An approximate Wiener-series expansion and a condition of stationarity for some quadratic time series models

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

During the last ten years or so there has been a great interest in nonlinear models and a number of nonlinear models have been developed. One of them, the bilinear model, has been studied extensively such as model fitting and parameter estimation, see SUBBA RAO and GABR (1984). Its Wiener—Ito representation and conditions of stationarity are given in the papers TERDIK (1985), TERDIK and SUBBA RAO (1987). In this paper we are using the multiple Wiener—Ito integrals to give the Wiener series expansion for a quadratic in the observation model which is a special case of the so-called state dependent model defined by PRIESTLEY (1985). The main difficulties are the determination of the Wiener kernels for the product of two nonlinear processes. That is why we are giving some explicit formulae for the product of multiple Wiener—Ito integrals. Using these formulae we are constructing an approximate transfer function system and the process defined by them will be close enough to the original solution in the mean square. Finally an assumption is given for the stationarity of a quadratic time series model.

Diagram formulae for the multiple Wiener—Ito integral

Let G be a spectral measure on $D=[-\pi, \pi]$ and $\overline{H}_G^n$ denote the real Hilbert space of complex-valued functions on D^n with properties $f(-\omega_{(n)})=\bar{f}(\omega_{(n)})$, where $\omega_{(n)}\in D^n$ and

$$\|f\|_{G^n}^2 = \int\limits_{D^n} |f(\omega_{(n)})|^2 G(d\omega_{(n)}) < \infty,$$

$$G(d\omega_{(n)}) = G(d\omega_1)\, G(d\omega_2) \dots G(d\omega_n).$$

The subspace H_G^n of $\overline{H}_G^n$ contains those $f\in\overline{H}_G^n$ that are invariant under permutations of their arguments, i.e. $f=\operatorname{sym} f$, where

$$\operatorname{sym} f(\omega_{(n)}) = \frac{1}{n!} \sum_{\mathscr{P}\in\mathscr{P}_n} f(\omega_{\mathscr{P}(n)}),$$

$\mathscr{P}_n$ denoting the group of all permutations of the set $\{1, 2, \dots, n\}$ and $\omega_{\mathscr{P}(n)}= =(\omega_{p_1}, \dots, \omega_{p_n})$. If $\mathscr{K}$ is a finite ordered subset of different natural numbers i.e.

$\mathcal{K}=(k_1, k_2, \ldots, k_n)$, $k_i<k_{i+1}$ then $\underline{\omega}_{\mathcal{K}}$ denotes the vector $(\omega_{k_1}, \omega_{k_2}, \ldots, \omega_{k_n})$, $\omega_{k_i}\in D$. Let $|\mathcal{K}|$ denote the number of the elements of $\mathcal{K}$ and $(\mathcal{K})_l$ is the l^{th} component of $\mathcal{K}$. If $\mathcal{K}$ and $\mathcal{L}$ are two sets of such type then $\mathcal{K}\cup\mathcal{L}$ and $\mathcal{K}\setminus\mathcal{L}$ are the sets of the same kind with respect to the set-theoretic union and difference of the sets $\mathcal{K}$, $\mathcal{L}$. (n) will denote the set $\{1, 2, \ldots, n\}$ and $(n, n+k)=\{n, n+1, \ldots, n+k\}$.

Let U_t, $t\in\mathbf{Z}$ be a discrete stationary Gaussian series with spectral representation

$$U_t = \int_D e^{it\omega}\, U(d\omega), \quad D = [-\pi, \pi],$$

spectral measure $G(d\omega)=E|U(d\omega)|^2$ and $EU_t=0$. The n-dimensional multiple Wiener—Ito integral is defined for the functions $f\in\overline{H}_G^n$ with respect to the Gaussian stochastic spectral measure $U(d\omega_{(n)})=U(d\omega_1)\,U(d\omega_2)\ldots U(d\omega_n)$. It will be denoted simply by

$$\int_{D^n} f(\omega_{(n)})\,U(d\omega_{(n)}), \quad D^n = [-\pi, \pi]^n$$

concerning the definition and the basic properties of the multiple Wiener—Ito integral we refer to the works of Dobrushin (1979), Major (1981), Rozanov (1981). Let $f\in H_G^n$ and $h\in H_G^1$. Then

$$\int_{D^n} f(\omega_{(n)})\,U(d\omega_{(n)}) \int_D h(\omega)\,U(d\omega) = \int_{D^{n+1}} f(\omega_{(n)})\,h(\omega_{(n+1)})\,U(d\omega_{(n+1)})+$$
$$+\sum_{k=1}^{n} \int_{D^{n-1}} \int_D f(\omega_{(n)}, \omega_k = \lambda)\,h(-\lambda)\,G(d\lambda)\,U(d\omega_{(n)\setminus\{k\}}).$$

