin{pmatrix}1\\1\\1\\ \vdots\\1\end{pmatrix},\quad a_1 = \begin{pmatrix}1\\1\\ \vdots\\1\\0\end{pmatrix},\quad a_2 = \begin{pmatrix}0\\0\\1\\ \vdots\\1\end{pmatrix}, \ldots, a_{2k-2} = \begin{pmatrix}0\\ \vdots\\0\\1\\1\end{pmatrix},\quad a_{2k-1} = \begin{pmatrix}1\\0\\0\\ \vdots\\0\end{pmatrix},$$

if $m = 2k$, and

$$a_{0,(2k+1)\times 1} = \begin{pmatrix}1\\1\\1\\ \vdots\\1\end{pmatrix},\quad a_1 = \begin{pmatrix}1\\1\\ \vdots\\1\\0\end{pmatrix},\quad a_2 = \begin{pmatrix}0\\0\\1\\ \vdots\\1\end{pmatrix}, \ldots, a_{2k-1,(2k+1)\times 1} = \begin{pmatrix}1\\1\\0\\ \vdots\\0\end{pmatrix},\quad a_{2k} = \begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix},$$

if $m=2k+1$. Denote by $x_1, \ldots, x_n$ the column-vectors of the matrix X. Then the equations of the homogeneous linear system (1) with mn equations and mn unknowns may be rearranged into the form

$$C\begin{pmatrix}x_1\\x_2\\ \vdots\\x_n\end{pmatrix} = 0_{mn\times 1}, \tag{4}$$

where C is a square matrix of order mn and

$$C = E\otimes B = \|E\ b_{ij}\|_{i,j=\overline{1,n}}. \tag{5}$$

Conversely, if the matrix C has the form (5) then (4) too can be rearranged into the form (1).

From the foregoing

$$Ea_j = a_j,\quad j = \overline{0, m-1}. \tag{6}$$

Let B have the eigenvalues $\beta_0=0$, $\beta_k\neq 0$ $(k=\overline{1, n-1})$ and let the corresponding linearly independent right eigenvectors be b_k $(k=\overline{0, n-1})$, that is

$$Bb_k = \beta_k b_k;\quad k = \overline{0, n-1}. \tag{7}$$

It is known from Egerváry's paper [1] that the eigenvalues of the matrix C are

$$1\cdot\beta_k,\quad k = \overline{0, n-1}, \tag{8}$$

but each of them has multiplicity m. The right eigenvectors belonging to the eigenvalues are the direct products

$$\boldsymbol{a}_j \otimes \boldsymbol{b}_k, \quad j = \overline{0, m-1}, \quad k = \overline{0, n-1}.$$

According to (8) the matrix (5) has the zero as an eigenvalue of multiplicity m. Thus the system (4) has m linearly independent solutions which are

$$\boldsymbol{a}_0 \otimes \boldsymbol{b}_0, \boldsymbol{a}_1 \otimes \boldsymbol{b}_0, \ldots, \boldsymbol{a}_{m-1} \otimes \boldsymbol{b}_0.$$

Consequently the complete solution of (4) is

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \gamma \cdot \boldsymbol{a}_0 \otimes \boldsymbol{b}_0 + \boldsymbol{\lambda} \otimes \boldsymbol{b}_0 \tag{9}$$

with

$$\boldsymbol{\lambda}_{m \times 1} = c_1 \boldsymbol{a}_1 + c_2 \boldsymbol{a}_2 + \ldots + c_{m-1} \boldsymbol{a}_{m-1}, \tag{10}$$

where $\gamma, c_1, c_2, \ldots, c_{m-1}$ are parameters, taking on arbitrary values independently of each other.

If we rewrite the vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ into matrix form, then the solution of the system (1) has the form

$$\boldsymbol{X} = \gamma \boldsymbol{a}_0 \boldsymbol{b}_0^* + \lambda \boldsymbol{b}_0^*,$$

i.e. the general solution of (1) can be given in the form (2).

Now we prove the validity of (3). Since $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{m-1}$ are linearly independent we can choose $m-1$ equations from the m equations of (10) so that the determinant of these equations with respect to the unknowns $c_1, \ldots, c_{m-1}$ should be different from zero. One can compute the unknowns $c_1, c_2, \ldots, c_{m-1}$ from the selected equations applying Cramer's rule. Substituting the obtained values into the equation (10) not yet used, we get the relation (3).

