A linear optimization problem and its probabilistic application

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Summary. A linear program is solved in closed form and applied to obtain a sharp upper bound for $\min P(A_{i_1}\cup\ldots\cup A_{i_m})$, where $A_1, \ldots, A_n$ are events in an arbitrary probability space satisfiyng $P(A_i)\leqq\alpha$ $(i=1, \ldots, n)$, $1\leqq m<n$, $\alpha\in[0, 1]$ and the minimum is taken with respect to all $(i_1, \ldots, i_m)$ such that $1\leqq i_1<\ldots<i_m\leqq n$. The bound is $1-\binom{n-m}{[\alpha n]}(mn(1-\alpha)-(n-[\alpha n])(m-1))/(n-[\alpha n])\binom{n}{[\alpha n]}$.

Let $\beta\in[0, 1]$, $m, n\in\mathbf{N}$, $m<n$. The following linear program (LP) will be solved in closed form:

$$\text{(1)}\qquad \text{Minimize} \sum_{k=0}^{n-m}\binom{n-m}{k}t_k =: F(t_0, \ldots, t_n)$$

subject to the conditions

$$\text{(2)}\qquad t_k\geqq 0 \quad\text{for}\quad k=0, \ldots, n,\ \sum_{k=0}^{n}\binom{n}{k}t_k=1,\quad \sum_{k=0}^{n}\binom{n-1}{k}t_k\geqq\beta.$$

The result is given by

Theorem 1. *The minimal value of the above* LP *is equal to*

$$\text{(3)}\qquad f(\alpha, m, n)=\frac{\binom{n-m}{[\alpha n]}}{(n-[\alpha n])\binom{n}{[\alpha n]}}(mn(1-\alpha)-(n-[\alpha n])(m-1)),$$

where $\alpha=1-\beta$ *and* $[\alpha n]$ *is the integer part of* αn. *It is attained at* $t^*=(t_0^*, \ldots, t_n^*)$, *where*

$$\text{(4)}\qquad t_i^*=\begin{cases}\binom{n}{[\alpha n]}^{-1}(1+[\alpha n]-\alpha n), & \text{if}\quad i=[\alpha n]\\ \binom{n}{[\alpha n]+1}^{-1}(\alpha n-[\alpha n]), & \text{if}\quad i=[\alpha n]+1\\ 0, & \text{if}\quad i\notin\{[\alpha n], [\alpha n]+1\}.\end{cases}$$

The LP under consideration occurs in the following problem of combinatorial probability theory: Let $A_1, \ldots, A_n$ be events in some probability space about which nothing is known except that $P(A_i) \leqq \alpha$ for $i=1, \ldots, n$. Find an upper bound for

$$g(\alpha, m, n) := \min_{1 \leqq i_1 < \ldots < i_m \leqq n} P(A_{i_1} \cup \ldots \cup A_{i_m}). \tag{5}$$

Theorem 2. *We have*

$$g(\alpha, m, n) \leqq 1 - f(\alpha, m, n). \tag{6}$$

For all values of α, m and n there is a probability space with events $A_1, \ldots, A_n$ satisfying $P(A_i) \leqq \alpha$ for $i=1, \ldots, n$ such that there is equality in (6).

The connection between certain LPs and bounds for probabilities of logical functions of events has been noted by Hailperin (1965). Related inequalities have been recently proved by Rüger (1981) and Morgenstern (1980). They consider n level α tests with critical regions $A_1, \ldots, A_n$ and look for upper and lower bounds for the level of the compound test "reject iff at least m tests reject". Our inequality can also be interpreted in this context. Related bounds for the probability of the occurrence of at least (or exactly) m of n events under different side conditions have been derived, e.g., by Erdős et al. (1983), Kwerel (1975) or, more recently, by Platz (1985) and Móri and Székely (1985); see also the references in this latter paper.

We need the following estimate for binomial coefficients.

Lemma. *Let $1 \leqq m < n$, $0 \leqq k \leqq n$, $0 \leqq i \leqq n$. Then*

$$mn\binom{n-m}{i}\binom{n-1}{k} \leqq (n-i)\binom{n}{i}\binom{n-m}{k} + (n-i)(m-1)\binom{n-m}{i}\binom{n}{k}. \tag{7}$$

Proof. We define $f(x) := n-x$, $g(x) := \prod_{j=0}^{m-1} (n-j-x)$, $x \in \mathbf{R}$.