We shall refer to this formula as diagram formula of first order. The rule of forming the product of an n-dimensional Wiener—Ito integral with a 2-dimensional one is called the diagram formula of second order. If $f\in\overline{H}_G^n$ and $h\in\overline{H}_G^2$ then

$$(1)\qquad \int_{D^n} f(\omega_{(n)})\,U(d\omega_{(n)}) \int_{D^2} h(\omega_{(2)})\,U(d\omega_{(2)}) = \int_{D^{n+2}} f(\omega_{(n)})h(\omega_{\{n+1, n+2\}})\,U(d\omega_{(n+2)})+$$
$$+\sum_{j=1}^{n} \int_{D^n} \int_D f(\omega_{(n)};\ \omega_j = \lambda)h(-\lambda, \omega_j)\,G(d\lambda)\,U(d\omega_{(n)})+$$
$$+\sum_{j=1}^{n} \int_{D^n} \int_D f(\omega_{(n)};\ \omega_j = \lambda)h(\omega_j, -\lambda)\,G(d\lambda)\,U(d\omega_{(n)})+$$
$$+\sum_{i\neq j} \int_{D^{n-2}} \int_{D^2} f(\omega_{(n)};\ \omega_{\{i,j\}} = \lambda_{(2)})\,h(-\lambda_{(2)})\,G(d\lambda_{(2)})\,U(d\omega_{(n)\setminus\{i,j\}}).$$

This formula proves to be simpler for the symmetric functions $f\in H_G^n$, $h\in H_G^2$ i.e.

$$\int_{D^n} f(\omega_{(n)})\,U(d\omega_{(n)}) \int_{D^2} h(\omega_{(2)})\,U(d\omega_{(2)}) = \int_{D^{n+2}} f(\omega_{(n)})\,h(\omega_{\{n+1, n+2\}})\,U(d\omega_{(n+2)})+$$
$$+2n \int_{D^n} \int_D f(\omega_{(n-1)}, \lambda)\,h(\omega_n, -\lambda)\,G(d\lambda)\,U(d\omega_{(n)})+$$
$$+n(n-1) \int_{D^{n-2}} \int_{D^2} f(\omega_{(n-2)}, \lambda_{(2)})\,h(-\lambda_{(2)})\,G(d\lambda_{(2)})\,U(d\omega_{(n-2)}).$$

One can prove the diagram formula of second order directly by the definition of the multiple Wiener—Ito integral. Another proof of (1) is based on the GHAW (Hermite) polynomials, see TERDIK (1988). As the system $\{\exp(i\sum_{k=1}^{n} t_k\lambda_k),\ t_k=0, \pm 1, \dots\}$ is complete in the space $\overline{H}_G^n$ so it is enough to prove (1) for these functions. The spectral representation for the GHAW polynomials of the series U_t is

$$A_n(U_{t_1}, \dots, U_{t_n}) = \int_{D^n} e^{i\Sigma t_k\lambda_k} U(d\lambda_{(n)}).$$

Let us fix the numbers $t_1, \dots, t_{n+2}$ and denote the polynomial $A_n(U_{t_1}, \dots, U_{t_n})$ by $A_n(U_{(n)})$ or simply A_n. Let us put the functions $\exp(i\sum_1^n t_k\lambda_k)$ into (1) and by the spectral representation we get

$$\text{(2)}\qquad A_n(U_{(n)})A_2(U_{\{n+1,n+2\}}) = A_{n+2}(U_{(n+2)})+$$

$$+\sum_{j=1}^{n} C(t_j-t_{n+1})A_n(U_{(n+2)\setminus\{j,n+1\}})+\sum_{j=1}^{n} C(t_j-t_{n+2})A_n(U_{(n+2)\setminus\{j,n+2\}})+$$

$$+\sum_{i\neq j} C(t_i-t_{n+1})C(t_j-t_{n+2})A_{n-2}(U_{(n)\setminus\{i,j\}})$$

where $C(t_i-t_j)=\operatorname{cov}(U_{t_i}, U_{t_j})$. To prove (2) we use the recursive formula

$$A_{n+1} = A_n\cdot U_{t+1}-\sum_{i=1}^{n} C(t_{n+1}-t_i)A_{n-1}(U_{(n)\setminus\{i\}})$$

and that

$$A_2(U_{\{n+1,n+2\}}) = U_{t_{n+1}}U_{t_{n+2}}-C(t_{n+1}-t_{n+2}).$$

Therefore

$$(A_n(U_{(n)})U_{t_{n+1}})U_{t_{n+2}}-C(t_{n+1}-t_{n+2})A_n(U_{(n)}) =$$

$$= A_{n+1}(U_{(n+1)})\cdot U_{t_{n+2}}+\sum_{i=1}^{n} C(t_{n+1}-t_i)U_{t_{n+2}}A_{n-1}(U_{(n)\setminus\{i\}})-$$

$$-C(t_{n+1}-t_{n+2})A_n(U_{(n)}) = A_{n+2}(U_{(n+2)})+\sum_{j=1}^{n} C(t_j-t_{n+2})A_n(U_{(n+1)\setminus\{j\}})+$$

$$+\sum_{j=1}^{n} C(t_{n+2}-t_j)A_n(U_{(n+2)\setminus\{n+2,j\}})+\sum_{i\neq j} C(t_{n+1}-t_i)C(t_{n+2}-t_j)A_{n-2}(U_{(n)\setminus\{i,j\}})$$

which proves (2). The method used here will be generalized for the diagram formula of m^{th} order.

