Some remarks on the twelfth problem of Hanna Neumann

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Introduction

In this paper we give a partial answer to the twelfth problem of HANNA NEUMANN from her book [1]. Namely we prove that the centres of c-generated, c-nilpotent relatively free groups F_c of some varieties are equal to the last term of their lower central series $\gamma_c(F_c)$ if and only if all torsion elements of these groups lie in $\gamma_c(F_c)$.

Our work consists of 3 paragraphs: the first is devoted to the 3-nilpotent varieties, the second to metabelian varieties of "small nilpotency" and the third to some 4-nilpotent varieties.

We use the standard notation for commutators i.e.

$$[x, y] = x^{-1}\cdot y^{-1}\cdot x\cdot y, \quad [x, ky] = [[x, (k-1)y], y] \quad \text{for} \quad k > 1.$$

$Z(G)$ denotes as usual the centre of G, $\gamma_c(G)$ the c^{th} term of its lower central series. For an arbitrary prime p we define the function $t_p\colon N \to N\cup 0$ as follows:

$$t_p(n) = \max\{\alpha\colon p^\alpha | n\}.$$

We assume that the reader is familiar with the book of H. Neumann. Other necessary facts are briefly recapitulated at the begining of each paragraph.

§ 1. The centres of 3-nilpotent groups

The following theorem gives the full classification of 3-nilpotent varieties i.e. of the varieties with the identity $[x_1, x_2, x_3, x_4]=1$. All incompletely bracketed commutators are to be read as "left-normed". Theorem: B. JONSSON and V. N. REMESLENNIKOV [9], [11].

There is a $1-1$ correspondence between the quadruples (m, n, p, q) satisfying the conditions:

1) $n\cdot \gcd(2, m)|m$,

2) $p|n$,

3) $q|p$,

4) $q\cdot \gcd(6, m)|m$,

5) $p|3q$,

and the 3-nilpotent varieties. In other words every 3-nilpotent variety is given by laws of the form

$$x^m = [x_1, x_2]^n = [x_1, x_2, x_3]^p = [x_1, x_2, x_2]^q = [x_1, x_2, x_3, x_4] = 1$$

with m, n, p, q satisfying the conditions as above.

Using this result we obtain the following

Theorem 1. *Let* $\mathbf{V}$ *be a 3-nilpotent variety defined by the quadruple* (m, n, p, q). *The centre* $Z(F_3)$ *of a 3-generated relatively free group from this variety coincides with the verbal subgroup given by the words:* $x_1^{\beta_1}$, $[x_1, x_2]^{\beta_2}$, $[x_1, x_2, x_3]$ *where* $\beta_1 = n$, $\beta_2 = p$ *except the case when* t_2 $n = t_2$, $q \neq 0$. *Then* $\beta_1 = 2n$ *and* $\beta_2 = p$.

Proof. Observe that the centre $Z(F_3)$ is verbal, because it is a fully invariant subgroup of the relatively free group F_3. Since the quotient group by the centre must lie in a variety of the same type and $\gamma_3(F_3)$ is obviously in $Z(F_3)$, we can assume that the generating words are of the form

$$x_1^{\beta_1}, \quad [x_1, x_2]^{\beta_2}, \quad x_1, x_2, x_3.$$

For $x_1^{\beta_1}$ and $[x_1, x_2]^{\beta_2}$ lying in the centre the following identities must hold in the factor group $F_3/Z(F_3)$:

$$(1) \qquad \begin{aligned} 1 &= [x_1^{\beta_1}, x_2] = [x_1, x_2]^{\beta_1} [x_1, x_2, x_2]^{\binom{\beta_1}{2}} \\ 1 &= \big[[x_1, x_2]^{\beta_2}, x_3\big] = [x_1, x_2, x_3]^{\beta_2}. \end{aligned}$$

Here we have used the identity:

$$(2) \qquad [x_1, x_2^k] = \prod_{l=1}^{k} [x_1, l x_2]^{\binom{k}{l}}$$

which holds in every metabelian group and which will be used later. From the Jonsson—Remeslennikov theorem we deduce that β_1 and β_2 must satisfy the following equations:

$$(3) \qquad \begin{aligned} \beta_1 &= k_1 n \\ \frac{\beta_1(\beta_1 - 1)}{2} &= k_2 q \\ \beta_2 &= k_3 p \\ k_4 \beta_2 &= \frac{\beta_1(\beta_1 - 1)}{2}. \end{aligned}$$

It is not difficult to see that $\beta_1 = n$ and $\beta_2 = p$ is the sought pair except the case when $t_2(n) = t_2(q) \neq 0$. Then we have $\beta_1 = 2n$ and the system (3) is satisfied since $t_2(p) = t_2(q)$ in view of the Jonsson—Remeslennikov theorem.