At (3) $\boldsymbol{a} \neq \boldsymbol{0}$. Indeed, for $\boldsymbol{a} = \boldsymbol{0}$ the relation $\boldsymbol{a}^* \boldsymbol{\lambda} = 0$ would be valid, i.e. the components of $\boldsymbol{\lambda}$ would be linearly dependent. Hence the system (4) would have at least $m+1$ solutions. This would contradict our assumption according which (4) has m solutions.

Putting (2) into (1) we obtain

$$\boldsymbol{EXB}^* = \gamma \boldsymbol{a}_0 (0 \ \boldsymbol{b}_0)^* + \lambda (0 \ \boldsymbol{b}_0)^*. \tag{13}$$

This means that (2) is indeed the solution of (1).

The conditions of the theorem are also sufficient in order that (2) be the complete solution of the system (1). It also follows from the previous proof that if the zero were a non-simple eigenvalue of $\boldsymbol{B}$ then the rank of $\boldsymbol{C}$ (defined by (5)) would be smaller or greater than $m(n-1)$. Then the general solution of (1) would contain as a proper part the solutions (2), or, conversely, the solution (2) would contain as a proper part the general solution of (1).

Let us introduce the following notations:

$$\boldsymbol{X} = \|x_{jk}\|_{j=\overline{1,m};\, k=\overline{1,n}}, \quad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)^*.$$

Then we obtain from Theorem 1 the following

Corollary 1. *The general solution of the matrix equation is given by the expression*

$$x_{jk} = \gamma + \lambda_j, \quad j = \overline{1, m}, \quad k = \overline{1, n} \tag{14}$$

*if and only if **B** has the zero as a simple eigenvalue and every component of the corresponding right eigenvector is equal to* 1 *and* **E** *has (with its eigenvalue* 1 *such) a right eigenvector which consists of only components* 1.

PROOF. Writing down the solution (2) elementwise we immediately get (14).

It is known that a given matrix with nonnegative elements is stochastic if and only if it has 1 as an eigenvalue, while the corresponding right eigenvector has all its components equal to 1. The eigenvalue 1 of a stochastic matrix has the greatest absolute value among the eigenvalues.

Corollary 2. *If $\boldsymbol{S}_2$ is a stochastic matrix of order n and* **E** *is the identity matrix of order m or order n, then the matrix equation*

$$\boldsymbol{E}\boldsymbol{X}_{m\times n}(\boldsymbol{E}-\boldsymbol{S}_2)^* = \boldsymbol{O}_{m\times n}$$

(briefly $\boldsymbol{X}(\boldsymbol{E}-\boldsymbol{S}_2)^*=\boldsymbol{O}$) has

$$x_{jk} = \gamma + \lambda_j \quad (j = \overline{1, m}; \quad k = \overline{1, n})$$

as its general solution if and only if the matrix $\boldsymbol{S}_2$ has 1 *as a simple eigenvalue.*

PROOF. This can be proved on the basis of Corollary 1. It is easy to see that the assumptions of Corollary 1 are valid on E and $-\boldsymbol{S}_2+\boldsymbol{E}$.

The condition (3) must be true for the cases of the previous two corollaries.

The stochastic matrix $\boldsymbol{S}_2$ of Corollary 2 may consist of positive elements columnwise equal, or it may have identical elements $\frac{1}{n}$. ([4], p. 284 and 285.)

3. A generalised form of the one-way analysis of the variance model

We consider the case when the numbers of observations are equal in each cell, that is the case when equal numbers of observations are made at each level of the single factor and this factor has a systematic effect on the result. The usual form of such a model is

$$\xi_{jk} = \gamma + \lambda_j + \varepsilon_{jk} \quad (j = \overline{1, m}; \; k = \overline{1, n}), \tag{15}$$

where $\sum_{j=1}^{m} \lambda_j = 0$, γ is the so-called overall mean, and λ_j is the differential effect of the j-th level of the factor. More precisely, λ_j corresponds to the j-th effect of the factor. The random variables ε_{jk} $(j=\overline{1, m};\ k=\overline{1, n})$ are generally assumed to be independent and normally distributed with parameters 0 and σ^2. The variance σ^2 is unknown. ξ_{jk} denotes the result of the k-th observation at the j-th level of the single factor.