Let $\mathscr{K}$ be a finite ordered set of natural numbers $k_1<k_2<\dots<k_l$. $\mathscr{P}\mathscr{K}$ means a permutation of $\mathscr{K}$ and $\omega_{\mathscr{P}\mathscr{K}}$ is the same permutation for $\omega_{\mathscr{K}}$. In case of $f\in\overline{H}_G^n$ and $h\in\overline{H}_G^m$, $\mathscr{K}\subseteq(n)$, $\mathscr{L}\subseteq(n+1, n+m)$ we define the function

$$f\langle\mathscr{P}\mathscr{K}\times\mathscr{L}\rangle h(\omega_{(n+m)\setminus\{\mathscr{K}\cup\mathscr{L}\}}) =$$

$$= \int_{D^{|\mathscr{K}|}} f(\omega_{(n)}, \omega_{\mathscr{K}} = \lambda_{\mathscr{K}})h(\omega_{(n+1,n+m)};\ \omega_{\mathscr{L}} = -\lambda_{\mathscr{P}\mathscr{K}})G(d\lambda_{\mathscr{K}})$$

where $|\mathscr{K}|=|\mathscr{L}|\leqq\min(m,n)$. If $|\mathscr{K}|=|\mathscr{L}|=0$ i.e. $\mathscr{K}$, $\mathscr{L}$ are empty sets then

$$f\langle\mathscr{P}\mathscr{K}\times\mathscr{L}\rangle h(\omega_{(n+m)})=f(\omega_{(n)})h(\omega_{(n+1,n+m)}).$$

It is clear that $f\langle\mathscr{P}\mathscr{K}\times\mathscr{L}\rangle h$ belongs to the $\overline{H}_G^{n+m-2|\mathscr{K}|}$. Under these assumptions we prove the following *diagram formula of m^{th} order* $(m\leqq n)$

$$\int_{D^n} f(\omega_{(n)})U(d\omega_{(n)})\int_{D^n} h(\omega_{(m)})U(d\omega_{(m)}) \tag{3}$$

$$=\sum_{l=0}^{m}\sum_{\substack{\mathscr{K}\subseteq(n)\\ \mathscr{L}\subseteq(n+1,n+m)\\ |\mathscr{K}|=|\mathscr{L}|=l}}\sum_{\mathscr{P}\mathscr{K}}\int_{D^{n+m-2l}} f\langle\mathscr{P}\mathscr{K}\times\mathscr{L}\rangle h(\omega_{(n+m)\setminus\{\mathscr{K}\cup\mathscr{L}\}})U(d\omega_{(n+m)\setminus\{\mathscr{K}\cup\mathscr{L}\}}).$$

We have seen earlier this formula for $m=1,2$. Now we put here the case $m=3\leqq n$ as an example,

$$\int_{D^n} f(\omega_{(n)})U(d\omega_{(n)})\int_{D^3} h(\omega_{(3)})U(d\omega_{(3)})$$

$$=\int_{D^{n+3}} f(\omega_{(n)})h(\omega_{n+1},\omega_{n+2},\omega_{n+3})U(d\omega_{(n+3)})+$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{3}\int_{D^{n+1}}\int_{D} f(\omega_{(n)},\omega_i=\lambda)h(\omega_{(n+1,n+3)};\ \omega_{n+j}=-\lambda)G(d\lambda)U(d_{(n+3)\setminus\{i,n+j\}})+$$

$$+\sum_{i<j}\sum_{k\neq l}\int_{D^{n-1}}\int_{D^2} f(\omega_{(n)},\omega_i=\lambda_1,\omega_j=\lambda_2)h(\omega_{(n+1,n+3)},\omega_{n+k}=-\lambda_1,\omega_{n+l}=-\lambda_2)\times$$
$$\times G(d\lambda_{(2)})U(d\omega_{(n+3)\setminus\{i,j,n+k,n+l\}})+$$

$$+\sum_{i<j<l}\sum_{3!}\int_{D^{n-3}}\int_{D^3} f(\omega_{(n)},\omega_i=\lambda_1,\omega_j=\lambda_2,\omega_l=\lambda_3)h(-\lambda_{(3)})G(d\lambda_{(3)})\times$$
$$\times U(d\omega_{(n)\setminus\{i,j,l\}}).$$

To prove (3) let us assume that it is true for any $m_1<m$ and n, and take the complete system $\{\exp i\sum t_k\lambda_k\}$. The GHAW polynomials can be applied and (3) has the form

$$A_n(U_{(n)})A_m(U_{(n+1,n+m)})=\sum_{l=0}\sum_{\substack{\mathscr{K}\subseteq(n)\\ \mathscr{L}\subseteq(n+1,n+m)\\ |\mathscr{K}|=|\mathscr{L}|=l}}\sum_{\mathscr{P}\mathscr{L}} C(t_{\mathscr{K}}-t_{\mathscr{P}\mathscr{L}})A_{n+m-2l}(U_{(n+m)\setminus\{\mathscr{K}\cup\mathscr{L}\}}) \tag{4}$$

where again $U_{(n)}=(U_{t_1},\dots,U_{t_n})$; $U_{(n+1,n+m)}=(U_{t_{n+1}},\dots,U_{t_{n+m}})$ and if $\mathscr{K}=(k_1,\dots,k_l)$ and $\mathscr{P}\mathscr{L}=(h_1,\dots,h_l)$ then

$$C(t_{\mathscr{K}}-t_{\mathscr{P}\mathscr{L}})=\prod_{i=1}^{l}\operatorname{cov}(U_{t_{k_i}},U_{t_{h_i}}).$$