Dividing (7) by $\binom{n}{k}$ shows that we have to prove

$$m\binom{n-m}{i} f(k) \leqq \frac{(n-i)\binom{n}{i}(n-m)!}{n!} g(k) + (n-i)(m-1)\binom{n-m}{i}. \tag{8}$$

g is convex in the interval $(-\infty, n-m+1)$. It is easily computed that both sides of (8) are equal for $k=i$ and for $k=i+1$. If $i \leqq n-m$, this implies (8) for $k \leqq n-m$. As f is monotone decreasing and $g(k)=0$ for $k > n-m$, (8) now also follows for $i \leqq n-m$ and $k=0, 1, \ldots, n$. But if $i > n-m$, (8) is trivially true.

Proof of Theorem 1. We have to transform F appropriately. Let $0 \leqq i \leqq n$. Then it is easily checked that

$$F(t_0, \ldots, t_n) = \sum_{\substack{k=0 \\ k \neq i, i+1}}^{n-m} \binom{n-m}{k} t_k + \binom{n-m}{i} t_i + \binom{n-m}{i+1} t_{i+1} = \tag{9}$$

$$= \sum_{\substack{k=0 \\ k \neq i, i+1}}^{n-m} \binom{n-m}{k} t_k + \frac{\binom{n-m}{i}}{\binom{n-1}{i}} m \left[\binom{n-1}{i} t_i + \binom{n-1}{i+1} t_{i+1} \right] -$$

$$- \frac{\binom{n-m}{i}}{\binom{n}{i}} (m-1) \left[\binom{n}{i} t_i + \binom{n}{i+1} t_{i+1} \right] =$$

$$= \sum_{\substack{k=0 \\ k \neq i, i+1}}^{n} \left[\binom{n-m}{k} - \frac{m \binom{n-m}{i} \binom{n-1}{k}}{\binom{n-1}{i}} + \frac{(m-1) \binom{n-m}{i} \binom{n}{k}}{\binom{n}{i}} \right] t_k +$$

$$+ \frac{\binom{n-m}{i}}{\binom{n-1}{i}} m \sum_{k=0}^{n-1} \binom{n-1}{k} t_k - \frac{\binom{n-m}{i}}{\binom{n}{i}} (m-1) \sum_{k=0}^{n} \binom{n}{k} t_k .$$

We denote by Q the polyhedron defined by the inequalities (2). By the lemma, the expression in square brackets on the right-hand side of (9) is nonnegative. Thus if $(t_0, \ldots, t_n) \in P$, (9) yields

$$F(t_0, \ldots, t_n) \leqq \frac{\binom{n-m}{i}}{\binom{n-1}{i}} m(1-\alpha) + \frac{\binom{n-m}{i}}{\binom{n}{i}} (m-1) \tag{10}$$

for every $i \in \{0, \ldots, n\}$. Now assume that there are a point $(t_0^*, \ldots, t_n^*) \in Q$ and a $i \in \{0, \ldots, n\}$ satisfying

$$\sum_{k=0}^{n-1} \binom{n-1}{k} t_k^* = 1 - \alpha \quad \text{and} \quad k \notin \{i, i+1\} \Rightarrow t_k^* = 0. \tag{11}$$

Then clearly F attains its maximum on Q in $(t_0^*, \ldots, t_n^*)$ and we have

$$f(\alpha, m, n) = F(t_0^*, \ldots, t_n^*) = \frac{\binom{n-m}{i}}{\binom{n-1}{i}} m(1-\alpha) + \frac{\binom{n-m}{i}}{\binom{n}{i}} (m-1). \tag{12}$$

The point t^* defined by (4) clearly satisfies (2) and (11). The theorem is proved.

PROOF OF THEOREM 2. Let $N := \{1, \dots, n\}$ and $P(A_i) \leqq \alpha$ for $i \in N$. If $T \subset N$, we set

$$A(T) := \bigcap_{j \in T} A_j \cap \bigcap_{j \in T^c} A_j^c.$$

We define a probability measure $\tilde{P}$ on the σ-algebra which is generated by the sets $A(T)$, $T \subset N$, by

$$\tilde{P}(A(T)) := \frac{1}{n!} \sum_{\sigma \in \gamma_n} P(A(\sigma(T))),$$

where γ_n is the set of all permutations $\sigma: N \to N$. We have

$$\tilde{P}(A_i) = \frac{1}{n!} \sum_{\sigma} \sum_{T: i \in T} P(A(\sigma(T))) = \frac{1}{n!} \sum_{\sigma} P(A_{\sigma(i)}) \leqq \alpha, \quad i \in N,$$