We introduce the following usual notations:

$$\bar{\xi}_{j\cdot} = \frac{1}{n} \sum_{k=1}^{n} \xi_{jk},$$

$$\bar{\bar{\xi}}_{\cdot\cdot} = \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} \xi_{jk}.$$

The differences $\bar{\xi}_{j\cdot} - \bar{\bar{\xi}}_{\cdot\cdot}$ $(j=\overline{1, m})$ are discrepancies between the levels of the factor. The differences $\xi_{jk} - \bar{\xi}_{j\cdot}$ $(k=\overline{1, n})$ are discrepancies within the j-th level of the systematic factor. The number of levels is m. The random variables $\xi_{jk} - \bar{\xi}_{j\cdot}$ are said to be random errors. It is easy to see that they have zero expectations.

From the decomposition (15) of the sample elements follows that the expectations of the random errors equal zero. Therefore we get from (15)

$$\mathbf{M}(\xi_{jk}) = \gamma + \lambda_j \quad (j = \overline{1, m};\ k = \overline{1, n}).$$

Taking into account (14)

$$\mathbf{M}(\xi_{jk}) = x_{jk} \quad (j = \overline{1, m};\ k = \overline{1, n}).$$

Now we define our generalized model, and with the aid of it we can prove the following theorem: If the expectations of the random errors are zero in the case of the one-way analysis of the variance model, then the sample elements can be written in the form (15).

Let $\boldsymbol{\xi}_{m\times n}$ be the matrix of the random variables ξ_{jk} $(j=\overline{1, m};\ k=\overline{1, n})$ which are defined by (15) and have expectations. Then

$$\boldsymbol{\xi} = \|\gamma\|_{m\times n} + \|\lambda_j\|_{j=\overline{1,m};\ k=\overline{1,n}} + \|\varepsilon_{jk}\|_{m\times n};$$

$$(16) \qquad \boldsymbol{\xi} = \gamma \begin{Vmatrix} 1 & 1 \dots 1 \\ 1 & 1 \dots 1 \\ \vdots & \\ 1 & 1 \dots 1 \end{Vmatrix} + \begin{Vmatrix} \lambda_1, & \lambda_1, & \dots, & \lambda_1 \\ \lambda_2, & \lambda_2, & \dots, & \lambda_2 \\ \vdots & & & \\ \lambda_m, & \lambda_m, & \dots, & \lambda_m \end{Vmatrix} + \begin{Vmatrix} \varepsilon_{11}, & \varepsilon_{12}, & \dots, & \varepsilon_{1n} \\ \varepsilon_{21}, & \varepsilon_{22}, & \dots, & \varepsilon_{2n} \\ \vdots & & & \\ \varepsilon_{m1}, & \varepsilon_{m2}, & \dots, & \varepsilon_{mn} \end{Vmatrix}.$$

Here $\mathbf{M}\{\|\varepsilon_{jk}\|\} = \boldsymbol{O}_{m\times n}$. If $\boldsymbol{a}_0$ denotes the column-vector of dimension m consisting only of components 1, $\boldsymbol{b}_0^*$ is the row-vector of dimension n consisting only of components 1, and $\boldsymbol{\lambda}$ is the m-dimensional column-vector of components $\lambda_1, \lambda_2, \dots, \lambda_m$, where λ_j corresponds to the j-th effect of the single factor $(j=\overline{1, m})$ then we obtain from (16)

$$(17) \qquad \mathbf{M}(\boldsymbol{\xi}) = \gamma \boldsymbol{a}_0 \boldsymbol{b}_0^* + \boldsymbol{\lambda} \boldsymbol{b}_0^*.$$

Let $\boldsymbol{S}_1$ be a stochastic matrix of order m having identical elements $\frac{1}{m}$. Let $\boldsymbol{S}_2$ be a stochastic matrix of order n consisting of identical elements $\frac{1}{n}$. Then

$$(18) \qquad \boldsymbol{S}_1 = \frac{1}{m} \boldsymbol{a}_0 \boldsymbol{a}_0^*, \quad \boldsymbol{S}_2 = \frac{1}{n} \boldsymbol{b}_0 \boldsymbol{b}_0^*.$$

Moreover let us define the following matrixvalued random variables of dimension $m\times n$:

(19) $$\eta_2 = \xi S_2^*, \quad \zeta = S_1 \xi S_2^*.$$

Then η_2 consists of row-wise identical elements. These elements are the means of the quantities which correspond to the levels of the factor. All elements of ζ are equal to $\bar{\bar{\xi}}_{..}$. Therefore we can write

(20) $$\eta_2 = \|\bar{\xi}_{j.}\|, \quad \zeta = \|\bar{\xi}_{..}\|.$$

The matrix of the random errors $\xi_{jk}-\bar{\xi}_{j.}$ $(j=\overline{1,m};\ k=\overline{1,n})$ can be written in the form