§ 2. $[c-2]$-isolated metabelian c-nilpotent varieties and centres of their c-generated relatively free groups

We start with some definitions. Let G be a nilpotent group, H its a subgroup, π a nonempty set of primes. A π-isolator of H is defined as the set $H_\pi=\{x\in G\colon x^m\in H\}$ where m is a π-number i.e. an integer having in its primary decomposition only primes from π. It is known, see HALL [7], that H_π is a subgroup of G. If $H=H_\pi$ we call the subgroup H π-isolated. If $1=1_\pi$, which means that in the group G there are no elements having π-numbers as their orders, then we call the group G π-isolated. Let $\underline{\underline{V}}$ be a variety of groups. If all relatively free groups of $\underline{\underline{V}}$ are π-isolated, then we call $\underline{\underline{V}}$ π-isolated too. Now let $[c-2]$ be the set of primes not greater than $c-2$. The following theorem was announced by Yu. A. BELOV [2] without proof. It follows from the results of W. BRISLEY [3], [4].

Theorem. *There is $1-1$ corespondence between the varieties of c-nilpotent metabelian groups whose free groups are $[c-2]$-isolated and $c+2$-tuples $\alpha_1, \dots, \alpha_{c+2}$ satisfying the conditions:*

1) $\alpha_{i+1}|\alpha_i \quad i=1, \dots, c+1$

2) $\alpha_{c-1}|(c-1)\alpha_c$

3) $\alpha_c\left|\binom{\alpha_1}{c-1}\right.$

4) $\alpha_{c+1}|c\alpha_{c+2}$

5) $\alpha_{c+2}\left|\binom{\alpha_1}{c}\right.$.

In other words the basis of such a variety is

$$x_1^{\alpha_1}=[x_1, x_2]^{\alpha_2}=[x_1, x_2, x_3]^{\alpha_3}=\dots=[x_1, \dots, x_{c-1}]^{\alpha_{c-1}}=$$
$$=[x_2, (c-2)x_1]^{\alpha_c}=[x_1, \dots, x_c]^{\alpha_{c+1}}=[x_2, (c-1)x_1]^{\alpha_{c+2}}=$$
$$=[x_1, x_2, \dots, x_{c+1}]=1,$$

with the same conditions as above. The centres of relatively free groups in these varieties are described in the following.

Theorem 2. *Let $\underline{\underline{V}}$ be a c-nilpotent metabelian variety defined by a $(c+2)$-tuple. The centre of a c-generated relatively free group F_c from this variety coincides with the verbal subgroup given by the words:*

(4) $$x_1^{\beta_1},\ [x, y]^{\beta_2}, \dots, [x, (c-2)y]^{\beta_{c-1}},\ [x_1, \dots, x_c]$$

where $\beta_1=\alpha_2$, $\beta_2=\alpha_3, \dots, \beta_{c-3}=\alpha_{c-2}$, $\beta_{c-2}=\alpha_{c-1}$, $\beta_{c-1}=\alpha_{c+1}$ except the case when $t_{(c-1)}(\alpha_2)=t_{(c-1)}(\alpha_{c+2})\neq 0$, then $\beta_1=(c-1)\alpha_2$, other β_i are as above.

PROOF. First of all observe that if there are elements of order n in an $[n-1]$-isolated group, then n must be a prime. The factor group $F_c/Z(F_c)$ of a nilpotent group of class c is at most $(c-1)$-nilpotent. A simple modification of Brisley's considerations [3], [4] gives us a basis of identities of a relatively free $c-1$-nilpotent, metabelian, $[c-2]$-isolated group in the form:

$$x^{\beta_1},\ [x, y]^{\beta_2},\ [x, y, y]^{\beta_3},\ \dots, [x, (c-2)y]^{\beta_{c-1}},$$

(5) $$[x, (c-3)y, z]^{\beta_c}.$$

The group $F_c/Z(F_c)$ is also $[c-2]$-isolated in view of the following result due to P. HALL [7]:

Let G be a locally nilpotent group without nontrivial π-elements (π is an arbitrary set of primes). If C is any centralizer or any term of the upper central series of G, then $C=C_\pi$.