Using the recursive formula for A_m and the diagram formula of first order we get

$$A_n(U_{(n)})A_m(U_{(n+1,n+m)}) = A_n(U_{(n)})U_{t_{n+1}}A_{m-1}(U_{(n+2,n+m)}) -$$

$$-\sum_{j=2}^{m} C(t_{n+1}-t_{n+j})A_n(U_{(n)})A_{m-2}(U_{(n+2,n+m)\setminus\{n+j\}}) =$$

$$= A_{n+1}(U_{(n+1)})A_{m-1}(U_{(n+2,n+m)}) +$$

$$+\sum_{j=1}^{n} C(t_{n+1}-t_j)A_{n-1}(U_{(n)\setminus\{j\}})A_{m-1}(U_{(n+2,m+n)}) -$$

$$-\sum_{j=2}^{m} C(t_{n+1}-t_{n+j})A_n(U_{(n)})A_{m-2}(U_{(n+2,n+m)\setminus\{n+j\}}).$$

Hence, (3) is true by induction for $m-1$, $m-2$ and any n, and therefore

$$A_n(U_{(n)})A_m(U_{(n+1,m+n)}) =$$

$$= \sum_{l=0}^{m-1} \sum_{\substack{\mathcal{K}\subseteq(n+1)\\ \mathcal{L}\subseteq(n+2,n+m)\\ |\mathcal{K}|=|\mathcal{L}|=l}} \sum_{\mathcal{P}\mathcal{L}} C(t_{\mathcal{K}}-t_{\mathcal{P}\mathcal{L}})A_{n+m-2l}(U_{(n+m)\setminus\{\mathcal{K}\cup\mathcal{L}\}}) +$$

$$+\sum_{j=1}^{n}\sum_{l=0}^{m-1} \sum_{\substack{\mathcal{K}\subseteq(n)\setminus\{j\}\\ \mathcal{L}\subseteq(n+2,n+m)\\ |\mathcal{K}|=|\mathcal{L}|=l}} \sum_{\mathcal{P}\mathcal{L}} C(t_{n+1}-t_j)C(t_{\mathcal{K}}-t_{\mathcal{P}\mathcal{L}})$$

$$A_{n+m-2(l+1)}(U_{(n+m)\setminus(\mathcal{K}\cup\mathcal{L}\cup\{j,n+1\})}) -$$

$$-\sum_{j=2}^{m}\sum_{l=0}^{m-2} \sum_{\substack{\mathcal{K}\subseteq(n)\\ \mathcal{L}\subseteq(n+2,n+m)\setminus\{n+j\}\\ |\mathcal{K}|=|\mathcal{L}|=l}} \sum_{\mathcal{P}\mathcal{L}} C(t_{n+1}-t_{n+j})C(t_{\mathcal{K}}-t_{\mathcal{P}\mathcal{L}})$$

$$A_{n+m-2(l+1)}(U_{(n+m)\setminus(\mathcal{K}\cup\mathcal{L}\cup\{n+1,n+j\})}).$$

Since the last term equals the first one when $k_l=n+1$, $l=1,2,\dots$ in it, and the second term corresponds to the case when $\mathcal{K}\subseteq(n)$, $\mathcal{L}\subseteq(n+1,n+m)$, $(\mathcal{L})_l=n+1$ which is missing from the first one, we have got (4).

The quadratic model and its approximate Wiener expansion

The ARMA model with quadratic terms i.e.

$$\sum_{k=0}^{P} a_k y_{t-k} = \sum_{k=0}^{Q} b_k w_{t-k} + \varepsilon \sum_{k,l=1}^{R} f_{k,l}\, y_{t-k}y_{t-l} + K, \tag{5}$$

$$a_0 = b_0 = 1$$

will be referred to as ARMAQ (P, Q, R) model where $Ey_t=0$ and $\{w_t\}$ is a Gaussian independent sequence with $Ew_t=0$, $Ew_t^2=\sigma^2$ and stochastic spectral measure $W(d\lambda)$. Since the product $y_{t-k}y_{t-l}$ is commutative we can assume that the polynomial $F(Z_1, Z_2)=\sum f_{k,l} Z_1^k Z_2^l$ is symmetric without any restriction of the general situation. In equation (5) the ε occurs as a parameter being as small as it is necessary and by the perturbation idea one can calculate the asymptotic Wiener expansion of the solution by considering a sequence of equations for the transfer functions. Consider a sequence $\{\varphi_n(\varepsilon)\}$ $n=1, 2, \dots$ of functions of ε such that $\varphi_{n+1}(\varepsilon)=\mathcal{O}(\varepsilon)\varphi_n(\varepsilon)$ as $\varepsilon\to 0$. Now we are looking for a stationary regular solution of (5) which is subordinated to the w_t in the form

$$(6) \qquad y_t = \sum_{n=1}^{\infty} \frac{\varphi_n(\varepsilon)}{n!} \int_{D^n} g_n(\lambda_{(n)}) e^{it\sum_1^n \lambda_j} W(d\lambda_{(n)}), \quad D^n = [-\pi, \pi]^n.$$

Let the transfer functions $g_n(\lambda_{(n)})$ be symmetric and denote by $\|g_n(\lambda_{(n)})\|$ their L^2 norm. The covariance function of y_t is

$$C_y(t-s) = \operatorname{cov}(y_t, y_s) =$$

$$= \sum_{n=1}^{\infty} \frac{\varphi_n^2(\varepsilon)}{n!} \int_{D^n} |g_n(\lambda_{(n)})|^2 e^{i(t-s)\sum_1^n \lambda_j} \left(\frac{\sigma^2}{2\pi}\right)^n d\lambda_{(n)}$$

and the variance is

$$C_y^{1/2}(0) = Vy_t = \left(\sum_{n=1}^{\infty} \frac{\varphi_n^2(\varepsilon)}{n!} \|g_n(\lambda_{(n)})\|_G^2\right)^{1/2}$$

where $G(d\lambda)=E|W(d\lambda)|^2=\sigma^2 d\lambda/2\pi$.