(21) $$\xi-\eta_2 = \|\xi_{jk}-\bar{\xi}_{j.}\|.$$

The matrix of the discrepancies between effects due to the levels of the systematic factor is

(22) $$\eta_2-\zeta = \|\xi_{j.}-\bar{\xi}_{..}\|.$$

Now we can formulate the following

Theorem 2. *The decomposition*

(23) $$\mathrm{M}(\xi) = \gamma a_0 b_0^* + \lambda b_0^*$$

is valid if and only if

(24) $$\mathrm{M}(\xi-\eta_2) = O_{m\times n}.$$

PROOF. The random error matrix (21) can be written in the form

$$\xi(E-S_2)^*$$

in consequence of (19). Therefore we obtain for the expectation of the random error matrix the following formula:

$$\mathrm{M}(\xi-\eta_2) = E\mathrm{M}(\xi)(E-S_2)^*.$$

So

$$E\mathrm{M}(\xi)(E-S_2)^* = O_{m\times n}$$

on the basis of (24). This means that the assumptions of Corollary 2 are valid. In consequence of this Theorem 2 is true, that is (23) satisfies (24) and conversely.

Theorem 3. *Let* (23) *be true. Then*

(25) $$\mathrm{M}(\eta_2-\zeta) = O$$

if and only if

(26) $$\lambda = c a_0,$$

where c is a numerical parameter.

PROOF. Applying (19) the expectation of the random error matrix can be written in the form

$$\mathrm{M}(\eta_2-\zeta) = (E-S_1)\mathrm{M}(\xi)S_2^*.$$

Taking into account (23) we get the decomposition

$$\mathrm{M}(\eta_2-\zeta) = \gamma(\boldsymbol{E}-\boldsymbol{S}_1)\,\boldsymbol{a}_0(\boldsymbol{S}_2\boldsymbol{b}_0)^* + (\boldsymbol{E}-\boldsymbol{S}_1)\,\boldsymbol{\lambda}(\boldsymbol{S}_2\boldsymbol{b}_0)^*.$$

Since $(\boldsymbol{E}-\boldsymbol{S}_1)\,\boldsymbol{a}_0=\boldsymbol{0}$ and $\boldsymbol{S}_2\boldsymbol{b}_0=\boldsymbol{b}_0$, we obtain

$$\mathrm{M}(\eta_2-\zeta) = (\boldsymbol{E}-\boldsymbol{S}_1)\boldsymbol{\lambda}\boldsymbol{b}_0.$$

Therefore (25) is valid if and only if

$$(\boldsymbol{E}-\boldsymbol{S}_1)\,\boldsymbol{\lambda} = \boldsymbol{0},$$

i.e. if $\boldsymbol{S}_1\boldsymbol{\lambda}=\boldsymbol{\lambda}$. The last equality means that $\boldsymbol{\lambda}$ is the right eigenvector of $\boldsymbol{S}_1$ belonging to the eigenvalue 1. But according to the assumption $\boldsymbol{S}_1$ has $\boldsymbol{a}_0$ as a right eigenvector belonging to the simple eigenvalue 1 of the matrix $\boldsymbol{S}_1$. Hence $\boldsymbol{S}_1\boldsymbol{\lambda}=\boldsymbol{\lambda}$ is true only if $\boldsymbol{\lambda}=c\boldsymbol{a}_0$, where c is a numerical parameter.

Conversely, in the case of $\boldsymbol{\lambda}=c\boldsymbol{a}_0$ we get from (17) the relation

$$\mathrm{M}(\xi) = (\gamma+c)\,\boldsymbol{a}_0\boldsymbol{b}_0^*, \tag{27}$$

which is a solution of (25).

Theorem 2 can be formulated in the following way:

If the stochastic matrix $\boldsymbol{S}_2$ has 1 *as a simple eigenvalue, then* $\mathrm{M}(\xi)$ *can be decomposed in the form* (23) *if and only if the expectation of the random error matrix is the zero matrix.*

Finally Theorem 3 is capable of the following equivalent formulation:

Let H_0 be the null hypothesis that the components of the column-vector $\boldsymbol{\lambda}$ — consisting of the quantities which represent the effects of the levels of the single systematic factor — *are equal.* According to Theorem 3 *this hypothesis is equivalent to the null hypothesis H_0' according which the expectation of the matrix of the discrepancies between the effects of the levels of the single factor equals the zero-matrix.*