On the other hand it follows from the result of GUPTA and NEWMAN ([6] lemma 2) that in a metabelian $(c-1)$-nilpotent group G, the identity being $[c-2]$-isolated $[x, (c-3)y, z]^\beta=1$ implies $(\gamma_{c-1}(G))^\beta=1$ i.e. $[x_1, \dots, x_{c-1}]^\beta=1$. So if $[[x, (c-2)y]^{\beta_{c-1}}, z]=[x, (c-2)y, z]^{\beta_{c-1}}=1$ then $[[x, c-3y, z]^{\beta_c}, t]= =[x, (c-3)y, z, t]^{\beta_c}=1$ which implies $\beta_{c-1}=\beta_c$ and consequently we can assume that the words which generate the centre are of the form (4).

We determine now the integers $\beta_1, \dots, \beta_{c-1}$ in the same manner as in § 1.

We have the identities:

$$1 = [y^{\beta_1}, x] = [y, x]^{\beta_1}[y, x, x]^{\binom{\beta_1}{2}} \dots [y, (c-1)x]^{\binom{\beta_1}{c-1}}$$

(6) $$1 = \big[[x, y]^{\beta_2}, z\big] = [x, y, z]^{\beta_2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$1 = \big[[x, (c-2)y]^{\beta_{c-1}}, z\big] = [x, (c-2)y, z]^{\beta_{c-1}}.$$

In the first equality of (6) we have used the identity (2). From (6) we deduce the following system of equations for $\beta_1 \dots \beta_{c-1}$:

$$\beta_1 = k_1\alpha_2$$

$$\frac{\beta_1(\beta_1-1)}{2} = k_2\alpha_3$$

$$\dots\dots\dots\dots\dots\dots$$

(7) $$\frac{\beta_1(\beta_1-1)\dots(\beta_1-c)}{c-1} = k_{c-1}\alpha_{c+2}$$

$$\beta_2 = l_1\alpha_3$$

$$\beta_3 = l_2\alpha_4$$

$$\dots\dots\dots\dots$$

$$\beta_{c-1} = l_{c-2}\alpha_{c+1}$$

$$l\beta_{c-1} = \binom{\beta_1}{c-1}.$$

The last equation of the system (7) follows from the condition 3 of Belov's theorem. It is easy to see that the smallest integer satisfying the system is $\beta_1=\alpha_2$ provided $t_{(c-1)}(\alpha_2)>t_{(c-1)}(\alpha_{c+2})$ or $t_{(c-1)}(\alpha_2)=t_{(c-1)}(\alpha_{c+2})=0$. If $t_{(c-1)}(\alpha_2)= =t_{(c-1)}(\alpha_{c+2})\neq 0$ we take $\beta_1=(c-1)\alpha_2$. The last equation of the system (7) is fulfilled in both cases because $t_{(c-1)}(\alpha_{c+1})=t_{(c-1)}(\alpha_{c+2})$ in view of 5 of Belov's theorem. Clearly we have $\beta_2=\alpha_3\dots\beta_{c-1}=\alpha_{c+1}$ and the proof is complete.

§ 3. The centres of 4-generated, relatively free groups of class four

The varieties of class at most four were fully described in the papers of P. FITZPATRICK and L. G. KOVACS. They have proved that the problem reduces to two cases: varieties whose free groups have no elements of order 2 and varieties whose free groups have no elements of odd order. Now we shall consider the first case i.e. (see [2]) isolated relatively free 4-nilpotent groups.

Theorem (FITZPATRICK and KOVACS [5]). *There is a $1-1$ correspondence between 2-isolated 4-nilpotent varieties and the 6-tuples (a, b, c, d, e, f), satisfying the conditions: $b|a$, $d|c$, $c|b$, $c|3d$, d being a common multiple of e and f, and if $3|a$ then $3d|a$. $a, b, \dots, f$ are natural numbers or 0.*

The basis of laws of such a variety is:

$$x^a=[x,y]^b=[x,y,z]^c=[x,y,y]^d=[x,y,y,x]^e=\big[[x,y],[z,t]\big]^f=$$

$$=[x,y,z,t]^{ef}=[x,y,z,t,u]=1.$$

Now we prove the following

Lemma. *In the varieties described above the following identities*

$$[x,y,y,y]^e=1 \tag{8}$$

$$[x,y,y,z]^\delta=1 \tag{9}$$

hold with e and δ being minimal and $\delta=ef$ or $=\frac{1ef}{3}$ when $3|f$.