and, if we set $M(i_1, \dots, i_m) := \{T \subset N | T \cap \{i_1, \dots, i_m\} \neq \emptyset\}$, for $1 \leqq i_1 < \dots < i_m \leqq n$,

$$\begin{aligned}\tilde{P}(A_{i_1} \cup \dots \cup A_{i_m}) &= \frac{1}{n!} \sum_{\sigma \in \gamma_n} \sum_{T \in M(i_1, \dots, i_m)} P(A(\sigma(T))) = \\ &= \frac{1}{n!} \sum_{\sigma \in \gamma_n} \sum_{S \in M(\sigma i_1, \dots, \sigma i_m)} P(A(S)) \geqq \\ &\geqq \min_{j_1 < \dots < j_m} \sum_{S \in M(j_1, \dots, j_m)} P(A(S)) = \\ &= \min_{j_1 < \dots < j_m} P(A_{j_1} \cup \dots \cup A_{j_m}).\end{aligned}$$

Thus,

$$\min_{i_1 < \dots < i_m} \tilde{P}(A_{i_1} \cup \dots \cup A_{i_m}) \geqq \min_{i_1 < \dots < i_m} P(A_{i_1} \cup \dots \cup A_{i_m}). \tag{13}$$

To prove (6) we can therefore restrict our attention to probability measures P and events $A_1, \dots, A_n$ for which $P(A(T))$ only depends on the number of elements card T of T. Define $t_k := P(A(T))$, if card $T = k$, $k = 0, 1, \dots, n$. Then

$$1 = \sum_{T \subset N} P(A(T)) = \sum_{k=0}^{n} \binom{n}{k} t_k \tag{14}$$

and for $j \in N$, $1 \leqq i_1 < \dots < i_j \leqq n$,

$$P(A_{i_1} \cup \dots \cup A_{i_j}) = 1 - \sum_{T: i_1, \dots, i_j \notin T} P(A(T)) = 1 - \sum_{k=0}^{n-j} \binom{n-j}{k} t_k. \tag{15}$$

By (14) and (15), the result follows immediately from theorem 1. All $(t_0, \dots, t_n) \in Q$ can obviously be represented as values $P(A(T))$ of a certain probability measure on a probability space with certain events $A_1, \dots, A_n$. Thus the inequality (6) is sharp.

Remark. An example of a probability space and events for which equality in (6) holds can be constructed as follows. Let $\Omega:=\{\omega_T|T\subset N\}$, where the ω_T are distinct elements, and

$$P(\{\omega_T\}):=t_i^*, \quad \text{if} \quad \text{card}\, T=i\in\{[\alpha n], [\alpha n]+1\}$$
$$P(\{\omega_T\}):=0, \quad \text{if} \quad \text{card}\, T\notin\{[\alpha n], [\alpha n]+1\}$$
$$A_i:=\{\omega_T|i\in T\}, \quad i=1, \dots, n.$$

Clearly the probability space can be reduced by eliminating points with probability 0 so that we obtain a space with $\binom{n}{[\alpha n]}+\binom{n}{[\alpha n]+1}$ points. If αn is integer, we can take the equidistribution on a set with $\binom{n}{\alpha n}$ elements.

References

[1] P. Erdös, J. Neveu and A. Rényi, An elementary inequality between the probability of events, *Math. Scand.* **13** (1963), 99—104.
[2] T. Hailperin, Best possible inequalities for the probability of a logical function of events, *Am. Math. Monthly* **72** (1965), 343—359.
[3] S. M. Kwerel, Most stringent bounds on aggregated probabilities of partially specified dependent probability systems, *J. Amer. Statist. Ass.* **70** (1975), 472—479.
[4] D. Morgenstern, Berechnung des maximalen Signifikanzniveaus des Testes „Lehne H_0 ab, wenn k unter n gegebenen Tests zur Ablehnung führen“, *Metrika* **27** (1980), 285—286.
[5] T. F. Móri and G. J. Székely, A note on the background of several Bonferroni—Galambos-type inequalities, *J. Appl. Prob.* **22** (1985), 836—843.
[6] O. Platz, A sharp upper probability bound for the occurrence of at least m of n events, *J. Appl. Prob.* **22** (1985), 978—981.
[7] B. Rüger, Scharfe untere und obere Schranken für die Wahrscheinlichkeit der Realisation von k unter n Ereignissen, *Metrika* **28** (1981), 71—77.

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