First we examine the product $y_t y_s$. If there exists the fourth momentum of y_t then the Wiener expansion of $y_t y_s$ can be written into the form

$$(7) \qquad y_t y_s = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{D^n} d_n(\lambda_{(n)}, \varepsilon, t, s) W(d\lambda_{(n)})$$

and the transfer function system d_n is determined by the transfer functions of y_t. We give the explicit formula for d_n by the help of the diagram formula. It is clear that $d_0=C_y(t-s)$. The first transfer function is

$$d_1(\lambda, \varepsilon, t, s) =$$

$$= \sum_{m=1}^{\infty} e^{is\lambda} \frac{\varphi_m(\varepsilon)\varphi_{m+1}(\varepsilon)}{m!(m+1)!} (m+1)\, m! \int_{D^m} g_{m+1}(\omega_{(m)}, \lambda) g_m(-\omega_{(m)})$$

$$e^{i(s-t)\Sigma\omega_j} G(d\omega_{(m)}) + e^{it\lambda} \sum_{m=1}^{\infty} \frac{\varphi_m(\varepsilon)\varphi_{m+1}(\varepsilon)}{m!(m+1)!} (m+1)! \int_{D^m} g_{m+1}(\omega_{(m)}, \lambda) g_m(-\omega_{(m)})$$

$$e^{i(t-s)\Sigma\omega_j} G(d\omega_{(m)}).$$

The variance of the first term in (7) can be estimated in the following way

$$V\Big(\int_D d_1(\lambda, \varepsilon, t, s) w(d\lambda)\Big) \leqq 2 \sum_{m=1}^{\infty} \frac{\varphi_m(\varepsilon)\varphi_{m+1}(\varepsilon)}{m!} \|g_{m+1}\|_G \|g_m\|_G \leqq 2\mathcal{O}(\varepsilon) K_1 C_y(0)$$

hence by the Cauchy inequality

$$\int_D \Big| \int_{D^m} g_{m+1}(\omega_{(m)}, \lambda) g_m(-\omega_{(m)}) G(d\omega_{(m)}) \Big|^2 G(d\lambda) \leqq$$

$$\leqq \int_{D^{m+1}} |g_{m+1}(\omega_{(m+1)})|^2 G(d\omega_{(m+1)}) \, \|g_m\|_G^2$$

and we assumed that $\|g_{m+1}\|_G / \|g_m\|_G \leqq K_1$ independently of m. The transfer function of second order of $y_t y_s$ is

$$\frac{1}{2!} d_2(\lambda_1, \lambda_2, \varepsilon, t, s) = \varphi^2(\varepsilon) g_1(\lambda_1) g_1(\lambda_2) e^{i(t\lambda_1 + s\lambda_2)} +$$

$$+ \sum_{m=1}^{\infty} e^{it(\lambda_1+\lambda_2)} \frac{\varphi_m(\varepsilon)\varphi_{m+2}(\varepsilon)}{m!\,(m+2)!} \frac{(m+2)(m+1)}{2} m!$$

$$\int_{D^m} g_{m+2}(\omega_{(m)}, \lambda_{(2)}) g_m(-\omega_{(m)}) e^{i(t-s)\Sigma\omega_j} G(d\omega_{(m)}) +$$

$$+ (\text{same terms changing } t \text{ and } s).$$

Put

$$\eta_2 = \varphi_1^2(\varepsilon) g_1(\lambda_1) g_1(\lambda_2) e^{i(t\lambda_1 + s\lambda_2)}$$

we get that the variance

$$V \int_{D^2} \left(\frac{d_2}{2!} - \eta_2 \right) W(d\lambda_{(2)}) \leqq 2^2 \mathcal{O}^2(\varepsilon) K_1^2 C_y(0).$$

In the general case we get the transfer function of $2n^{\text{th}}$ order

$$\frac{1}{2n!} d_{2n}(\lambda_{(2n)}, \varepsilon, t, s) = \tag{8}$$

$$= \sum_{l=1}^{2n-1} \frac{\varphi_l(\varepsilon)}{l!} g(\lambda_{(l)}) e^{it\sum_1^l \lambda_j} \frac{\varphi_{2n-l}(\varepsilon)}{(2n-l)!} g_{2n-l}(\lambda_{(l+1,2n)}) e^{is\sum_{l+1}^{2n} \lambda_j} +$$

$$+ \sum_{r=1}^{n} \sum_{m=n-r+1}^{\infty} e^{i(t\Sigma\lambda_{(r+n)} + s\Sigma\lambda_{(r+n+1,2n)}} \frac{\varphi_m(\varepsilon)\varphi_{m+2r}(\varepsilon)}{m!(m+2r)!}$$

$$\int_{D^{m+r-n}} g_{m+r-n}(\omega_{(m+r-n)}, \lambda_{(r+n)}) g_m(-\omega_{(m+r-n)}, \lambda_{(r+n+1,2n)})$$

$$e^{i(t-s)\Sigma\omega_j} G(d\omega_{(m+r-n)}) \cdot (m+r-n)! \binom{m+2r}{m+r-n} \binom{m}{m+r-n} + (\text{same term}).$$