PROOF. By the substitution xy for x in $[x,y,y,x]^e=1$ we have $[x,y,y,y]^e$ $[x,y,y,x]^e=1$ which implies $[x,y,y,y]^e=1$. Suppose now $[x,y,y,y]^m=1$. By substituting xy for y we obtain $[x,y,y,x]^m[x,y,x,y]^m=1$. Now using the equation $[x,y,z,t]=[x,y,t,z]\big[[x,y],[z,t]\big]$ holding in every group with commuting commutators of weight $\geqq 3$ we have $[x,y,y,x]^{2m}=$ so, for lack of elements of even order we get $[x,y,y,x]^m=1$ and thus $m\geqq c$, which ends the proof of (8). To prove (9) observe that the fact $[x,y,y,z]^\delta=1$ with δ-minimal implies $\delta=ef$ or $\delta=\frac{1}{3}ef$ follows from the proof of the theorem of Heineken (see HUPPERT [8] III 6, 9). Let now V denote the verbal subgroup generated by the word $[x, y, y, z]$. We prove that $V \cap F'' = F^3$, where F is a relatively free 4-generated group from a variety under consideration. Indeed

$$(10) \qquad 1 = [x, y, y, z] = [z, x, y, y]\big[[x, y], [y, z]\big].$$

On the other hand, using the Jacobi identity for $[x, y, z, y]$ we have

$$(11) \qquad 1 = [x, z, y, y,][z, y, x, y]\big[[x, y], [y, z]\big].$$

Multiplying (11) by (10) in the square we obtain

$$\big[[x, y], [y, z]\big]^3 = 1.$$

MAC DONALD [10] has shown that the commutator $\big[[x, y], [y, z]\big]$ generates verbally the second derived subgroup in a group without elements of even order.

On the other hand from (10) follows that the verbal subgroups generated by the commutators $[z, x, y, y]$ and $\big[[x, y], [y, z]\big]$ are equal modulo V. So the above mentioned result of Heineken makes $F'' \subset V$ impossible.

So if $3 | f$ then $[x, y, y, z]^{\frac{ef}{3}} = 1$.

Now we can prove the main theorem concerning the description of 4-generated relatively free 2-isolated 4-nilpotent groups.

Theorem 3. *The centres of 4-nilpotent 2-isolated relatively free groups on four letters coincide with the verbal subgroups given by the words:*

$$x^{\beta_1},\ [x, y]^{\beta_2},\ [x, y, z]^{\beta_3},\ [x, y, y]^{\beta_4},\ [x, y, z, t]$$

with $\beta_1 = b$ except the case when $t_3(b) = t_3(e) \neq 0$ or $t_3(b) = t_3(ef) \neq 0$ and $3 | f$, then $\beta_1 = 3b \cdot \beta_2 = c$ $\beta_3 = ef$ and $\beta_4 = ef$ except the case when $3 | f$, then $\beta_4 = \frac{1}{3} ef$.

PROOF. As in § 2 we observe that the factor group $F_4 | Z(F_4)$ is [2]-isolated in view of Hall's cited lemma. Leter we shall use the same arguments as in preceding paragraphs. There are identities:

$$1 = [x, y]^{\beta_1} [x, y, y]^{\binom{\beta_1}{2}} [x, y, y, y]^{\binom{\beta_1}{3}}$$

$$1 = [x, y, z]^{\beta_2}$$

$$1 = [x, y, z, t]^{\beta_3}$$

$$1 = [x, y, y, z]^{\beta_4}$$

which imply the system of equations:

$$\beta_1 = k_1 b$$

$$\frac{\beta_1(\beta_1-1)}{2} = k_2 d$$

(12)
$$\frac{\beta_1(\beta_1-1)(\beta_1-2)}{2\cdot 3} = k_3 e$$

$$\beta_2 = k_u c$$

$$\beta_3 = k_5 ef$$

$$\beta_4 = k_6 ef$$

or

$$\beta_4 = k_6' \frac{ef}{3}$$

in the case if $3|f$.

We can observe that the minimal natural numbers satisfying this system are as in theorem 3. So the proof is complete.

Corollary. In the considered c-nilpotent varieties c-generated relatively free groups F_c have centers equal to $\gamma_c(F_c)$ if and only if all torsion elements lie in $\gamma_c(F_c)$.

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