The minimum dyadical representation of $\mathrm{M}(\xi)$

The dyad generated by the nonzero element a_{jk} of $\boldsymbol{A}_{m\times n}=\|a_{jk}\|$ is defined by the expression $\boldsymbol{o}_k\boldsymbol{s}_j^*/a_{jk}$ or in another form by the expression

$$\boldsymbol{A}\boldsymbol{e}_k\boldsymbol{e}_j^*\boldsymbol{A}/\boldsymbol{e}_j^*\boldsymbol{A}\boldsymbol{e}_k,$$

where $\boldsymbol{A}=\|\boldsymbol{o}_1, \boldsymbol{o}_2, \ldots, \boldsymbol{o}_n\|$ and

$$\boldsymbol{A} = \left\| \begin{matrix} \boldsymbol{s}_1^* \\ \boldsymbol{s}_2^* \\ \vdots \\ \boldsymbol{s}_m^* \end{matrix} \right\|,$$

$\boldsymbol{o}_k$ is the k-th column-vector of $\boldsymbol{A}$ with dimension m, $\boldsymbol{s}_j^*$ is the j-th row-vector of $\boldsymbol{A}$ with dimension n, $\boldsymbol{e}_k$ is the k-th column-vector of the unit matrix of order n, $\boldsymbol{e}_j^*$ is the j-th row-vector of the identity matrix of order m.

The representation of a matrix is called minimum dyadical if it is the sum of the minimum number of dyads. The recursive formula for the dyads of the minimum dyadical representation in the case of A is

$$A^{(k+1)} = A^{(k)} - \frac{A^{(k)} e_k e_k^* A^{(k)}}{e_k^* A^{(k)} e_k},$$

where $A^{(1)}=A$ and $A^{(s+1)}=O$ if the procedure comes to an end after s steps. Summing the former equalities and ordering the obtained formula we get the following dyadical decomposition of A:

$$A = \sum_{k=1}^{s} \frac{A^{(k)} e_k e_k^* A^{(k)}}{e_k^* A^{(k)} e_k}.$$

Let us introduce the notation $u_k v_k^*$ for the k-th dyad of the former representation $(k=\overline{1, s})$. In this case the minimum dyadical decomposition of A is

$$A = \sum_{k=1}^{s} u_k v_k^*.$$

The following theorem can easily be proved. The rank of a matrix equals the number of the dyads occurring in the minimum dyadical representation of the matrix.

Further details in connection with this method can be found in [2].

Now we give the minimum dyadical decomposition of $\mathbf{M}(\xi)$. On the basis of (15)

$$\mathbf{M}(\xi) = \|\gamma+\lambda_j\|_{m\times n}. \tag{28}$$

If $\gamma+\lambda_{j_i}\neq 0$, then the minimum dyadical representation of $\mathbf{M}(\xi)$ is

$$\mathbf{M}(\xi) = \frac{1}{\gamma+\lambda_{ji}} \begin{pmatrix} \gamma+\lambda_1 \\ \gamma+\lambda_2 \\ \vdots \\ \gamma+\lambda_m \end{pmatrix} (\gamma+\lambda_{ji}, \ldots, \gamma+\lambda_{ji})_{1\times n}.$$

This can be written in the form

$$\mathbf{M}(\xi) = \begin{pmatrix} \gamma+\lambda_1 \\ \gamma+\lambda_2 \\ \vdots \\ \gamma+\lambda_m \end{pmatrix} (1, 1, \ldots, 1)_{1\times n}. \tag{29}$$

From (29) we get

$$\mathbf{M}(\xi) = \gamma \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{m\times 1} (1, 1, \ldots, 1)_{1\times n} + \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} (1, 1, \ldots, 1)_{1\times n}.$$

With our former notations

$$\mathbf{M}(\xi) = \gamma a_0 b_0^* + \lambda b_0^*.$$

The minimum dyadical decomposition of a matrix was also applied in our earlier papers [7] and [8].

Formula (29) shows that the rank of $\mathbf{M}(\xi)$ is 1. This fact can be seen also from (28).

Remark 1. In the case of $m=n=1$ Theorem 2 states that in the one-way analysis of variance model with fixed effects an arbitrary sample element can be written in the form

$$\xi_{jk} = \gamma + \lambda_j + \varepsilon_{jk}$$

if and only if $\mathbf{M}(\xi_{jk} - \bar{\xi}_{j.}) = 0$. This means that the decomposition of a sample element is valid if and only if the expectation of the random error is zero. (The meaning of the quantities γ, λ_j and ε_{jk} can be found in formula (15).)

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