If we put

$$\eta_{2n} = \sum_{l=0}^{2n-1} \frac{\varphi_l(\varepsilon)\varphi_{2n-l}(\varepsilon)}{l!(2n-l)!} g_l(\lambda_{(l)}) g_{2n-l}(\lambda_{(l+1,2n)}) e^{i(t\sum_1^l \lambda_j + s\sum_{l+1}^{2n} \lambda_j)}$$

then we get the following estimate for the variance

$$V \int_{D^{2n}} \left(\frac{d_{2n}}{2n!} - \eta_{2n}\right) W(d\lambda_{(2n)}) \leqq$$

$$\leqq 2(2n)! \sum_{r=1}^{n} \sum_{m=n-r+1}^{\infty} \frac{\varphi_m(\varepsilon)\|g_m\|_G \varphi_{m+2r}(\varepsilon)\|g_{m+2r}\|_G}{(n-r)!(n+r)!(m+r-n)!} \leqq 2^{2n}(\mathcal{O}(\varepsilon))^{2n} K_1^{2n} C_y(0),$$

taking into consideration that

$$\sum_{r=1}^{n} \frac{(2n)!}{(n-r)!(n+r)!} \sim 2^{2n-1}$$

and

$$\varphi_m \sim \varphi_{m+r-n} \mathcal{O}^{n-r}; \quad \varphi_{m+2r} \sim \varphi_{m+r-n} \mathcal{O}^{2r+n-2}.$$

An analogous result is valid for the terms of $2n+1^{\text{th}}$ order as well i.e.

$$V\left(\int_{D^{2n+1}} \left(\frac{d_{2n+1}}{(2n+1)!} - \eta_{2n+1}\right) W(d\lambda_{(2n+1)})\right) \leqq 2^{2n+1} \mathcal{O}^{2n+1}(\varepsilon) K_1^{2n+1} C_y(0).$$

The following theorem is proved

Theorem 1. *Let the stationary process y_t be defined by*

$$y_t = \sum_{n=1}^{\infty} \frac{\varphi_n(\varepsilon)}{n!} \int_{D^n} g_n(\lambda_{(n)}) e^{it\sum \lambda_j} W(d\lambda_{(n)})$$

where

$$\varphi_{m+1}(\varepsilon) = \mathcal{O}(\varepsilon)\varphi_m(\varepsilon) \quad \text{and} \quad \|g_{m+1}\|_G / \|g_m\|_G \leqq K_1$$

for every m. Moreover let us assume that there exists the fourth momentum of y_t. Then for every ε for which $\mathcal{O}(\varepsilon) < (2K_1)^{-1}$ the following inequality holds

$$V\left(y_t y_s - \sum_{m=1}^{\infty} \int_{D^m} \varphi_m(\lambda_{(m)}, \varepsilon, t, s) W(d\lambda_{(m)})\right) \leqq \mathcal{O}(\varepsilon) C_y(0) \frac{2K_1}{1 - 2\mathcal{O}(\varepsilon) K_1}$$

where

$$\eta_m(\lambda_{(m)}, \varepsilon, t, s) = \sum_{l=1}^{m-1} \frac{\varphi_l(\varepsilon)\varphi_{m-l}(\varepsilon)}{l!(m-l)!} g_l(\lambda_{(l)}) g_{m-l}(\lambda_{(l+1,m)}) e^{i(t\sum_{j=1}^{l} \lambda_j + s\sum_{j=l+1} \lambda_j)}.$$

Let us now turn to the quadratic model (5) and ask for a solution of type (6). If n is fixed then we get an equation for the n^{th} transfer function $g_n(\lambda_{(n)})$. Let us begin with $n=1$

$$\varphi_1(\varepsilon) A(e^{-i\lambda}) g_1(\lambda) = B(e^{-i\lambda}) + \varepsilon \xi_1(\lambda, \varepsilon)$$

where

$$A(Z) = \sum a_k Z^k, \quad B(Z) = \sum b_k Z^k, \quad F(Z_1, Z_2) = \sum f_{kl} Z^k Z_2^l, \quad |F| = \sum |f_{k,l}|$$

and

$$\xi_1 = \sum_{k,l=1}^{R} f_{k,l} d_1(\lambda, \varepsilon, -k, -l)$$

If ε tends to zero we see that $\varphi_1(\varepsilon)=1$ therefore

$$g_1(\lambda)=\frac{B(e^{-i\lambda})}{A(e^{-i\lambda})}+\varepsilon\frac{\xi_1(\lambda,\varepsilon)}{A(e^{-i\lambda})}.$$

We shall show later on that $\varphi_n(\varepsilon)$ can be chosen as

$$\varepsilon^{n-1}\quad\text{so}\quad \varphi_{n+1}(\varepsilon)=\mathcal{O}(\varepsilon)\varphi_n(\varepsilon)\quad\text{and}\quad \mathcal{O}(\varepsilon)=\varepsilon.$$

From this follows that

$$\left\|g_1(\lambda)-\frac{B(e^{-i\lambda})}{A(e^{-i\lambda})}\right\|_G\leqq \varepsilon^2 2K_1C_y(0)\frac{|F|}{|A|}$$

where

$$|F|=\sum_{k,l=1}^{R}|f_{k,l}|\quad\text{and}\quad |A|=\prod_{j=1}^{P}(1-|\alpha_j|),$$

α_j denotes the roots of the polynomial

$$A_1(z)=\sum_{k=1}^{P}a_kz^{P-k}.$$

These are assumed to be less then 1.

Let us consider the case of the second order transfer function. From (5) one gets

$$A(e^{-i(\lambda_1+\lambda_2)})\varphi_2(\varepsilon)\frac{g_2(\lambda_{(2)})}{2!}=\varepsilon F(e^{-i\lambda_1},e^{-i\lambda_2})g_1(\lambda_1)g_1(\lambda_2)+\varepsilon\xi_2(\lambda_{(2)},\varepsilon)$$

therefore $\varphi_2(\varepsilon)=\varepsilon$ and

$$\frac{g_2(\lambda_{(2)})}{2!}=\varepsilon\frac{F(e^{-i\lambda_1},e^{-i\lambda_2})}{A(e^{-i(\lambda_1+\lambda_2)})}g_1(\lambda_1)g_1(\lambda_2)+\varepsilon\frac{\xi_2(\lambda_{(2)},\varepsilon)}{A(e^{-i(\lambda_1+\lambda_2)})}.$$

Hence

$$\xi_2(\lambda_{(2)},\varepsilon)=\sum_{k,l=1}^{R}f_{k,l}\left(\frac{1}{2!}d_2(\lambda_{(2)},\varepsilon,-k,-l)-g_1(\lambda_1)g_1(\lambda_2)e^{-i(k\lambda_1+l\lambda_2)}\right).$$

We can approximate $g_2(\lambda_{(2)})$ by the first term i.e.

$$\left\|\frac{g_2(\lambda_{(2)})}{2!}-\varepsilon\frac{F(e^{-i\lambda_1},e^{-i\lambda_2})}{A(e^{-i(\lambda_1+\lambda_2)})}g_1(\lambda_1)g_1(\lambda_2)\right\|_G\leqq\varepsilon^2 2^2K_1^2C_y(0)\frac{|F|}{2!|A|}.$$

Let now n be odd. Then by the equation (5) and from (8) we get

$$A(e^{-i\Sigma\lambda_j})\varphi_n(\varepsilon)\frac{g_n(\lambda_{(n)})}{n!}=$$

$$=\varepsilon^{n-1}\sum_l\frac{g_l(\lambda_{(l)})}{l!}\frac{g_{n-l}(\lambda_{(l+1,n)})}{(n-l)!}F\Big(e^{-i\sum_1^l\lambda_j},e^{-i\sum_{l+1}^n\lambda_j}\Big)+\varepsilon\xi_n(\lambda_{(n)},\varepsilon)$$

where $\xi_n(\lambda_{(n)},\varepsilon)$ is defined by (8) and $F(Z_1,Z_2)$. By the inductive assumption that $\varphi_k(\varepsilon)=\varepsilon^{k-1}$, $k=1,\dots,n-1$, this implies $\varphi_n(\varepsilon)=\varepsilon^{n-1}$. The transfer function of n^{th} order is determined by the transfer functions of order less than n up to the $\mathcal{O}(\varepsilon^2)$ i.e.

$$\left\| \frac{g_n(\lambda_{(n)})}{n!} - \sum_{l=1}^{n-1} \frac{g_l(\lambda_{(l)})}{l!} \frac{g_{n-l}(\lambda_{(l+1,n)})}{(n-l)!} \frac{F(e^{-i\sum_1^l \lambda_j}, e^{-i\sum_{l+1}^n \lambda_j})}{A(e^{-i\sum_1^n \lambda_j})} \right\| \leqq \varepsilon^2 \frac{2^n K_1^n}{n!} C_y(0) \frac{|F|}{|A|}.$$

Let us now denote the new transfer functions by g_n^*. They are determined by induction

$$g_1^*(\lambda) = \frac{B(e^{-i\lambda})}{A(e^{-i\lambda})}$$

and

$$\frac{g_n^*(\lambda_{(n)})}{n!} = \sum_{l=1}^{n-1} \frac{g_l^*(\lambda_{(l)})}{l!} \frac{g_{n-l}^*(\lambda_{(l+1,n)})}{(n-l)!} \frac{F(e^{-i\sum_1^l \lambda_j}, e^{-i\sum_{l+1}^n \lambda_j})}{A(e^{-i\sum_1^n \lambda_j})}. \tag{9}$$

If we define a process by the help of the transfer functions g_n^* in the following way

$$y_t^* = \sum_{n=1}^{\infty} \varepsilon^{n-1} \int_{D^n} \frac{g_n^*}{n!}(\lambda_{(n)}) e^{it\sum_1^n \lambda_j} W(d\lambda_{(n)}) \tag{10}$$

and ε is small enough then it will be close to the solution y_t of the equation (5) i.e.

$$V(y_t - y_t^*) \leqq \varepsilon^2 \frac{2K_1}{1-2\varepsilon K_1} C_y(0) \frac{|F|}{|A|}. \tag{11}$$

Thus we have got

Theorem 2. *Let all the roots of the characteristic polynomial of the* AR *part of the equation* (5) *be inside the unit circle. If there exists a stationary solution* y_t *of equation* (5) *in the form* (6) *with the transfer functions* g_n *such that* $\|g_{n+1}\|_G / \|g_n\|_G < K_1$ *and* $\varphi_n(\varepsilon) = \varepsilon^{n-1}$ *then the stationary process* y_t^* *defined by* (10) *is an* $\mathcal{O}(\varepsilon^2)$ *approximate solution uniformly in t in the sense* (11) *for every* ε *less than* $(2K_1)^{-1}$.

A condition for stationarity

In the previous section we assumed that there exists a stationary solution for the quadratic ARMAQ (P, Q, R) model (5)

$$\sum_{k=0}^{P} a_k y_{t-k} = \sum_{k=0}^{Q} b_k w_{t-k} + \varepsilon \sum_{k,l=1}^{R} f_{k,l} y_{t-k} y_{t-l} + K, \quad a_0 = b_0 = 1.$$

Now we start with the definition of the approximative transfer functions g_n^* and we examine the square mean convergence of their series. Let

$$g_1^*(\lambda) = \frac{B(\lambda)}{A(\lambda)}$$

and by recursion

$$g_n^*(\lambda_{(n)}) = \sum_{l=1}^{n-1} \frac{g_l^*(\lambda_{(l)})}{l!} \frac{g_{n-l}^*(\lambda_{(l+1,n)})}{(n-l)!} \frac{F(\sum_{j=1}^{l} \lambda_j, \sum_{j=l+1}^{n} \lambda_j)}{A(\sum_{j=1}^{n} \lambda_j)}$$

where

$$A(\lambda) = \sum_{k=0}^{P} a_k e^{-i\lambda k}, \quad B(\lambda) = \sum_{k=0}^{Q} b_k e^{-ik\lambda}$$

$$F(\lambda, \mu) = \sum_{k,l=1}^{R} f_{k,l} e^{-i(k\lambda+l\mu)}.$$

To get an assumption for the existence of the variation of the approximative solution y_t^* we need the following estimation for the square norm of $\operatorname{sym} g_n^*$.

Lemma. *Let*

$$F_0 = \sum_{k,l=1}^{R} |f_{k,l}|, \quad |A| = \prod_{j=1}^{P} (1-|\alpha_j|)$$

and g_n be defined as above, where α_j denotes the root of the polynomial

$$A_0(Z) = \sum_{k=0}^{P} a_k Z^{P-k}.$$

Then

$$\|\operatorname{sym} g_n^*(\lambda_{(n)})\|_G^2 \leqq \left[\frac{F_0}{|A|}\right]^{2(n-1)} \|g_1^*\|_G^{2n} \varrho^{2n}$$

where $\varrho = \sup_n \frac{2^n}{\sqrt{n!}} = \frac{8}{\sqrt{6}}$.

Proof. In case $n=2$

$$g_2^*(\lambda_{(2)}) = \frac{B(\lambda_1)B(\lambda_2)}{A(\lambda_1)A(\lambda_2)} \cdot \frac{F(\lambda_1, \lambda_2)}{A(\lambda_1, \lambda_2)}$$

and

$$\|\operatorname{sym} g_2^*\|_G^2 = \|g_2^*\|_G^2 \leqq 2\left[\frac{F_0}{|A|}\right]^2 \|g_1\|_G^4.$$

For $n=3$

$$g_3^*(\lambda_{(3)}) = \prod_{j=1}^{3} \frac{B(\lambda_j)}{A(\lambda_j)} \cdot \frac{1}{A(\sum_{j=1}^{3} \lambda_j)}$$

$$\cdot \frac{F(\lambda_1, \lambda_2) F(\lambda_1+\lambda_2, \lambda_3)}{2!\, A(\lambda_1+\lambda_2)} + \frac{F(\lambda_2, \lambda_3) F(\lambda_1, \lambda_2+\lambda_3)}{A(\lambda_2+\lambda_3)}$$

so

$$\|\text{sym } g_3^*\|_G \leq \frac{F_0}{|A|} \frac{1}{\sqrt{3!}} \left[\binom{3}{2} \|g_2\|_G \|g_1\|_G + \binom{3}{2} \|g_2\|_G \|g_1\|_G \right] \leq$$

$$\leq \left[\frac{F_0}{|A|}\right]^2 \|g_1\|^3 \cdot \frac{2^3}{\sqrt{3!}} \frac{2^2}{\sqrt{2!}}.$$

The proof for arbitrary n will be the same as in the case $n=4$.

$$\|\text{sym } g_4^*\|_G \frac{F_0}{|A|} \left(\binom{4}{1} \|g_1^*\|_G \|g_3^*\|_G + \binom{4}{2} \|g_2^*\|_G^2 + \binom{4}{3} \|g_1^*\|_G \|g_3^*\|_G \right) \frac{1}{\sqrt{4!}} \leq$$

$$\leq \left[\frac{F_0}{|A|}\right]^3 \|g_1^*\|_G^4 \frac{2^4}{\sqrt{4!}} \varrho^2.$$

Theorem 3. *Let us assume that all the roots of the polynomial A_0 are inside the unit circle, the solution y_t of the* ARMAQ (P, Q, R) *model* (5) *depends only on w_{t-s} $s=0, 1, 2, \ldots$, moreover its transfer function g_n fulfils $\|g_n\|_G / \|g_{n-1}\|_G \leq K_1$ and $\varepsilon \leq (2K_1)^{-1}$. Then for the existence of a stationary solution for the* ARMAQ (P, Q, R) *model* (5) *it is suffcient that*

$$\varepsilon \varrho \frac{F_0}{|A|} \|g_1\|_G < 1.$$

FOR THE PROOF of this theorem it is enough to consider the inequality

$$V(y_t) \leq V(y_t^*) + V(y_t - y_t^*)$$

where

$$y_t^* = \sum_{n=1}^{\infty} \varepsilon^{n-1} \frac{1}{n!} \int_{D^n} g_n^*(\lambda_{(n)}) e^{it \sum_{j=1}^{n} \lambda_j} W(d\lambda_{(n)}),$$

and to take into consideration (11) and the previous lemma.

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